FOGLIO 5 – Svolgimento

ES. 1

•) $$\mathrm{Ind}_{\bar\gamma}(\bar z) = \frac{1}{2\pi i}\int_a^b \frac{\bar\gamma'(t)}{\bar\gamma(t)-\bar z}\,dt = \overline{\left(-\frac{1}{2\pi i}\int_a^b \frac{\gamma'(t)}{\gamma(t)-z}\,dt\right)}$$
$$= \overline{-\mathrm{Ind}_\gamma(z)} = -\mathrm{Ind}_\gamma(z)$$

•) $$\mathrm{Ind}_{a\gamma+b}(az+b) = \frac{1}{2\pi i}\int_\alpha^\beta \frac{a\gamma'(t)}{(a\gamma(t)+b)-(az+b)}\,dt$$
$$= \frac{1}{2\pi i}\int_\alpha^\beta \frac{\gamma'(t)}{\gamma(t)-z}\,dt = \mathrm{Ind}_\gamma(z).$$

ES. 2

$$\int_{|z|=1} \frac{1}{z^2+2z}\,dz = ?$$

Scriviamo l'integrando come somma di frazioni semplici.

$$\frac{1}{z(z+2)} = \frac{A}{z} + \frac{B}{z+2}$$

$$A(z+2) + Bz = 1$$

$$\longrightarrow \begin{cases} A+B=0 \\ 2A=1 \end{cases} \longrightarrow \begin{cases} A = 1/2 \\ B=-1/2 \end{cases}$$

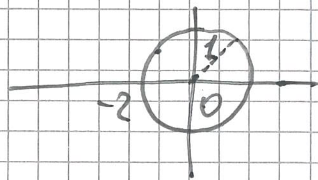
quindi $$\frac{1}{z(z+2)} = \frac{1/2}{z} - \frac{1/2}{z+2}$$

Segue che

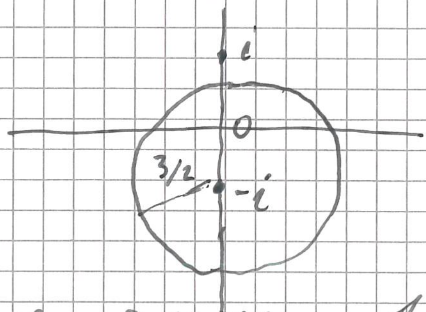
$$\int_{|z|=1} \frac{1}{z^2+2z}\,dz = \frac{1}{2}\int_{|z|=1} \frac{1}{z}\,dz - \frac{1}{2}\underbrace{\int_{|z|=1} \frac{1}{z+2}\,dz}_{=0}$$

(con $\gamma(t)=e^{it}$, $t\in[0,2\pi]$)

$$= \frac{1}{2}\cdot 2\pi i \cdot \mathrm{Ind}_\gamma(0) = \pi i$$



$$\int_{|z+i|=3/2} \frac{1}{z^4+z^2}\,dz = \int_{|z+i|=3/2} \frac{1}{z^2(z+i)(z-i)}\,dz = \,?$$



Scrivo l'integrando come somma di frazioni semplici.

$$\frac{1}{z^2(z+i)(z-i)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z+i} + \frac{D}{z-i}$$

$$Az(z+i)(z-i) + B(z+i)(z-i) + Cz^2(z-i) + Dz^2(z+i) = 1$$

$$(A+C+D)z^3 + (B-iC+iD)z^2 + Az + B = 1$$

$$\begin{cases} A+C+D=0 \\ B-iC+iD=0 \\ A=0 \\ B=1 \end{cases} \longrightarrow \begin{cases} A=0 \\ B=1 \\ C+D=0 \\ C-D=-i \end{cases} \longrightarrow \begin{cases} A=0 \\ B=1 \\ C=-i/2 \\ D=i/2 \end{cases}$$

quindi

$$\frac{1}{z^2(z+i)(z-i)} = \frac{1}{z^2} - \frac{i}{2}\,\frac{1}{z+i} + \frac{i}{2}\,\frac{1}{z-i}$$

e quindi

$$\int_{|z+i|=3/2} \frac{1}{z^2(z+i)(z-i)}\,dz = \underbrace{\int_{|z+i|=3/2} \frac{1}{z^2}}_{=0} - \frac{i}{2}\int_{|z+i|=3/2} \frac{1}{z+i}\,dz + \underbrace{\frac{i}{2}\int_{|z+i|=3/2} \frac{1}{z-i}\,dz}_{=0}$$

$$= -\frac{i}{2}\,2\pi i\,\frac{1}{2\pi i}\int_{|z+i|=3/2} \frac{1}{z+i}\,dz$$

$$= -\frac{i}{2}\,2\pi i\,\underbrace{\mathrm{Ind}_{|z+i|=3/2}(-i)}_{=1} = \pi$$

$\boxed{\text{ES. 3}}$

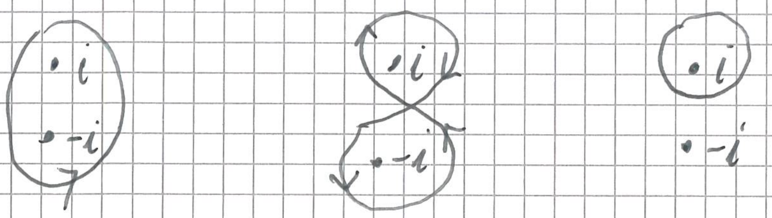
$$\int_\gamma \frac{1}{1+z^2}\,dz \qquad \gamma \text{ curva in } \mathbb{C}\setminus\{\pm i\}$$

$$= \frac{i}{2}\int_\gamma \frac{1}{z+i}\,dz - \frac{i}{2}\int_\gamma \frac{1}{z-i}\,dz$$

$$= \frac{i}{2}\,2\pi i\,\left(\mathrm{Ind}_\gamma(-i) - \mathrm{Ind}_\gamma(i)\right)$$

$$= -\pi\,(\underbrace{k+h}_{k,h\in\mathbb{Z}}) = \pi j \qquad j \in \mathbb{Z}.$$

a seconda di quante volte γ gira intorno a i e $-i$



etc.

Analogamente

$$\int_\gamma \frac{1}{z(z^2-1)}\,dz = -\int_\gamma \frac{1}{z}\,dz + \frac{1}{2}\int_\gamma \frac{1}{z+1}\,dz + \frac{1}{2}\int_\gamma \frac{1}{z-1}\,dz$$

$$= -2\pi i \,\mathrm{Ind}_\gamma(0) + \pi i \,\mathrm{Ind}_\gamma(1) + \pi i \,\mathrm{Ind}_\gamma(-1)$$

$$= -2\pi i k + \pi i h + \pi i l \qquad k, h, l \in \mathbb{Z}$$

$$= \pi i m \quad , \qquad m \in \mathbb{Z}.$$

ES. 4

$\gamma(s) = e^{ims} + r e^{iks} \qquad s \in [0, 2\pi]$

$0 < r \quad , \quad r \neq 1 \quad ; \quad m, k \in \mathbb{Z}$

$z_0 = 0$

$\mathrm{Ind}_\gamma(0) = m$ se $r < 1$

$\mathrm{Ind}_\gamma(0) = k$ se $r > 1$

Infatti:

se $r < 1$, $|\gamma(s) - e^{ims}| = |r e^{iks}| = r < 1 = |e^{ims}|$

se $r > 1$, $|\gamma(s) - r e^{iks}| = |e^{ims}| = 1 < r = |r e^{iks}|$

e si applica il lemma del cane al guinzaglio.