3

RADIATION FROM MOVING CHARGES

3.1 RETARDED POTENTIALS OF SINGLE MOVING CHARGES: THE LIÉNARD-WIECHART POTENTIALS

Consider a particle of charge q that moves along a trajectory $\mathbf{r}=\mathbf{r}_0(t)$. Its velocity at any time is then $\mathbf{u}(t)=\dot{\mathbf{r}}_0(t)$. The charge and current densities are given by

$$\rho(\mathbf{r},t)=q\delta(\mathbf{r}-\mathbf{r}_0(t)), \tag{3.1a}$$

$$\mathbf{j}(\mathbf{r},t)=q\mathbf{u}(t)\delta(\mathbf{r}-\mathbf{r}_0(t)). \tag{3.1b}$$

The δ-function has the property of localizing the charge and current; we also obtain the proper total charge and current by integrating over volume:

$$q=\int\rho(\mathbf{r},t)d^3\mathbf{r},$$

$$q\mathbf{u}=\int\mathbf{j}(\mathbf{r},t)d^3\mathbf{r}.$$

Let us calculate the retarded potentials [Eq. (2.67)] due to these charge and current densities. We use the scalar potential as an example:

$$\phi(\mathbf{r},t)=\int d^3\mathbf{r}'\int dt'\frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta(t'-t+|\mathbf{r}-\mathbf{r}'|/c), \tag{3.2}$$

using the property of the δ-function. Substitution of Eq. (3.1a) for the charge density and integration over $\mathbf{r}'$ yields

$$\phi(\mathbf{r},t)=q\int\delta(t'-t+|\mathbf{r}-\mathbf{r}_0(t')|/c)\frac{dt'}{|\mathbf{r}-\mathbf{r}_0(t')|}.$$

This is now an integral over the single variable t'. We now introduce the notations

$$\mathbf{R}(t')=\mathbf{r}-\mathbf{r}_0(t'),\qquad R(t')=|\mathbf{R}(t')|. \tag{3.3}$$

We then have

$$\phi(\mathbf{r},t)=q\int R^{-1}(t')\delta(t'-t+R(t')/c)dt', \tag{3.4a}$$

$$\mathbf{A}(\mathbf{r},t)=\frac{q}{c}\int\mathbf{u}(t')R^{-1}(t')\delta(t'-t+R(t')/c)dt', \tag{3.4b}$$

where we have performed the identical integrations for $\mathbf{A}$. Equations (3.4) are useful forms for the potentials, but they may be simplified still further. Note that the argument of the δ-function vanishes for a value of $t'=t_{\text{ret}}$ given by

$$c(t-t_{\text{ret}})=R(t_{\text{ret}}). \tag{3.5}$$

Let us change variables from t' to $t''=t'-t+[R(t')/c]$, which implies that

$$dt''=dt'+\frac{1}{c}\dot{R}(t')dt'.$$

Since $R^2(t')=\mathbf{R}^2(t')$, it follows that $2R(t')\dot{R}(t')=-2\mathbf{R}(t')\cdot\mathbf{u}(t')$, where $\dot{\mathbf{R}}(t')=-\mathbf{u}(t')$. We also define the unit vector $\mathbf{n}$ by

$$\mathbf{n}=\frac{\mathbf{R}}{R}$$

Finally, we obtain

$$dt'' = \left[1 - \frac{1}{c}\mathbf{n}(t')\cdot\mathbf{u}(t')\right]dt',$$

$$\phi(\mathbf{r},t) = q\int R^{-1}(t')\left[1 - \frac{1}{c}\mathbf{n}(t')\cdot\mathbf{u}(t')\right]^{-1}\delta(t'')dt''.$$

Now the integration over the δ-function can be performed by setting $t''=0$, or equivalently by setting $t'=t_{\text{ret}}$. This yields

$$\phi(\mathbf{r},t) = \frac{q}{\kappa(t_{\text{ret}})R(t_{\text{ret}})}$$

where we have used the notation

$$\kappa(t') = 1 - \frac{1}{c}\mathbf{n}(t')\cdot\mathbf{u}(t'). \tag{3.6}$$

Then, with the brackets denoting retarded times, we have

$$\phi = \left[\frac{q}{\kappa R}\right] \tag{3.7a}$$

$$\mathbf{A} = \left[\frac{q\mathbf{u}}{c\kappa R}\right]. \tag{3.7b}$$

These are called the *Liénard–Wiechart potentials*. These potentials differ from those of static electromagnetic theory in two ways: First, there is the factor $\kappa = 1-(\mathbf{n}\cdot\mathbf{u}/c)$. This factor becomes very important at velocities close to that of light, where it tends to concentrate the potentials into a narrow cone about the particle velocity. It is related to the *beaming effect* found in the Lorentz transformation of photon direction of propagation. (See Chapter 4.)

The second difference is that the quantities are all to be evaluated at the retarded time t_{ret}. We have already discussed the meaning of this. The major consequence of retardation is that it makes it possible for a particle to radiate. The potentials roughly fall off as $1/r$ so that differentiation to find the fields would give a $1/r^2$ decrease if this differentiation acted solely on the $1/r$ factor. As we show in the following section retardation allows an implicit dependence on position to occur via the definition of retarded time, and differentiation with respect to this dependence carries the $1/r$ behavior of the potentials into the fields themselves. We have seen that this allows radiation energy to flow to infinite distances.

3.2 THE VELOCITY AND RADIATION FIELDS

The differentiations of the potentials to obtain the fields are straightforward but lengthy, and we omit details (see Jackson, §14.1). The results are as follows: If we want the fields at point r at time t we first must determine the retarded position and time of the particle r_{ret} and t_{ret}. At this time the particle has velocity $\mathbf{u}=\dot{\mathbf{r}}_0(t_{\text{ret}})$ and acceleration $\dot{\mathbf{u}}=\ddot{\mathbf{r}}_0(t_{\text{ret}})$. We introduce the notation

$$\boldsymbol{\beta}\equiv\frac{\mathbf{u}}{c}, \qquad \kappa\equiv 1-\mathbf{n}\cdot\boldsymbol{\beta}. \tag{3.8}$$

Then the fields are

$$\mathbf{E}(\mathbf{r},t)=q\left[\frac{(\mathbf{n}-\boldsymbol{\beta})(1-\beta^2)}{\kappa^3 R^2}\right]+\frac{q}{c}\left[\frac{\mathbf{n}}{\kappa^3 R}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\right], \tag{3.9a}$$

$$\mathbf{B}(\mathbf{r},t)=\left[\mathbf{n}\times\mathbf{E}(\mathbf{r},t)\right]. \tag{3.9b}$$

Note from Figure 3.1 that at time t the particle is at some point further along its path, but only the conditions at the retarded time determine the fields at point r at time t. The magnetic field is always perpendicular to both $\mathbf{E}$ and $\mathbf{n}$.

