FOGLIO 2 - Svolgimento

ES. 1 Si utilizza la proprietà, supposta nota,
$$\left\| \int_a^b u(t)\,dt \right\|_{eucl.} \leq \int_a^b \| u(t) \|_{eucl.}\, dt$$
per la norma euclidea in $\mathbb{R}^n$.

$$\left| \int_\gamma f(z)\,dz \right| = \left| \int_{t_0}^{t_1} f(\gamma(t))\,\dot\gamma(t)\,dt \right|$$
$$\leq \int_{t_0}^{t_1} | f(\gamma(t))\,\dot\gamma(t) |\,dt =$$
$$= \int_{t_0}^{t_1} | f(\gamma(t))|\,|\dot\gamma(t)|\,dt \leq \int_{t_0}^{t_1} C\,|\dot\gamma(t)|\,dt$$
$$= C\,\ell(\gamma).$$

ES. 2

$$\gamma(t) = t e^{it} \qquad \dot\gamma = e^{it} + i t e^{it}$$

(i) $$\int_\gamma \bar z\,dz = \int_0^\pi t e^{-it}(e^{it} + ite^{it})\,dt$$
$$= \int_0^\pi (t + it^2)\,dt = \frac{1}{2}t^2 + \frac{1}{3}it^3 \Big|_0^\pi$$
$$= \frac{1}{2}\pi^2 + i\frac{1}{3}\pi^3$$

(ii) $$\int_\gamma z\,dz = \int_0^\pi t e^{it}(e^{it} + ite^{it})\,dt$$
$$= \int_0^\pi (t e^{2it} + i t^2 e^{2it})\,dt$$
$$= \int_0^\pi (t\cos 2t - t^2 \sin 2t)\,dt +$$
$$+ i\int_0^\pi (t \sin 2t + t^2 \cos 2t)\,dt = \ldots$$

(iii) $$\int_\gamma |z|\,dz = \int_0^\pi t\,(e^{it} + i t e^{it})\,dt =$$

$$= \int_0^\pi (t\cos t - t^2 \sin t)\,dt + i\int_0^\pi (t \sin t + t^2 \cos t)\,dt$$

$$= \dots$$

Es. 3

$$\gamma(t) = 3t^2 + 2t^3 i \qquad \dot\gamma(t) = 6(t + t^2 i)$$

$$|\dot\gamma(t)| = 6\sqrt{t^2 + t^4} = 6\,|t|\sqrt{1+t^2}$$

(i) $$\ell(\gamma) = \int_0^1 |\dot\gamma(t)|\,dt = \int_0^1 6t\sqrt{1+t^2}\,dt = 3\int_0^1 \sqrt{1+x}\,dx$$

$$= 3\cdot\frac{2}{3}(1+x)^{3/2}\Big|_0^1 = 2(2^{3/2} - 1)$$

$$= 2(2\sqrt{2} - 1)$$

(ii) $$\int_\gamma x\,dz = \int_0^1 3t^2\, 6(t + t^2 i)\,dt$$

$$= 18\int_0^1 t^3\,dt + i\,18\int_0^1 t^4\,dt$$

$$= \frac{9}{2} + i\,\frac{18}{5}$$

(iii) $$\int_\gamma z\,dz = \int_0^1 (3t^2 + 2t^3 i)\,6(t + t^2 i)\,dt$$

$$= \int_0^1 (18t^3 - 12t^5)\,dt + i \int_0^1 (18t^4 + 12t^4)\,dt$$

$$= \left(\frac{9}{2} - 2\right) + i\, 4 = \frac{5}{2} + 4i$$

$\boxed{\text{ES. 4}}$ Col calcolo dei residui è immediato.

