FOGLIO 7 - svolgimento

ES. 1

$$\int_0^{+\infty} \frac{x^2}{x^4+6x^2+13}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{x^2}{x^4+6x^2+13}\,dx$$

Determiniamo i poli e i rispettivi residui di

$$\frac{z^2}{z^4+6x^2+13}$$

pongo $z^2 = w$

$$w^2+6w+13=0$$

$$w_{1,2} = \frac{6 \pm i\sqrt{52-36}}{2} = \frac{-6 \pm i\sqrt{16}}{2} = \frac{-6 \pm 4i}{2}$$

$$= \begin{cases} -3+2i \\ -3-2i \end{cases}$$

$$w_1 = \sqrt{13}\left(-\frac{3}{\sqrt{13}} + \frac{2}{\sqrt{13}}i\right) = \sqrt{13}\, e^{i\vartheta^*}$$

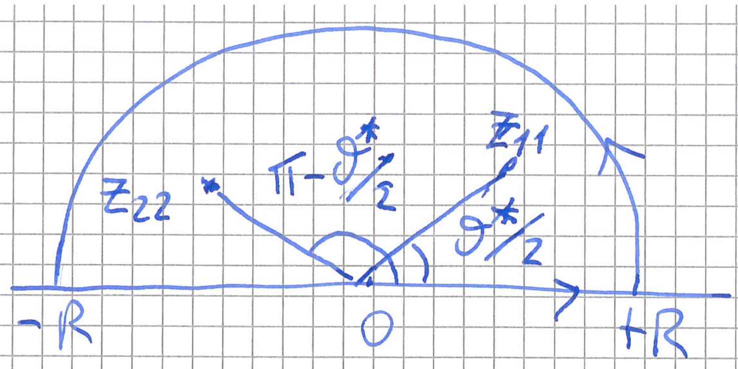
$$\vartheta^* \in \left]-\frac{\pi}{2}, \pi\right[$$

$$w_2 = \sqrt{13}\left(-\frac{3}{\sqrt{13}} - \frac{2}{\sqrt{13}}i\right) = \sqrt{13}\, e^{-i\vartheta^*}$$

Quindi i poli sono:

$$z_{11} = \sqrt[4]{13}\, e^{i\vartheta^*/2}, \quad z_{12} = \sqrt[4]{13}\, e^{i(\vartheta^*/2+\pi)}$$

$$z_{21} = \sqrt[4]{13}\, e^{-i\vartheta^*/2}, \quad z_{22} = \sqrt[4]{13}\, e^{i(-\vartheta^*/2+\pi)}$$



$$\int_{-\infty}^{+\infty} \frac{z^2}{z^4+6z^2+13}\,dz = 2\pi i\left(\mathrm{Res}(f, z_{11}) + \mathrm{Res}(f, z_{22})\right)$$

$$\mathrm{Res}(f, z_{11}) = \left.\frac{z^2}{4z^3+12z}\right|_{z=z_{11}}$$

$$= \frac{z_{11}}{4z_{11}^2+12} =$$

$$= \frac{\sqrt[4]{13}\, e^{i\vartheta^*/2}}{4(-3+2i)+12}$$

$$= \frac{1}{8i} \sqrt[4]{13}\, e^{i\vartheta^*/2}$$

$$\mathrm{Res}(f, z_{22}) = \frac{z_{22}}{4z_{22}^2+13}$$

$$= \frac{\sqrt[4]{13}\, e^{i(\vartheta^*/2-\pi)}}{4(-3-2i)+12}$$

$$= \frac{1}{8i} \sqrt[4]{13}\, e^{-i\vartheta^*/2}$$

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{x^2}{x^4+6x^2+13}\,dx = 2\pi i \cdot \frac{1}{8i}\sqrt[4]{13}\left(e^{i\vartheta^*/2}+e^{-i\vartheta^*/2}\right) \quad 3$$