The electric field appears above as composed of two terms: the first, the *velocity field*, falls off as $1/R^2$ and is just the generalization of the Coulomb law to moving particles: for $u \ll c$ this becomes precisely Coulomb's law. When the particle moves with constant velocity it is only this term that contributes to the fields. A remarkable fact in this case is that the electric field always points along the line toward the *current* position of the particle. This follows from the fact that the displacement to

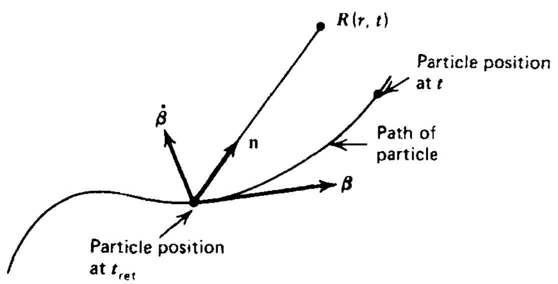

***Figure 3.1** Geometry for calculation of the radiation field at R from the position of the radiating particle at the retarded time.*

the field point from the retarded point is $\mathbf{n}c\bar{t}$, where $\bar{t}=t-t_{\rm ret}$ is the light travel time. In the same time the particle undergoes a displacement $\boldsymbol{\beta}c\bar{t}$. The displacement between the field point and the current position is thus $(\mathbf{n}-\boldsymbol{\beta})c\bar{t}$, which is seen to be the direction of the velocity field in Eq. (3.9a).

The second term, the *acceleration field*, falls off as $1/R$, is proportional to the particle's acceleration and is perpendicular to $\mathbf{n}$. This electric field, along with the corresponding magnetic field, constitutes the *radiation field*:

$$\mathbf{E}_{\rm rad}(\mathbf{r},t)=\frac{q}{c}\left[\frac{\mathbf{n}}{\kappa^3 R}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\right], \tag{3.10a}$$

$$\mathbf{B}_{\rm rad}(\mathbf{r},t)=[\mathbf{n}\times\mathbf{E}_{\rm rad}]. \tag{3.10b}$$

Note that $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{n}$ form a right-hand triad of mutually perpendicular vectors, and that $|\mathbf{E}_{\rm rad}|=|\mathbf{B}_{\rm rad}|$. These properties are consistent with the radiation solutions of the source-free Maxwell equations.

Figure 3.2 demonstrates geometrically how an acceleration can give rise to a transverse field that decreases as $1/R$, rather than the $1/R^2$ decrease of a nonaccelerated charge. The particle originally moved with constant velocity along the x-axis and stopped at $x=0$ at time $t=0$. At $t=1$ the field outside of a radius c is radial and points to the position where the particle would have been had there been no deceleration, since no information of the latter has yet propagated to this distance. On the other hand, the field inside radius c is "informed" and is radially directed to the true position of the particle. There is only one way these two fields can be

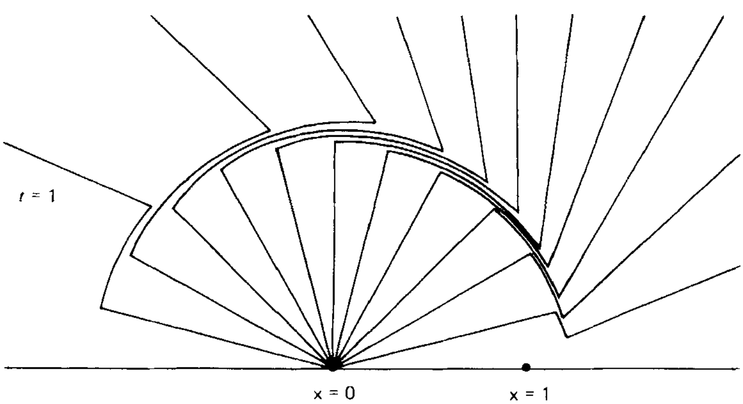

***Figure 3.2** Graphical demonstration of the 1/R acceleration field. Charged particle moving at uniform velocity in positive x direction is stopped at x = 0 and t = 0.*

connected that is consistent with Gauss's law and flux conservation: it is graphically illustrated in the figure. It can be seen that a transition zone (whose radial thickness is the time interval over which the deceleration occurs) propagates outward. In this zone the field is almost transverse and is much stronger (closely packed flux lines) than the radial fields outside the zone. Further geometrical arguments can be used to show that the field intensity in this zone is proportional to $1/ct = 1/R$. If one looks at an annular ring centered on and perpendicular to the line of travel, containing all the flux lines in the passing wavefront, then the thickness of the ring is constant (light travel distance during acceleration time), and the radius of the ring varies as R. Since the total number of flux lines is conserved, the strength of the field varies as $1/R$.

A useful result is obtained by considering the energy per unit frequency per unit solid angle corresponding to the radiation field of a single particle [cf. Eqs. (3.10a) and (2.33)]:

$$\frac{dW}{d\omega\, d\Omega} = \frac{c}{4\pi^2}\left|\int [R\mathbf{E}(t)] e^{i\omega t} dt\right|^2 \tag{3.11a}$$

$$= \frac{q^2}{4\pi^2 c}\left|\int \left[\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\kappa^{-3}\right] e^{i\omega t} dt\right|^2 \tag{3.11b}$$

where the expression in the brackets is evaluated at the retarded time $t' = t - R(t')/c$. Now, changing variables from t to t' in the integral, $dt = \kappa\, dt'$, and using the expansion $R(t') \approx |\mathbf{r}| - \mathbf{n}\cdot\mathbf{r}_0$, valid for $|\mathbf{r}_0| \ll |\mathbf{r}|$, we have

$$\frac{dW}{d\omega\, d\Omega} = \frac{q^2}{4\pi^2 c}\left|\int \mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\kappa^{-2} \exp\left[i\omega\left(t' - \mathbf{n}\cdot\mathbf{r}_0(t')/c\right)\right] dt'\right|^2. \tag{3.12}$$

Finally, we may integrate Eq. (3.12) by parts to obtain an expression involving only $\boldsymbol{\beta}$. Using the identity $\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\kappa^{-2} = d/dt'\,\kappa^{-1}\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})$, Eq. (3.12) becomes

$$\frac{dW}{d\omega\, d\Omega} = (q^2\omega^2/4\pi^2 c)\left|\int \mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta}) \exp\left[i\omega\left(t' - \mathbf{n}\cdot\mathbf{r}_0(t')/c\right)\right] dt'\right|^2. \tag{3.13}$$

3.3 RADIATION FROM NONRELATIVISTIC SYSTEMS OF PARTICLES

Using the above formulas we could discuss many radiation processes involving moving charges, including particles moving relativistically. However, the interpretation of many of these results would be made easier after the section on special relativity. Therefore, for the moment, we shall specialize the discussion to nonrelativistic particles, that is, the case

$$|\beta| = \frac{u}{c} \ll 1.$$

Let us compare the order of magnitude of the two fields E_{rad} and E_{vel}: taking the leading terms we obtain

$$\frac{E_{\text{rad}}}{E_{\text{vel}}} \sim \frac{R\dot{u}}{c^2}. \tag{3.14a}$$

Now, if we focus on the particular Fourier component of frequency ν, or if the particle has a characteristic frequency of oscillation ν, then $\dot{u} \sim u\nu$, and Eq. (3.14a) becomes

$$\frac{E_{\text{rad}}}{E_{\text{vel}}} \sim \frac{Ru\nu}{c^2} = \frac{u}{c}\frac{R}{\lambda}. \tag{3.14b}$$

Thus for field points inside the "near zone", $R \lesssim \lambda$, the velocity field is stronger than the radiation field by a factor $\gtrsim c/u$; whereas for field points sufficiently far in the "far zone," $R \gg \lambda(c/u)$, the radiation field dominates and increases its domination linearly with R.