Con gli strumenti sviluppati finora si fa così:

Intanto osserviamo che $\forall\ 0 < r < \varepsilon$,

$$\int_{|z|=\varepsilon} \frac{dz}{f(z)} = \int_{|z|=r} \frac{dz}{f(z)} \qquad \text{(faremo tendere } r \to 0\text{)}$$

osserviamo poi che $f(z) = f'(0)z + \varphi(z)z$,

con $\lim_{|z|\to 0} \varphi(z) = 0$.

ove si ha

$$\int_{|z|=r} \frac{dz}{f(z)} = \int_{|z|=r} \frac{1}{f'(0)z + \varphi(z)z}\,dz$$

$$= \int_{|z|=r} \frac{1}{f'(0)z}\,dz + \int_{|z|=r} \left(\frac{1}{f'(0)z + \varphi(z)z} - \frac{1}{f'(0)z}\right) dz$$

$$= \frac{2\pi i}{f'(0)} - \int_{|z|=r} \frac{\varphi(z)}{z\,f'(0)\left(f'(0) + \varphi(z)\right)}\,dz$$

Ora si ha

$$\left| \int_{|z|=r} \frac{\varphi(z)}{z f'(0) (f'(0) + \varphi(z))} dz \right|$$

$$\leq \left| \int_0^{2\pi} \frac{\varphi(re^{it}) \, ir e^{it}}{re^{it} f'(0) (f'(0) + \varphi(re^{it}))} dt \right|$$

$$\leq \int_0^{2\pi} \frac{C_r \cdot r}{r \, |f'(0)| (|f'(0)| - C_r)} dt \xrightarrow{r \to 0} 0$$

dove $C_r = \max_{|z| \leq r} |f(z)|$

orché $\lim_{r \to 0} C_r = 0$

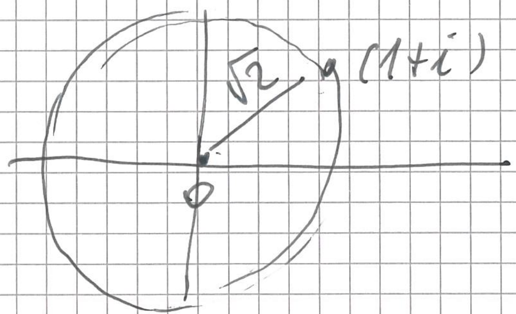
Quindi

$$\int_{|z|=\varepsilon} \frac{1}{f(z)} dz = \frac{2\pi i}{f'(0)}$$

$\boxtimes$

ES. 5

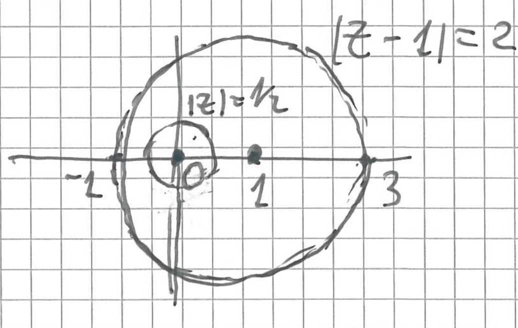
La serie $\sum_{n=0}^{\infty} c_n z^n$ è la serie di Taylor di f in 0. Pertanto, per il teorema sullo sviluppo in serie delle funzioni olomorfe, il raggio di convergenza è quello del più grande disco centrato in 0 e contenuto nel dominio della funzione.



$$\Rightarrow \quad R = \sqrt{2}$$

ES. 6

$$\int_{|z-1|=2} z^{-4} \sin z \, dz$$



Osserviamo che

$$z^{-4} \sin z = z^{-4} \left(z - \frac{1}{3!} z^3 + z^5 f(z) \right)$$

con f olomorfa in $\mathbb{C}$.

Quindi

$$z^{-4}\sin z = \underbrace{\frac{1}{z^3}}_{\substack{\text{ammette}\\ \text{primitiva}\\ -\frac{1}{4}\frac{1}{z^4}}} - \frac{1}{3!}\frac{1}{z} + \underbrace{z f(z)}_{\text{olomorfa}}$$

Quindi

$$\int_{|z-1|=2} z^{-4}\sin z\, dz = -\frac{1}{3!}\int_{|z-1|=2}\frac{1}{z}\,dz$$

$$= -\frac{1}{3!}\int_{|z|=\frac{1}{2}}\frac{1}{z}\,dz$$

$$= -\frac{1}{3!}\,2\pi i$$