$$= \frac{\pi}{4}\sqrt[4]{13}\cos\left(\vartheta^*/2\right)$$

$$= \frac{\pi}{4}\sqrt[4]{13}\sqrt{\frac{1+\cos\vartheta^*}{2}}$$

$$= \frac{\pi}{4}\sqrt[4]{13}\sqrt{\frac{1-3/\sqrt{13}}{2}}$$

$$= \frac{\pi}{4}\sqrt[4]{13}\sqrt{\frac{\sqrt{13}-3}{2\sqrt{13}}}$$

$$= \frac{\pi}{4}\sqrt{\frac{13-3\sqrt{13}}{2\sqrt{13}}}$$

$$= \frac{\pi}{4}\sqrt{\frac{13\sqrt{13}-39}{26}} = \frac{\pi}{4}\sqrt{\frac{\sqrt{13}-3}{2}}$$

$\boxed{\text{ES. 2}}$

$$\int_0^{+\infty}\frac{\cos x}{x^2+a^2}\,dx \quad , \quad a>0.$$

devo calcolare

$$\int_{-\infty}^{+\infty}\frac{e^{ix}}{x^2+a^2}\,dx$$

e quindi considerare la funzione

$\dfrac{e^{iz}}{z^2+a^2}$. I poli sono $\pm ia$.

quindi $$\int_{-\infty}^{+\infty}\frac{e^{ix}}{x^2+a^2}\,dx = 2\pi i \operatorname{Res}\left(\frac{e^{iz}}{x^2+a^2}, ia\right)$$

$$= 2\pi i \left. \frac{e^{iz}}{2\cancel{z}} \right|_{\cancel{z}=ia} = 2\pi i \frac{e^{-a}}{2ia} = \frac{\pi e^{-a}}{a}$$ 4

e quindi

$$\int_0^{+\infty} \frac{\cos x}{x^2+a^2}\, dx = \frac{\pi e^{-a}}{2a}$$

$\boxed{\text{ES. 3}}$

$$\int_0^{+\infty} \frac{x \sin x}{x^2+a^2}\, dx \qquad a>0$$

devo calcolare $\displaystyle\int_{-\infty}^{+\infty} \frac{x e^{ix}}{x^2+a^2}\, dx$

e quindi considerare la funzione

$$\frac{z e^{iz}}{z^2+a^2}.$$

I poli sono $\pm ia$

quindi

$$\int_{-\infty}^{+\infty} \frac{x \, e^{ix}}{x^2+a^2}\, dx = 2\pi i \operatorname{Res}\left(\frac{z e^{iz}}{z^2+a^2}, ia \right)$$

$$= 2\pi i \left. \frac{z e^{iz}}{2z} \right|_{z=ia} = \frac{2\pi i e^{-a}}{2} = \pi i e^{-a}$$

$$\Longrightarrow \int_{-\infty}^{+\infty} \frac{x \sin x}{x^2+a^2} = \pi e^{-a}$$

$$\Longrightarrow \int_0^{+\infty} \frac{x \sin x}{x^2+a^2}\, dx = \frac{\pi e^{-a}}{2}.$$

ES 4

$$\int_0^{+\infty} \frac{\sin x}{x}\,dx$$

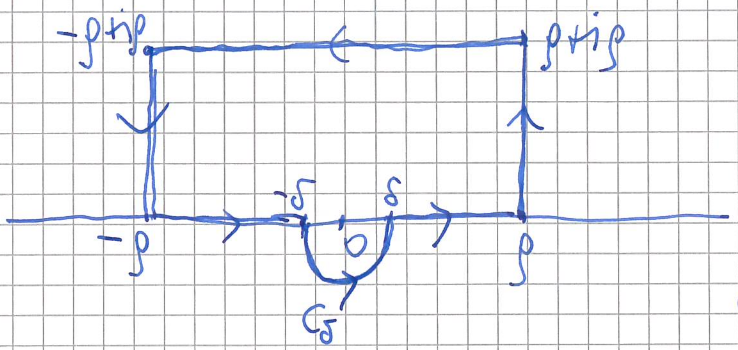
l'integrando è pari

Inoltre $\frac{\sin x}{x} \xrightarrow{x \to 0} 1$

L'idea è che $\displaystyle\int_0^{+\infty} \frac{\sin x}{x}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\sin x}{x}\,dx$

e calcolare $\displaystyle\int_{-\infty}^{+\infty} \frac{e^{ix}}{x}\,dx$.