Larmor's Formula

When $\beta \ll 1$ we can simplify equations (3.10) to

$$\mathbf{E}_{\text{rad}} = \left[(q/Rc^2)\mathbf{n} \times (\mathbf{n} \times \dot{\mathbf{u}}) \right] \tag{3.15a}$$

$$\mathbf{B}_{\text{rad}} = \left[\mathbf{n} \times \mathbf{E}_{\text{rad}} \right]. \tag{3.15b}$$

This is illustrated in Fig. 3.3, which has been drawn in the plane of $\mathbf{n}$ and $\dot{\mathbf{u}}$. We note that $\mathbf{E}_{\text{rad}}$ is also in this plane in the orientation indicated, and $\mathbf{B}_{\text{rad}}$ is into the plane of the diagram. The magnitudes of $\mathbf{E}_{\text{rad}}$ and $\mathbf{B}_{\text{rad}}$ are

$$|\mathbf{E}_{\text{rad}}| = |\mathbf{B}_{\text{rad}}| = \frac{q\dot{u}}{Rc^2}\sin\Theta. \tag{3.16}$$

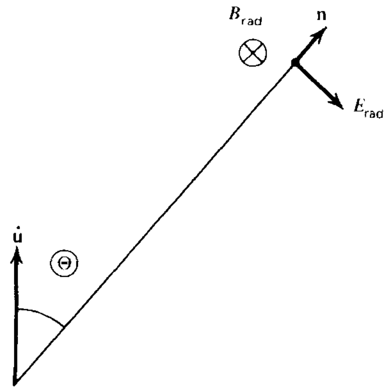

***Figure 3.3** Electric and magnetic radiation field configurations for a slowly moving particle. The direction of $\mathbf{B}_{\rm rad}$ is into the page.*

The Poynting vector is in the direction of **n** and has the magnitude

$$S = \frac{c}{4\pi} E_{\rm rad}^2 = \frac{c}{4\pi} \frac{q^2 \dot{u}^2}{R^2 c^4} \sin^2\Theta. \tag{3.17}$$

This corresponds to an outward flow of energy, along the direction **n**. We can put this into the form of an emission coefficient. The energy dW emitted per unit time into solid angle $d\Omega$ about **n** can be evaluated by multiplying the Poynting vector (erg s^{-1} cm^{-2}) by the area $dA = R^2 d\Omega$ represented by Ω at the field point:

$$\frac{dW}{dt\, d\Omega} = \frac{q^2 \dot{u}^2}{4\pi c^3} \sin^2\Theta. \tag{3.18}$$

We may obtain the total power emitted into all angles by integrating this over solid angles:

$$\begin{aligned} P = \frac{dW}{dt} &= \frac{q^2 \dot{u}^2}{4\pi c^3} \int \sin^2\Theta\, d\Omega \\ &= \frac{q^2 \dot{u}^2}{2c^3} \int_{-1}^{1} (1-\mu^2)\, d\mu. \end{aligned}$$

Thus we have *Larmor's formula* for emission from a single accelerated charge q:

$$P = \frac{2q^2 \dot{u}^2}{3c^3}. \tag{3.19}$$

There are several points to notice about Eqs. (3.18) and (3.19):

1. The power emitted is proportional to the square of the charge and the square of the acceleration.
2. We have the characteristic dipole pattern $\propto \sin^2\Theta$: no radiation is emitted along the direction of acceleration, and the maximum is emitted perpendicular to acceleration.
3. The instantaneous direction of $\mathbf{E}_{\rm rad}$ is determined by $\dot{\mathbf{u}}$ and $\mathbf{n}$. If the particle accelerates along a line, the radiation will be 100% linearly polarized in the plane of $\dot{\mathbf{u}}$ and $\mathbf{n}$.

The dipole approximation

When there are many particles with positions $\mathbf{r}_i$, velocities $\mathbf{u}_i$, and charges q_i, $i=1,2\ldots N$, we can find the radiation field at large distances by simply adding together the $\mathbf{E}_{\rm rad}$ from each particle. However, there is a complication here, because the above expressions for the radiation fields refer to conditions at *retarded times*, and these retarded times will differ for each particle. Another way of stating the complication is that we must keep track of the phase relations between the various pieces of the radiating system introduced by retardation.

There are situations, however, in which it is possible to ignore this difficulty. Let the typical size of the system be L, and let the typical time scale for changes within the system be τ. If τ is much longer than the time it takes light to travel a distance L, $\tau \gg L/c$, then the differences in retarded time across the source are negligible. We may also characterize τ as the time scale over which significant changes in the radiation field $\mathbf{E}_{\rm rad}$ occur, and this in turn determines the typical characteristic frequency of the emitted radiation. Calling this frequency ν, we write

$$\nu \approx \frac{1}{\tau}.$$

Combining this with the above we obtain

$$\frac{c}{\nu} \gg L,$$

or

$$\lambda \gg L, \tag{3.20}$$

that is, the differences in retarded times can be ignored when the size of the system is small compared to a wavelength.

We may also characterize τ as the time a particle takes to change its motion substantially. Letting l be a characteristic scale of the particle's orbit and u be a typical velocity, then $\tau \sim l/u$. The condition $\tau \gg L/c$ then implies $u/c \ll l/L$. But since $l < L$, this is simply equivalent to the nonrelativistic condition

$$u \ll c.$$

We may therefore consistently use the nonrelativistic form of the radiation fields for these problems. With the above conditions met we can write

$$\mathbf{E}_{\text{rad}} = \sum_i \frac{q_i}{c^2} \frac{\mathbf{n} \times (\mathbf{n} \times \dot{\mathbf{u}}_i)}{R_i}. \tag{3.21}$$

Let R_0 be the distance from some point in the system to the field point (see Fig. 3.4). Since the differences in the actual R_i are negligible as $R_0 \to \infty$, we have

$$\mathbf{E}_{\text{rad}} = \frac{\mathbf{n} \times (\mathbf{n} \times \ddot{\mathbf{d}})}{c^2 R_0}, \tag{3.22a}$$

where the *dipole moment* is

$$\mathbf{d} = \sum_i q_i \mathbf{r}_i. \tag{3.22b}$$

The right-hand side of Eqs. (3.22) must still be evaluated at a retarded time, but this time can be evaluated using any point within the region, say, the point used to define R_0.

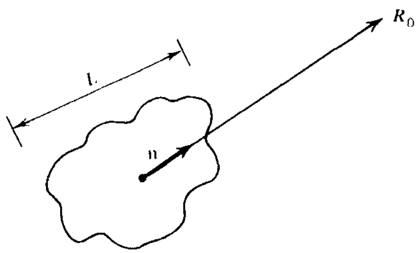

***Figure 3.4** Radiation from a medium of size L.*

As before, we find

$$\frac{dP}{d\Omega} = \frac{\ddot{\mathbf{d}}^2}{4\pi c^3} \sin^2 \Theta, \tag{3.23a}$$

$$P = \frac{2\ddot{\mathbf{d}}^2}{3c^3}. \tag{3.23b}$$

This is called the *dipole approximation* and is a generalization of the formulas [Eqs. (3.18) and (3.19)] for a single nonrelativistic particle. The instantaneous polarization of $\mathbf{E}$ lies in the plane of $\ddot{\mathbf{d}}$ and $\mathbf{n}$ (see Fig. 3.5).

