da Rudin, "Real and complex analysis"

PROOF For $n = 1, 2, 3, \ldots$, put

$$V_n = D(\infty; n) \cup \bigcup_{a \notin \Omega} D\left(a; \frac{1}{n}\right) \tag{1}$$

and put $K_n = S^2 - V_n$. [Of course, $a \neq \infty$ in (1).] Then K_n is a closed and bounded (hence compact) subset of Ω, and $\Omega = \bigcup K_n$. If $z \in K_n$ and $r = n^{-1} - (n+1)^{-1}$, one verifies easily that $D(z; r) \subset K_{n+1}$. This gives (*a*). Hence Ω is the union of the interiors W_n of K_n. If K is a compact subset of Ω, then $K \subset W_1 \cup \cdots \cup W_N$ for some N, hence $K \subset K_N$.

Finally, each of the discs in (1) intersects $S^2 - \Omega$; each disc is connected; hence each component of V_n intersects $S^2 - \Omega$; since $V_n \supset S^2 - \Omega$, no component of $S^2 - \Omega$ can intersect two components of V_n. This gives (*c*). ////

13.4 Sets of Oriented Intervals Let Φ be a finite collection of oriented intervals in the plane. For each point p, let $m_I(p)[m_E(p)]$ be the number of members of Φ that have initial point [end point] p. If $m_I(p) = m_E(p)$ for every p, we shall say that Φ is *balanced*.

If Φ is balanced (and nonempty), the following construction can be carried out.

Pick $\gamma_1 = [a_0, a_1] \in \Phi$. Assume $k \geq 1$, and assume that distinct members $\gamma_1, \ldots, \gamma_k$ of Φ have been chosen in such a way that $\gamma_i = [a_{i-1}, a_i]$ for $1 \leq i \leq k$. If $a_k = a_0$, stop. If $a_k \neq a_0$, and if precisely r of the intervals $\gamma_1, \ldots, \gamma_k$ have a_k as end point, then only $r - 1$ of them have a_k as initial point; since Φ is balanced, Φ contains at least one other interval, say γ_{k+1}, whose initial point is a_k. Since Φ is finite, we must return to a_0 eventually, say at the nth step.

Then $\gamma_1, \ldots, \gamma_n$ join (in this order) to form a closed path.

The remaining members of Φ still form a balanced collection to which the above construction can be applied. It follows that the members of Φ can be so numbered that they form finitely many closed paths. The sum of these paths is a cycle. The following conclusion is thus reached.

If $\Phi = \{\gamma_1, \ldots, \gamma_N\}$ is a balanced collection of oriented intervals, and if

$$\Gamma = \gamma_1 \dotplus \cdots \dotplus \gamma_N$$

then Γ is a cycle.

13.5 Theorem *If K is a compact subset of a plane open set Ω $(\neq \varnothing)$, then there is a cycle Γ in $\Omega - K$ such that the Cauchy formula*

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta \tag{1}$$

holds for every $f \in H(\Omega)$ and for every $z \in K$.

PROOF Since K is compact and Ω is open, there exists an $\eta > 0$ such that the distance from any point of K to any point outside Ω is at least 2η. Construct a grid of horizontal and vertical lines in the plane, such that the distance between any two adjacent horizontal lines is η, and likewise for the vertical lines. Let $Q_1, \ldots, Q_m$ be those squares (closed 2-cells) of edge η which are formed by this grid and which intersect K. Then $Q_r \subset \Omega$ for $r = 1, \ldots, m$.

If a_r is the center of Q_r and $a_r + b$ is one of its vertices, let γ_{rk} be the oriented interval

$$\gamma_{rk} = [a_r + i^k b, a_r + i^{k+1} b] \tag{2}$$

and define

$$\partial Q_r = \gamma_{r1} \dotplus \gamma_{r2} \dotplus \gamma_{r3} \dotplus \gamma_{r4} \qquad (r = 1, \ldots, m). \tag{3}$$

It is then easy to check (for example, as a special case of Theorem 10.37, or by means of Theorems 10.11 and 10.40) that

$$\mathrm{Ind}_{\partial Q_r}(\alpha) = \begin{cases} 1 & \text{if } \alpha \text{ is in the interior of } Q_r, \\ 0 & \text{if } \alpha \text{ is not in } Q_r. \end{cases} \tag{4}$$

Let Σ be the collection of all γ_{rk} $(1 \le r \le m, 1 \le k \le 4)$. It is clear that Σ is balanced. Remove those members of Σ whose opposites (see Sec. 10.8) also belong to Σ. Let Φ be the collection of the remaining members of Σ. Then Φ is balanced. Let Γ be the cycle constructed from Φ, as in Sec. 13.4.

If an edge E of some Q_r intersects K, then the two squares in whose boundaries E lies intersect K. Hence Σ contains two oriented intervals which are each other's opposites and whose range is E. These intervals do not occur in Φ. Thus Γ is a cycle in $\Omega - K$.

The construction of Φ from Σ shows also that

$$\mathrm{Ind}_\Gamma(\alpha) = \sum_{r=1}^{m} \mathrm{Ind}_{\partial Q_r}(\alpha) \tag{5}$$

if α is not in the boundary of any Q_r. Hence (4) implies

$$\mathrm{Ind}_\Gamma(\alpha) = \begin{cases} 1 & \text{if } \alpha \text{ is in the interior of some } Q_r, \\ 0 & \text{if } \alpha \text{ lies in no } Q_r. \end{cases} \tag{6}$$

If $z \in K$, then $z \notin \Gamma^*$, and z is a limit point of the interior of some Q_r. Since the left side of (6) is constant in each component of the complement of Γ^*, (6) gives

$$\mathrm{Ind}_\Gamma(z) = \begin{cases} 1 & \text{if } z \in K, \\ 0 & \text{if } z \notin \Omega. \end{cases} \tag{7}$$

Now (1) follows from Cauchy's theorem 10.35. ////