C'è però un problema: $\frac{e^{ix}}{x}$ ha un polo in 0. Allora prendiamo un cammino fatto così:



$\gamma_{\rho,\delta}$

dove $C_\delta = \delta e^{i\vartheta}$, $\vartheta \in [-\pi, 0]$.

Si ha che

$$\int_{\gamma_{\rho,\delta}} \frac{e^{iz}}{z} = 2\pi i \,\mathrm{Res}\left(\frac{e^{iz}}{z}, 0\right) = 2\pi i \, e^{iz}\Big|_{z=0} = 2\pi i$$

$$= \int_{[-\rho,\delta]} + \int_{C_\delta} + \int_{[\delta,\rho]} + \underbrace{\int_{[\rho,\rho+i\rho]} + \int_{[\rho+i\rho,-\rho+i\rho]} + \int_{[-\rho+i\rho,\rho]}}_{\longrightarrow 0}$$

per $\rho \to +\infty$

Abbiamo

$$\int_{C_\delta} \frac{e^{iz}}{z}\,dz = \int_{C_\delta} \frac{1}{z}\left(1 + iz + \frac{1}{2}(iz)^2 + \cdots\right) dz$$

$$= \int_{C_\delta} \left[\frac{1}{z} + h(z)\right] dz \qquad h \text{ olomorfa su } \mathbb{C}$$

$$= \int_{C_\delta} \frac{1}{z}\,dz + \int_{C_\delta} h(z)\,dz$$

$$= \int_{-\pi}^{0} \frac{1}{\delta e^{i\theta}}\, i\delta e^{i\theta}\,d\theta + \int_{C_\delta} h(z)\,dz$$

$$= \pi i + \underbrace{\int_{C_\delta} h(z)\,dz}_{\to 0 \text{ se } \delta \to 0}$$

Segue che

$$\int_{-\infty}^{+\infty} \frac{e^{ix}}{x} + \pi i = 2\pi i$$

da cui $\displaystyle\int_{-\infty}^{+\infty} \frac{e^{ix}}{x} = \pi i$

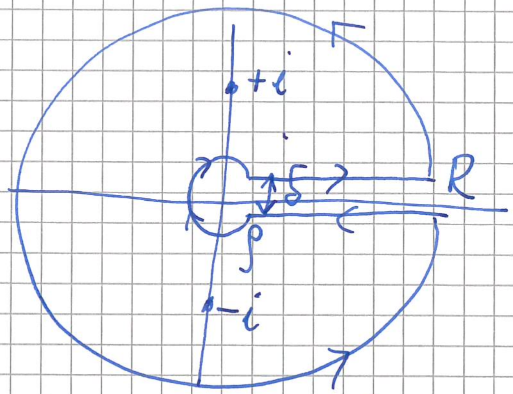
e quindi $\displaystyle\int_{-\infty}^{+\infty} \frac{\sin x}{x} = \pi$, $\displaystyle\int_{0}^{+\infty} \frac{\sin x}{x} = \frac{\pi}{2}$

(N.B. : è compatibile con l'esercizio precedente, se $a \to 0$).

ES. 5

$$\int_0^{+\infty} \frac{\sqrt{x}}{1+x^2}\, dx$$

Prendo un cammino così:



$\gamma_{R,\rho,\delta}$

$$\sqrt{z} = e^{\frac{1}{2}\log z} = e^{\frac{1}{2}(\log|z| + i \operatorname{Arg} z)}$$