As an application of the preceding analysis, let us consider the spectrum of radiation in the dipole approximation. For simplicity we assume that $\mathbf{d}$ always lies in a single direction. Then from Eq. (3.22a), we have

$$E(t) = \ddot{d}(t) \frac{\sin \Theta}{c^2 R_0}, \tag{3.24}$$

where $E(t)$ and $d(t)$ are the magnitudes of $\mathbf{E}(t)$ and $\mathbf{d}(t)$, respectively. The

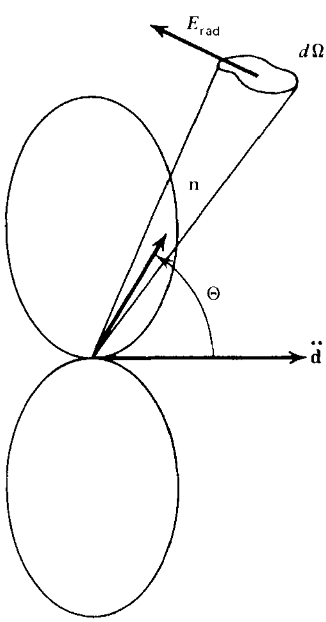

Figure 3.5 ***Geometry and emission pattern for dipole radiation.***

Fourier transform of $d(t)$ can be defined so that

$$d(t)=\int_{-\infty}^{\infty} e^{-i\omega t}\hat{d}(\omega)\,d\omega.$$

Then we have the relations

$$\ddot{d}(t)=-\int_{-\infty}^{\infty}\omega^2\hat{d}(\omega)e^{-i\omega t}\,d\omega, \tag{3.25a}$$

$$\hat{E}(\omega)=-\frac{1}{c^2R_0}\omega^2\hat{d}(\omega)\sin\Theta. \tag{3.25b}$$

For the energy per unit solid angle per frequency range and for the total energy per frequency range we have, using Eqs. (2.33), (3.25), and $dA=R_0^2 d\Omega$,

$$\frac{dW}{d\omega\, d\Omega}=\frac{1}{c^3}\omega^4|\hat{d}(\omega)|^2\sin^2\Theta, \tag{3.26a}$$

$$\frac{dW}{d\omega}=\frac{8\pi\omega^4}{3c^3}|\hat{d}(\omega)|^2. \tag{3.26b}$$

These formulas describe an interesting property of dipole radiation, namely, that the spectrum of the emitted radiation is related directly to the frequencies of oscillation of the dipole moment. This property is not true for particles with relativistic velocities.

The general multipole expansion

In the above treatment of the dipole approximation we have argued only qualitatively. We would like to be slightly more explicit and indicate the features of the general case. Since **E** and **B** are simply related well outside of the source, we may consider the vector potential **A** to contain all of the necessary information. Consider a Fourier analysis of the sources and fields [cf. Eq. (2.3)]:

$$\mathbf{j}_\omega(\mathbf{r})=\int \mathbf{j}(\mathbf{r},t)e^{i\omega t}\,dt, \tag{3.27a}$$

$$\mathbf{A}_\omega(\mathbf{r})=\int \mathbf{A}(\mathbf{r},t)e^{i\omega t}\,dt. \tag{3.27b}$$

Then, using the equation analogous to Eq. (3.2) for the vector potential

$$\mathbf{A}(\mathbf{r},t)=\frac{1}{c}\int d^3\mathbf{r}'\int dt'\frac{\mathbf{j}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\,\delta(t'-t+|\mathbf{r}-\mathbf{r}'|/c),$$

and taking the Fourier transform of this equation, using Eqs. (3.27), we obtain

$$\mathbf{A}_\omega(\mathbf{r}) = \frac{1}{c}\int \frac{\mathbf{j}_\omega(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} e^{ik|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}', \tag{3.28}$$

where $k \equiv \omega/c$. Note that our equations now relate single Fourier components of $\mathbf{j}$ and $\mathbf{A}$.

Let us choose an origin of coordinates inside the source of size L. Then, at field points such that $r \gg L$, we have the approximation

$$|\mathbf{r}-\mathbf{r}'| \approx r - \mathbf{n}\cdot\mathbf{r}', \tag{3.29}$$

where $\mathbf{n}$ points toward the field point $\mathbf{r}$ and where $r \equiv |\mathbf{r}|$. Substituting Eq. (3.29) into (3.28), we obtain

$$\mathbf{A}_\omega(\mathbf{r}) \approx (e^{ikr}/cr)\int \mathbf{j}_\omega(\mathbf{r}') e^{-ik\mathbf{n}\cdot\mathbf{r}'} d^3\mathbf{r}'. \tag{3.30}$$

The factor $\exp(ikr)$ outside the integral expresses the effect of retardation from the source as a whole. The factor $\exp(-ik\mathbf{n}\cdot\mathbf{r}')$ inside the integral expresses the *relative* retardation of each element of the source. In our slow-motion approximation, $kL \ll 1$. Thus, expanding the exponential in the integral:

$$\mathbf{A}_\omega(\mathbf{r}) = \frac{e^{ikr}}{cr}\sum_{n=0}^{\infty}\frac{1}{n!}\int \mathbf{j}_\omega(\mathbf{r}')(-ik\mathbf{n}\cdot\mathbf{r}')^n d^3\mathbf{r}'. \tag{3.31}$$

Equation (3.31) is clearly an expansion in the small dimensionless parameter $kL = 2\pi L/\lambda$. The *dipole approximation* results from taking just the first term in the expansion ($n=0$):

$$\mathbf{A}_\omega(\mathbf{r})|_{\text{dipole}} = \frac{e^{ikr}}{cr}\int \mathbf{j}_\omega(\mathbf{r}') d^3\mathbf{r}'. \tag{3.32}$$

The *quadrupole* term is the second term in the expansion ($n=1$):

$$\mathbf{A}_\omega(\mathbf{r})|_{\text{quad}} = \frac{-ike^{ikr}}{cr}\int \mathbf{j}_\omega(\mathbf{r}')(\mathbf{n}\cdot\mathbf{r}') d^3\mathbf{r}'. \tag{3.33}$$

Although it is true that the frequencies present in the vector potential (and hence in the radiation) are identical to those in the current density, it should be pointed out that these frequencies may differ from the frequencies of particle orbits in the source. For example, in the case of a particle orbiting in a circle with angular frequency ω_0, the function $\mathbf{j}_\omega(\mathbf{r})$ actually contains frequencies not only at ω_0 but also at all harmonics $2\omega_0$, $3\omega_0 \ldots$. In the dipole approximation only ω_0 contributes, in the quadrupole approximation only $2\omega_0$ contributes, and so on (see problem 3.7).

3.4 THOMSON SCATTERING (ELECTRON SCATTERING)

An important application of the dipole formula is to the process in which a free charge radiates in response to an incident electromagnetic wave. If the charge oscillates at nonrelativistic velocities, $v \ll c$, then we may neglect magnetic forces, since $E = B$ for an electromagnetic wave. Thus the force due to a *linearly polarized wave* is

$$\mathbf{F} = e\boldsymbol{\epsilon} E_0 \sin\omega_0 t, \tag{3.34}$$

where e is the charge and $\boldsymbol{\epsilon}$ is the **E**-field direction. (See Fig. 3.6.) From Eq. (3.34), we have

$$m\ddot{\mathbf{r}} = e\boldsymbol{\epsilon} E_0 \sin\omega_0 t.$$

In terms of the dipole moment, $\mathbf{d} = e\mathbf{r}$, we have

$$\ddot{\mathbf{d}} = \frac{e^2 E_0}{m}\boldsymbol{\epsilon}\sin\omega_0 t,$$

$$\mathbf{d} = -\left(\frac{e^2 E_0}{m\omega_0^2}\right)\boldsymbol{\epsilon}\sin\omega_0 t,$$

