ASCOLI-ARZELA

Theorem 3.6.4 *Let $\mathcal{K}$ be a family of functions, uniformly bounded and equicontinuous on a compact metric space X. Then any sequence $\{f_n\}$ of functions of $\mathcal{K}$ has a subsequence that is uniformly convergent in X to a continuous function.*

Proof. Let $\{x_m\}$ be a dense sequence in X. Since the sequence $\{f_n(x_1)\}$ is bounded, there is a subsequence $\{f_{n,1}(x_1)\}$ that is convergent. Next, the sequence $\{f_{n,1}(x_2)\}$ is bounded. Hence it has a subsequence $\{f_{n,2}(x_2)\}$ that is convergent. We proceed in this way step by step. In the kth step, we extract a convergent subsequence $\{f_{n,k}(x_k)\}$ of the bounded sequence $\{f_{n,k-1}(x_k)\}$. Consider now the diagonal sequence $\{f_{n,n}\}$ of the double sequence $\{f_{n,k}\}$, and write $g_n = f_{n,n}$. Then $\{g_n(x_k)\}$ is convergent for every k, since, except for the first k terms, it is a subsequence of $\{f_{n,k}(x_k)\}$.

We shall prove that $\{g_n\}$ is uniformly convergent in X. Since the family $\{g_n\}$ is equicontinuous, for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|g_n(x) - g_n(y)| < \varepsilon \qquad \text{whenever } \rho(x,y) < \delta,$$

for all x,y in X and $1 \le n < \infty$. For any $x \in X$ we now write

$$|g_n(x) - g_m(x)| \le |g_n(x) - g_n(x_k)| + |g_n(x_k) - g_m(x_k)| + |g_m(x_k) - g_m(x)|,$$

where x_k is such that $\rho(x,x_k) < \delta$. Then

$$|g_n(x) - g_m(x)| < 2\varepsilon + |g_n(x_k) - g_m(x_k)|. \tag{3.6.4}$$

We now claim that there is a finite number of the points x_k, say $x_1,\ldots,x_h$, such that for any $x \in X$ there is a point x_k with $1 \le k \le h$ such that $\rho(x,x_k) < \delta$. Indeed, take a finite $(\delta/2)$-covering of X by balls $B_1,\ldots,B_p$ and choose in each ball B_j a point x_{α_j} from the sequence $\{x_m\}$. Then the claim above holds with $h = \max(\alpha_1,\ldots,\alpha_p)$.

For each k, $1 \le k \le h$, there is a positive integer n_k such that

$$|g_m(x_k) - g_n(x_k)| < \varepsilon \qquad \text{if } m \ge n \ge n_k.$$

Using this in (3.6.4), we get $|g_n(x) - g_m(x)| < 3\varepsilon$ if $m \ge n \ge \bar{n}$, where $\bar{n} = \max(n_1,\ldots,n_h)$. Thus $\{g_n\}$ is uniformly convergent. Denote by $f(x)$ the uniform limit of $\{g_n(x)\}$. Then, for any $\varepsilon > 0$,

$$|f(x) - f(y)| \le |f(x) - g_n(x)| + |g_n(x) - g_n(y)| + |g_n(y) - f(y)|$$
$$< 2\varepsilon + |g_n(x) - g_n(y)|$$

if n is sufficiently large. We now fix n. The uniform continuity of g_n then implies that $|g_n(x) - g_n(y)| < \varepsilon$ if $\rho(x,y) < \delta$. Hence $|f(x) - f(y)| < 3\varepsilon$ if $\rho(x,y) < \delta$. Thus, $f(x)$ is continuous, and the proof of the theorem is complete.

da A. Friedman, FOUNDATIONS OF MODERN ANALYSIS