$\log z$ ramo principale di logaritmo su $\mathbb{C} \setminus [0,+\infty[$, $\operatorname{Arg} \in]0, 2\pi[$

$$\int_{\gamma_{R,\rho,\delta}} \frac{\sqrt{z}}{1+z^2} = 2\pi i \left(\operatorname{Res}\left(\frac{\sqrt{z}}{z^2+1}, +i\right) + \operatorname{Res}\left(\frac{\sqrt{z}}{z^2+1}, -i\right) \right)$$

$$= 2\pi i \left(\left.\frac{\sqrt{z}}{2z}\right|_{z=+i} + \left.\frac{\sqrt{z}}{2z}\right|_{z=-i} \right)$$

$$= 2\pi i \left(\frac{\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}}{2i} + \frac{-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}}{-2i} \right)$$

$$= \frac{2\pi i}{2i} \left(\underbrace{\frac{\sqrt{2}}{2}}_{\text{sì}} + \cancel{i\frac{\sqrt{2}}{2}} + \underbrace{\frac{\sqrt{2}}{2}}_{\text{sì}} - \cancel{i\frac{\sqrt{2}}{2}} \right)$$

$$= \pi\sqrt{2}$$

Ora osserviamo che

$$\int_{\gamma_{R,\rho,\delta}} \frac{\sqrt{z}}{1+z^2} \underset{\text{per } \delta\to 0}{=} \underbrace{\int_{|z|=\rho} \frac{\sqrt{z}}{1+z^2}\,dz}_{\xrightarrow[\rho\to 0]{} 0} + \underbrace{\int_{|z|=R} \frac{\sqrt{z}}{1+z^2}\,dz}_{\xrightarrow[R\to+\infty]{} 0} + \int_{-\rho}^{R} \frac{\sqrt{x}}{1+x^2}\,dx - \int_{-\rho}^{R} \frac{\sqrt{x}}{1+x^2} e^{\pi i}\,dx$$

$$\Rightarrow (1-e^{\pi i}) \int_0^{+\infty} \frac{\sqrt{x}}{1+x^2} = \pi\sqrt{2}$$

$$\Rightarrow \int_0^{+\infty} \frac{\sqrt{x}}{1+x^2}\,dx = \frac{\pi\sqrt{2}}{2}$$

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$\boxed{\text{ES. 6}}$

$$\int_0^{\pi/2} \frac{1}{1+\sin^2 x}\,dx = \frac{1}{4}\int_0^{2\pi} \frac{1}{1+[(e^{ix}-e^{-ix})/2i]^2} \frac{1}{ie^{ix}}\, ie^{ix}\,dx$$

$$= \frac{1}{4}\oint_{|z|=1} \frac{1}{iz} \frac{1}{1+\left(z-\frac{1}{z}\right)^2\left(-\frac{1}{4}\right)}\,dz$$

$$= \oint_{|z|=1} \frac{1}{iz} \frac{1}{4-\left(z-\frac{1}{z}\right)^2}\,dz$$

$$= \oint_{|z|=1} \frac{1}{iz} \frac{z^2}{4z^2-(z^2-1)^2}\,dz$$

$$= \oint_{|z|=1} \frac{1}{i} \frac{z}{(2z-z^2+1)(2z+z^2-1)}\,dz$$

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$$= \frac{1}{i} 2\pi i \left(\operatorname{Res}(f, 1-\sqrt{2}) + \operatorname{Res}(f, -1+\sqrt{2}) \right)$$

$z^2 + 2z - 1 = 0$

$z = \frac{-2 \pm \sqrt{8}}{2}$

$= -1 \pm \sqrt{2}$

$z^2 - 2z - 1 = 0$

$z = \frac{2 \pm \sqrt{8}}{2}$

$= 1 \pm \sqrt{2}$

$$= 2\pi \left[\frac{z}{8z - 4(z^2-1)z} \bigg|_{z=1-\sqrt{2}} + \frac{z}{8z - 4(z^2-1)z} \bigg|_{z=-1+\sqrt{2}} \right]$$

$$= 2\pi \left[\frac{1}{8 - 4(1 - 2\sqrt{2} + 2 - 1)} + \frac{1}{8 - 4(1 - 2\sqrt{2} + 2 - 1)} \right]$$

$$= 2\pi \left[\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} \right] = \pi\sqrt{2}$$