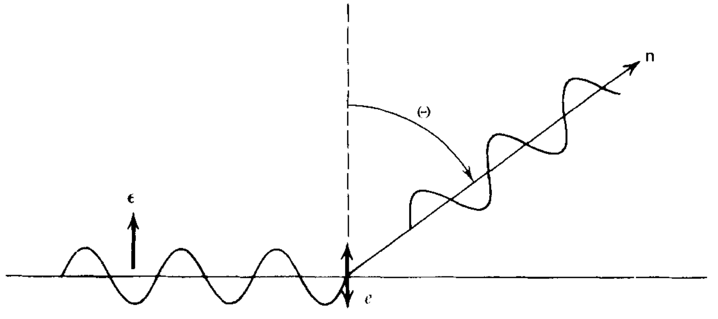

Figure 3.6 Scattering of polarized radiation by a charged particle.

which describes an oscillating dipole of amplitude

$$\mathbf{d}_0=\frac{e^2E_0}{m\omega_0^2}\boldsymbol{\epsilon}.$$

From our previous results of Eqs. (3.23), we can write the time-averaged power as

$$\frac{dP}{d\Omega}=\frac{e^4E_0^2}{8\pi m^2c^3}\sin^2\Theta, \tag{3.35a}$$

$$P=\frac{e^4E_0^2}{3m^2c^3}, \tag{3.35b}$$

where the time average of $\sin^2\omega_0 t$ gives a factor $\frac{1}{2}$. Note that the incident flux is $\langle S\rangle=(c/8\pi)E_0^2$. Defining the *differential cross section* $d\sigma$ for scattering into $d\Omega$ we have

$$\frac{dP}{d\Omega}=\langle S\rangle\frac{d\sigma}{d\Omega}=\frac{cE_0^2}{8\pi}\frac{d\sigma}{d\Omega}. \tag{3.36}$$

Therefore, we have the relation

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{polarized}}=\frac{e^4}{m^2c^4}\sin^2\Theta=r_0^2\sin^2\Theta, \tag{3.37}$$

where

$$r_0\equiv\frac{e^2}{mc^2}. \tag{3.38}$$

The quantity r_0 gives a measure of the "size" of the point charge, assuming its rest energy mc^2 is purely electromagnetic in origin. For an electron r_0 is called the *classical electron radius* and has a value $r_0=2.82\times 10^{-13}$ cm. The total cross section can be found by integrating over solid angle, using $\mu\equiv\cos\Theta$,

$$\sigma=\int\frac{d\sigma}{d\Omega}d\Omega=2\pi r_0^2\int_{-1}^{1}(1-\mu^2)d\mu.$$

This gives the result

$$\sigma=\frac{8\pi}{3}r_0^2. \tag{3.39}$$

(Alternatively, one can obtain σ from $P=\langle S\rangle\sigma$.)

For an electron $\sigma = \sigma_T$ = *Thomson cross section* $= 0.665 \times 10^{-24}$ cm^2. The above scattering process is then called *Thomson scattering* or *electron scattering*.

Note that the total and differential cross sections above are frequency independent, so that the scattering is equally effective at all frequencies. However, this is really only valid for sufficiently low frequencies, so that a classical description is valid. At high frequencies, where the energy of emitted photons $h\nu$ becomes comparable to or larger than mc^2, then the quantum mechanical cross sections must be used; this occurs for X-rays of energies $h\nu \gtrsim 0.511$ MeV for electron scattering (see Chapter 7). Also, for sufficiently intense radiation fields the electron moves relativistically; then the dipole approximation ceases to be valid.

We note that the scattered radiation is linearly polarized in the plane of the incident polarization vector $\boldsymbol{\epsilon}$ and the direction of scattering $\mathbf{n}$.

It is easy to get the differential cross section for scattering of *unpolarized radiation* by recognizing that an unpolarized beam can be regarded as the independent superposition of two linear-polarized beams with perpendicular axes. Let us choose one such beam along $\boldsymbol{\epsilon}_1$, which is in the plane of the incident and scattered directions, and the second along $\boldsymbol{\epsilon}_2$, perpendicular to this plane. (See Fig. 3.7.) Let Θ be the angle between $\boldsymbol{\epsilon}_1$ and $\mathbf{n}$. Note that the angle between $\boldsymbol{\epsilon}_2$ and $\mathbf{n}$ is $\pi/2$. We also have introduced the angle $\theta = \pi/2 - \Theta$, which is the angle between the scattered wave and incident wave. Now the differential cross section for unpolarized radiation is the average of the cross sections for scattering of linear-polarized radiation

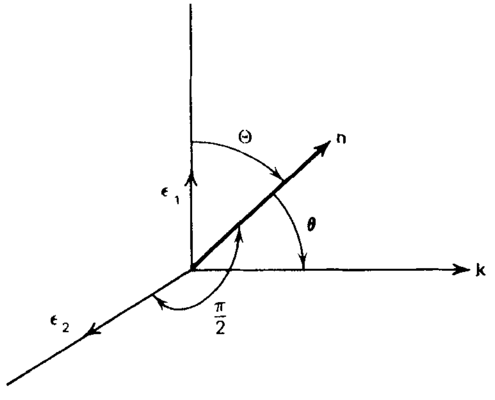

Figure 3.7 Geometry for scattering unpolarized radiation.

through angles Θ and $\pi/2$. Thus we have the result

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} &= \frac{1}{2}\left[\left(\frac{d\sigma(\Theta)}{d\Omega}\right)_{\text{pol}} + \left(\frac{d\sigma(\pi/2)}{d\Omega}\right)_{\text{pol}}\right] \\ &= \frac{1}{2}r_0^2(1+\sin^2\Theta) \\ &= \frac{1}{2}r_0^2(1+\cos^2\theta),\end{aligned} \tag{3.40}$$

which depends only on the angle between the incident and scattered directions, as it should for unpolarized radiation.

There are several features of electron scattering of unpolarized radiation which we now point out:

1. Forward-backward symmetry: The scattering cross section, Eq. (3.40), is symmetric under the reflection $\theta \to -\theta$.
2. Total cross section: The total scattering cross section of unpolarized incident radiation is the same as that for polarized incident radiation $\sigma_{\text{unpol}} = \sigma_{\text{pol}} = (8\pi/3)r_0^2$. This is because the electron at rest has no net direction intrinsically defined.
3. Polarization of scattered radiation: The two terms in Eq. (3.40) clearly refer to intensities in two perpendicular directions in the plane normal to $\mathbf{n}$, since they arise from the two perpendicular components of the incident wave. Since the polarized intensities in the plane and perpendicular to the plane of scattering are in the ratio $\cos^2\theta$: 1, the degree of polarization of the scattered wave is [cf. Eq. (2.57)]

$$\Pi = \frac{1-\cos^2\theta}{1+\cos^2\theta}. \tag{3.41}$$

   Since $\Pi \geqslant 0$, we have the interesting result that electron scattering of a completely unpolarized incident wave produces a scattered wave with some degree of polarization, the degree depending on the viewing angle with respect to the incident direction. If we look along the incident direction ($\theta = 0$) we see no net polarization, since, by symmetry, all directions in the plane are equivalent. If we look perpendicular to the incident wave ($\theta = \pi/2$) we see 100% polarization, since the electron's motion is confined to a plane normal to the incident direction.

3.5 RADIATION REACTION

The energy radiated away by an accelerating charge must come from the particle's own energy or from the agency maintaining the particle's energy.

