①

f holomorphe sur $D(0,R)$

$0 < r < R$

Si $|z| < r$,

$$f(z) = \frac{1}{2\pi i} \oint_{\partial D(0,r)} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

$$= \frac{1}{2\pi i} \oint_{\partial D(0,r)} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \int_{\partial D(0,r)} \frac{f(\zeta)}{\zeta - z^*}\, d\zeta$$

donc $z^* = \frac{r^2}{\bar{z}}$

$$|z^*| = \frac{r^2}{|\bar{z}|} > \frac{r^2}{r} = r$$

$$= \frac{1}{2\pi i} \oint_{\partial D(0,r)} \left(\frac{1}{(\zeta - z)} - \frac{1}{\zeta - z^*} \right) f(\zeta)\, d\zeta$$

$$= \frac{1}{2\pi i} \oint_{\partial D(0,r)} \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - z^*} \right) f(\zeta) \Big|_{\zeta = re^{it}} \cdot ire^{it}\, dt$$

$$= \frac{1}{2\pi r} \oint_{\partial D(0,R)} \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - z^*} \right) \zeta\, f(\zeta)\, |d\zeta|$$

Ora

$$\frac{1}{\zeta - z} - \frac{1}{\zeta - z^*} = \frac{z - z^*}{(\zeta - z)(\zeta - z^*)}$$

$$= \frac{z - \frac{r^2}{\bar{z}}}{(\zeta - z)\left(\zeta - \frac{r^2}{\bar{z}}\right)}$$

$$= \frac{|z|^2 - r^2}{(\zeta - z)(\bar{z}\zeta - r^2)}$$

$$= \frac{|z|^2 - r^2}{(\zeta - z)(\bar{z}\zeta - \bar{\zeta}\zeta)}$$

$$= \frac{-|z|^2 + r^2}{\zeta\,|\zeta - z|^2}$$

$\Longrightarrow$

$$f(z) = \frac{1}{2\pi r} \oint_{\partial D(0,r)} \frac{-|z|^2 + r^2}{|\zeta - z|^2} f(\zeta)\,|d\zeta|$$

$$f(z) = \frac{1}{2\pi r} \oint_{\partial D(0,r)} \frac{r^2 - |z|^2}{|\zeta - z|^2} f(\zeta)\,|d\zeta|$$

Sia $u \in C^2(D(0,R)) \cap C(\overline{D(0,R)})$ armonica.

Sia v armonica coniugata di u in $D(0,R)$. Sia $f = u + iv$.

Allora $\forall\ 0 < r < R$, $\forall\ z$, $|z| < r$,

$$f(z) = \frac{1}{2\pi r} \oint_{\partial D(0,r)} \frac{r^2 - |\zeta|^2}{|\zeta - z|^2} f(\zeta)\,|d\zeta|$$

e quindi

$$u(z) = \frac{1}{2\pi r} \oint_{\partial D(0,r)} \frac{r^2 - |\zeta|^2}{|\zeta - z|^2} u(\zeta)\,|d\zeta|$$

Per $r \to R$, si ottiene

$$\boxed{u(z) = \frac{1}{2\pi R} \int_{\partial D(0,R)} \frac{R^2 - |\zeta|^2}{|\zeta - z|^2} u(\zeta)\,|d\zeta|}$$