We conclude that there must be a force acting on a particle by virtue of the radiation it produces. This is called the *radiation reaction force*. The full treatment of this effect from first principles involves the calculation of the force on one part of the charge by the fields of another part, including retardation within the particle itself. Throughout the calculation the size of the particle is kept as nonzero. Afterwards the size can be set to zero, or at least to some small value such as r_0. Here we derive the main result using energy considerations alone.

We first delineate those regimes in which radiation reaction may be considered as a *perturbation* on the particle's motion. Let T be the time interval over which the kinetic energy of the particle is changed substantially by the emission of radiation. Then from Eq. (3.19), with $a=\dot{u}$,

$$T\sim\frac{mv^2}{P_{rad}}\sim\frac{3mc^3}{2e^2}\left(\frac{v}{a}\right)^2,$$

where m is the mass of the particle, and v its velocity. We estimate $v/a\sim t_p$ as the typical orbital time scale for the particle. Then the condition $T/t_p\gg 1$ requires that $t_p\gg\tau$, where, for an electron,

$$\tau\equiv\frac{2e^2}{3mc^3}\sim 10^{-23}s. \tag{3.42}$$

Thus as long as we are considering processes that occur on a time scale much longer than τ, we can treat radiation reaction as a perturbation. It should be noticed that τ is the time for radiation to cross a distance comparable to the classical electron radius, the "size" of the electron: [cf. Eq. (3.38)]

$$\tau\sim\frac{r_0}{c}.$$

We can infer the formula for the radiation reaction force from elementary considerations of energy balance. When the radiation reaction force is relatively small, we may sensibly define the force as a term added onto the existing external force, such that the energy radiated must be compensated for by the work done against the radiation reaction force. Thus we are tempted to set

$$-\mathbf{F}_{\rm rad}\cdot\mathbf{u}=\frac{2e^2\dot{u}^2}{3c^3}. \tag{3.43}$$

However, one can see that there is no $\mathbf{F}_{\text{rad}}$ that can instantaneously satisfy this equation: $\mathbf{F}_{\text{rad}}$ cannot depend on $\mathbf{u}$, because this would imply a preferred frame relative to which $\mathbf{u}$ is measured. But then one side of Eq. (3.43) explicitly depends on $\mathbf{u}$ whereas the other does not, a contradiction. The best we can do is to satisfy this equation in some average sense, the remaining energy fluctuations being taken up in the nonradiation fields. Integrate the above equation over a time interval t_1 to t_2, with $(t_2-t_1)\gg\tau$. Integrating by parts, we obtain:

$$
\begin{aligned}
-\int_{t_1}^{t_2}\mathbf{F}_{\text{rad}}\cdot\mathbf{u}\,dt &= \frac{2e^2}{3c^3}\int_{t_1}^{t_2}\dot{\mathbf{u}}\cdot\dot{\mathbf{u}}\,dt \\
&= \frac{2e^2}{3c^3}\left[\dot{\mathbf{u}}\cdot\mathbf{u}\Big|_{t_1}^{t_2}-\int_{t_1}^{t_2}\ddot{\mathbf{u}}\cdot\mathbf{u}\,dt\right].
\end{aligned} \tag{3.44}
$$

If we assume that the initial and final states are the same (so that the nonradiation fields are the same and do not contribute to the energy difference) or that $\dot{\mathbf{u}}\cdot\mathbf{u}(t_1)=\dot{\mathbf{u}}\cdot\mathbf{u}(t_2)$, the first term on the right-hand side of Eq. (3.44) vanishes, leaving

$$-\int_{t_1}^{t_2}\left(\mathbf{F}_{\text{rad}}-\frac{2e^2\ddot{\mathbf{u}}}{3c^3}\right)\cdot\mathbf{u}\,dt=0.$$

Thus we take

$$\mathbf{F}_{\text{rad}}=\frac{2e^2\ddot{\mathbf{u}}}{3c^3}=m\tau\ddot{\mathbf{u}}, \tag{3.45}$$

where Eq. (3.45) now represents the radiation force in some *time-averaged*, approximate sense.

This formula for the radiation reaction force depends on the derivative of acceleration, that is, the third derivative of position. This increases the degree of the equation of motion of a particle and can lead to some nonphysical behavior if not used properly and consistently.

For example, the equation of motion for a particle with applied force $\mathbf{F}$ is

$$m(\dot{\mathbf{u}}-\tau\ddot{\mathbf{u}})=\mathbf{F}.$$

Suppose $\mathbf{F}=0$; then a solution is the obvious

$$\mathbf{u}=\text{constant},$$

which is also the physically correct solution. However, there is also another solution

$$\mathbf{u}=\mathbf{u}_0 e^{t/\tau},$$

which rapidly becomes exceedingly large ("runaway" solution). We must exclude such solutions from consideration. We note that they violate the restriction on the motion that it not change on a time scale short compared to τ. Furthermore, $\dot{\mathbf{u}}\cdot\mathbf{u}(t_2)\neq\dot{\mathbf{u}}\cdot\mathbf{u}(t_1)$. We can thus argue that such solutions are spurious, on the mathematical grounds that they violate the assumptions on which the equations were based.

3.6 RADIATION FROM HARMONICALLY BOUND PARTICLES

Undriven Harmonically Bound Particles

A particle that is harmonically bound to a center of force (i.e., $\mathbf{F}=-k\mathbf{r}=-m\omega_0^2\mathbf{r}$) will oscillate sinusoidally with frequency ω_0. Such a system, although rarely found in nature, is interesting because it gives the only possible *classical* model of a spectral line. Many of the quantum results are stated against the framework of this model ("oscillator strengths," "classical damping widths"). Since there is always a small damping of the oscillations by the radiation reaction force, the oscillation will not be purely harmonic. We assume that $\omega_0\tau\ll 1$, so that the radiation reaction formula is valid. If the oscillations are along the x axis, [cf. Eq. (3.45)]

$$-\tau\dddot{x}+\ddot{x}+\omega_0^2 x=0. \tag{3.46}$$

This is a third-order differential equation with constant coefficients. Since the term involving the third derivative is small, a convenient approximation is that the motion will be harmonic to first order, with $x(t)\propto\cos(\omega_0 t+\phi_0)$. Therefore, we approximate the damping implied by the third derivative by a damping in the first derivative, through

$$\dddot{x}\approx-\omega_0^2\dot{x}. \tag{3.47}$$

This approximation preserves an important feature of damping: it is expressed as an odd number of time derivatives and is therefore not time reversible. Therefore, our equation becomes

$$\ddot{x}+\omega_0^2\tau\dot{x}+\omega_0^2 x=0. \tag{3.48}$$

This may be solved by assuming $x(t)$ has the form $e^{\alpha t}$, where α is found from

$$\alpha^2+\omega_0^2\tau\alpha+\omega_0^2=0,$$

which has the solution

$$\alpha=\pm i\omega_0-\tfrac{1}{2}\omega_0^2\tau+0\left(\omega_0^3\tau^2\right)$$

when expanded in powers of $\omega_0\tau$. Taking as initial conditions for $t=0$, $x(0)=x_0$, $\dot{x}(0)\approx 0$, we have

$$\begin{aligned} x(t)&=x_0 e^{-\Gamma t/2}\cos\omega_0 t \\ &=\tfrac{1}{2}x_0\left(e^{-\frac{1}{2}\Gamma t+i\omega_0 t}+e^{-\frac{1}{2}\Gamma t-i\omega_0 t}\right), \end{aligned} \tag{3.49}$$

where

$$\Gamma\equiv\omega_0^2\tau=\frac{2e^2\omega_0^2}{3mc^3}. \tag{3.50}$$

The Fourier transform of $x(t)$ is, [cf. Eq. (2.27)],

$$\hat{x}(\omega)=\frac{1}{2\pi}\int_0^\infty x(t)e^{i\omega t}dt=\frac{x_0}{4\pi}\left[\frac{1}{\Gamma/2-i(\omega+\omega_0)}+\frac{1}{\Gamma/2-i(\omega-\omega_0)}\right]. \tag{3.51}$$

This becomes large in the vicinity of $\omega=\omega_0$ and $\omega=-\omega_0$. Since we are ultimately interested only in positive frequencies, and only in regions in which the values become large, let us make the approximations

$$\begin{aligned} \hat{x}(\omega)&\approx\frac{x_0}{4\pi}\frac{1}{\Gamma/2-i(\omega-\omega_0)}, \\ |\hat{x}(\omega)|^2&=\left(\frac{x_0}{4\pi}\right)^2\frac{1}{(\omega-\omega_0)^2+(\Gamma/2)^2}. \end{aligned} \tag{3.52}$$

The energy radiated per unit frequency is then [cf. Eq. (3.26b)]

$$\frac{dW}{d\omega}=\frac{8\pi\omega^4}{3c^3}\frac{e^2x_0^2}{(4\pi)^2}\frac{1}{(\omega-\omega_0)^2+(\Gamma/2)^2}. \tag{3.53}$$

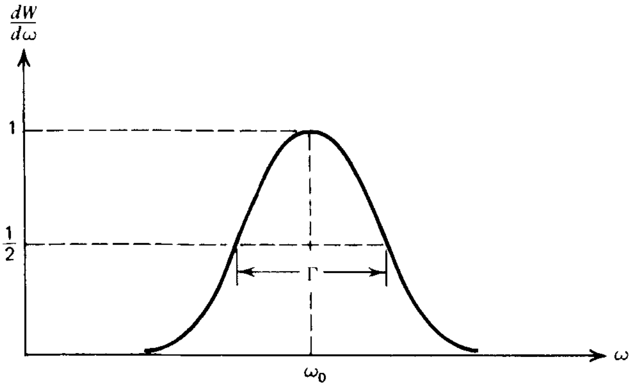

Figure 3.8 ***Power spectrum for an undriven, harmonically bound particle damped by radiation reaction.***

Equation (3.53) gives the frequency spectrum typical of a "decaying oscillator." Note that this has a sharp maximum in the neighborhood of $\omega=\omega_0$, since $\Gamma/\omega_0 \ll 1$. This is illustrated in Fig. 3.8, where it is seen that Γ is the full width at half maximum (FWHM).

Using the definition of Γ and $k \equiv m\omega_0^2 =$ spring constant, we can write Eq. (3.53) in the form

$$\frac{dW}{d\omega}=\left(\tfrac{1}{2}kx_0^2\right)\frac{\Gamma/2\pi}{(\omega-\omega_0)^2+(\Gamma/2)^2}. \tag{3.54}$$

The first factor gives the initial potential energy of the particle (energy stored in spring). The second factor gives the distribution of the radiated energy over frequency. The integral over ω can be performed easily, if we note that the range of integration can be taken as infinite, since the function is confined essentially to a small region about ω_0:

$$\int_{-\infty}^{\infty}\frac{\Gamma/2\pi}{(\omega-\omega_0)^2+(\Gamma/2)^2}\,d\omega=\frac{1}{\pi}\tan^{-1}\left[\frac{2(\omega-\omega_0)}{\Gamma}\right]_{-\infty}^{\infty}=1.$$

Thus we find that

$$W=\int_0^{\infty}\frac{dW}{d\omega}\,d\omega=\tfrac{1}{2}kx_0^2 \tag{3.55}$$

is the total emitted energy, as it should by conservation of energy.

The profile of the emitted spectrum,

$$\frac{\Gamma/2\pi}{(\omega-\omega_0)^2+(\Gamma/2)^2}, \tag{3.56}$$

is known as a *Lorentz profile.*

The classical line breadth $\Delta\omega=\Gamma$ for electronic oscillators is a universal constant when expressed in terms of wavelength:

$$\begin{aligned}\Delta\lambda &= 2\pi c\frac{\Delta\omega}{\omega^2}\\ &= 2\pi c\tau = 1.2\times 10^{-4}\mathring{A},\end{aligned} \tag{3.57}$$

where

$$1\ \mathring{A} \equiv 10^{-8}\ \text{cm}.$$

Driven Harmonically Bound Particles

We have just computed the radiation from the *free* oscillations of a harmonic oscillator. Now, we wish to consider *forced* oscillations, when the forcing is due to an incident beam of radiation. This will give the *scattered* radiation from the incident beam. Let us now write

$$m\ddot{x} = -m\omega_0^2 x + m\tau\dddot{x} + eE_0\cos\omega t, \tag{3.58}$$

where the last term is the force due to a sinusoidally varying incident field. Here we have left the radiation reaction term as a third derivative. With the usual trick of representing x by a complex variable, we have

$$\ddot{x} - \tau\dddot{x} + \omega_0^2 x = \frac{eE_0}{m}e^{i\omega t}, \tag{3.59}$$

where we take the real part of x. The steady-state solution of this equation is

$$x = x_0 e^{i\omega t} \equiv |x_0| e^{i(\omega t+\delta)}, \tag{3.60a}$$

where

$$x_0 = -\left(\frac{eE_0}{m}\right)\left(\omega^2-\omega_0^2-i\omega_0^3\tau\right)^{-1} \tag{3.60b}$$

$$\delta = \tan^{-1}\left(\frac{\omega^3\tau}{\omega^2-\omega_0^2}\right) \tag{3.60c}$$

Note that there is a *phase shift* in the response of the particle displacement to the driving force, caused by the odd time derivative damping term. For $\omega > \omega_0$ the particle "leads" the driving force and for $\omega < \omega_0$ it "lags." Taking the real part of x we see that we have an oscillating dipole of charge e and amplitude $|x_0|$ with frequency ω. The time-averaged total power radiated is therefore

$$P = \frac{e^2|x_0|^2\omega^4}{3c^3} = \frac{e^4E_0^2}{3m^2c^3}\frac{\omega^4}{\left(\omega^2-\omega_0^2\right)^2+\left(\omega_0^3\tau\right)^2}. \tag{3.61}$$

Dividing Eq. (3.61) by the time-average Poynting vector $\langle S\rangle = (c/8\pi)E_0^2$, we obtain the cross section for scattering as a function of frequency:

$$\sigma(\omega) = \sigma_T\frac{\omega^4}{\left(\omega^2-\omega_0^2\right)^2+\left(\omega_0^3\tau\right)^2}. \tag{3.62}$$

Here σ_T is the Thomson cross section. Three interesting regimes for ω can be identified (see Fig. 3.9):

1—$\omega \gg \omega_0$. In this case $\sigma(\omega) \to \sigma_T$, the value for free electrons. This is to be expected, since at high incident energies the binding becomes negligible.

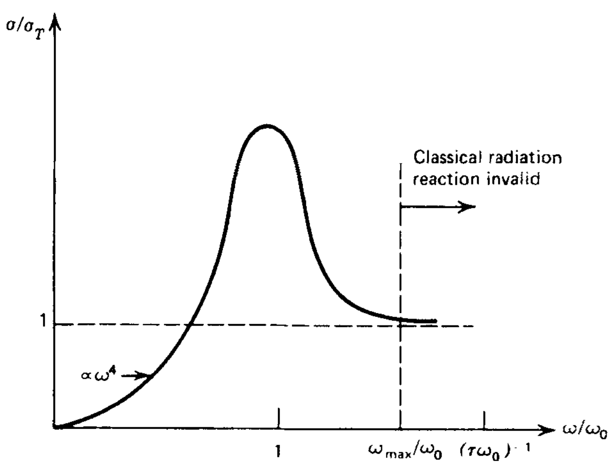

***Figure 3.9** Scattering cross section for a driven, harmonically bound particle as a function of the driving frequency. Here ω_0 and σ_T are the natural frequency and Thomson cross section, respectively.*

2—$\omega \ll \omega_0$. Here we have

$$\sigma(\omega) \to \sigma_T \left(\frac{\omega}{\omega_0} \right)^4. \tag{3.63}$$

This case corresponds to the electron responding directly to the incident field with no inertial effects, so that $kx \approx eE$. (Since $\omega \ll \omega_0$, the electric field appears nearly static and produces a nearly static force.) The dipole moment is then directly proportional to the incident field and therefore is describable in terms of a static *polarizability*. In such cases the scattered radiation will always go as ω^4, and the scattering is called *Rayleigh scattering*. It is responsible for the blue color of the sky and the red color of the sun at sunrise and sunset, because it favors the scattering of higher frequency (bluer) light.

3—$\omega \approx \omega_0$. This case is dominated by the closeness of $\omega^2 - \omega_0^2$ to zero. Thus we write

$$\omega^2 - \omega_0^2 = (\omega - \omega_0)(\omega + \omega_0)$$

and leave the factor $(\omega - \omega_0)$, but in every other appearance of ω we set $\omega = \omega_0$. This leads to the approximation

$$\sigma(\omega) \approx \frac{\pi \sigma_T}{2\tau} \frac{\Gamma/2\pi}{(\omega - \omega_0)^2 + (\Gamma/2)^2},$$

using $\Gamma = \omega_0^2 \tau$. With the definitions of σ_T and τ, this can be written

$$\sigma(\omega) = \frac{2\pi^2 e^2}{mc} \frac{\Gamma/2\pi}{(\omega - \omega_0)^2 + (\Gamma/2)^2}. \tag{3.64}$$

In the neighborhood of the resonance the shape of the scattering cross section is the same as the emission from the free oscillations of the oscillator [cf. Eq. (3.56)]. This can easily be explained, since the free oscillations can be excited by a pulse of radiation, $E(t) \propto \delta(t)$. The spectrum of this pulse is independent of ω (white spectrum), so that the free oscillations may be regarded as the scattering of a white spectrum, yielding emission proportional to the scattering cross section.

An interesting result obtains from integrating $\sigma(\omega)$ over ω:

$$\int_0^\infty \sigma(\omega)\, d\omega = \frac{2\pi^2 e^2}{mc} \tag{3.65a}$$

or in terms of frequency ν,

$$\int_0^\infty \sigma(\nu)\,d\nu = \frac{\pi e^2}{mc}. \tag{3.65b}$$

In evaluating this integral we have apparently neglected a divergence, since the cross section actually approaches σ_T for large ω. This may be justified as follows: the radiation reaction formula is only valid for $\omega\tau \ll 1$, so that we must cut off the integral at a $\omega_{\max}$ such that $\omega_{\max} \ll 1/\tau$. The contribution to the integral from the Thomson limit is less than

$$\int_0^{\omega_{\max}} \sigma_T\,d\omega = \sigma_T \omega_{\max}.$$

This is negligible in comparison to the value of the integral in Eq. (3.65a), since $\sigma_T \omega_{\max} \ll \sigma_T/\tau = 4\pi e^2/mc$.

In the quantum theory of spectral lines we obtain similar formulas, which are conveniently stated in terms of the above classical results as

$$\int_0^\infty \sigma(\nu)\,d\nu = \frac{\pi e^2}{mc} f_{nn'}, \tag{3.66}$$

where $f_{nn'}$ is called the *oscillator strength* or *f-value* for the transition between states n and n' (see Chapter 10).

PROBLEMS

3.1—A pulsar is conventionally believed to be a rotating neutron star. Such a star is likely to have a strong magnetic field, B_0, since it traps lines of force during its collapse. If the magnetic axis of the neutron star does not line up with the rotation axis, there will be magnetic dipole radiation from the time-changing magnetic dipole, $\mathbf{m}(t)$. Assume that the mass and radius of the neutron star are M and R, respectively; that the angle between the magnetic and rotation axes is α; and that the rotational angular velocity is ω.

a. Find an expression for the radiated power P in terms of ω, R, B_0 and α.

b. Assuming that the rotational energy of the pulsar is the ultimate source of the radiated power, find an expression for the slow-down time scale, τ, of the pulsar.

c. For $M = 1M_{\odot} \equiv 2\times 10^{33}$ g, $R = 10^6$ cm, $B_0 = 10^{12}$ gauss, $\alpha = 90°$, find P and τ for $\omega = 10^4\ \mathrm{s}^{-1}$, $10^3\ \mathrm{s}^{-1}$; $10^2\ \mathrm{s}^{-1}$. (The highest rate, $\omega = 10^4\ \mathrm{s}^{-1}$, is believed to be typical of newly formed pulsars.)

3.2—A particle of mass m and charge e moves at constant, nonrelativistic speed v_1 in a circle of radius a.

a. What is the power emitted per unit solid angle in a direction at angle θ to the axis of the circle?

b. Describe qualitatively and quantitatively the polarization of the radiation as a function of the angle θ.

c. What is the spectrum of the emitted radiation?

d. Suppose a particle is moving nonrelativistically in a constant magnetic field B. Show that the frequency of circular motion is $\omega_B = eB/mc$, and that the total emitted power is

$$P = \tfrac{2}{3} r_0^2 c (v_{\perp}/c)^2 B^2,$$

and is emitted solely at the frequency ω_B. (This nonrelativistic form of synchrotron radiation is called *cyclotron* or *gyro* radiation).

e. Find the differential and total cross sections for Thomson scattering of circularly polarized radiation. Use these results to find the cross sections for unpolarized radiation.

3.3—Two oscillating dipole moments (radio antennas) $\mathbf{d}_1$ and $\mathbf{d}_2$ are oriented in the vertical direction and are a horizontal distance L apart. They oscillate in phase at the same frequency ω. Consider radiation at angle θ with respect to the vertical and in the vertical plane containing the two dipoles.

a. Show that

$$\frac{dP}{d\Omega} = \frac{\omega^4 \sin^2\theta}{8\pi c^3}\left(d_1^2 + 2d_1 d_2 \cos\delta + d_2^2\right),$$

where

$$\delta \equiv \frac{\omega L \sin\theta}{c}.$$

b. Thus show directly that when $L \ll \lambda$, the radiation is the same as from a single oscillating dipole of amplitude $d_1 + d_2$.

3.4—An optically thin cloud surrounding a luminous object is estimated to be 1 pc in radius and to consist of ionized plasma. Assume that electron scattering is the only important extinction mechanism and that the luminous object emits unpolarized radiation.

a. If the cloud is unresolved (angular size smaller than angular resolution of detector), what is the net polarization observed?

b. If the cloud is resolved, what is the polarization direction of the observed radiation as a function of position on the sky? Assume only single scattering occurs.

c. If the central object is clearly seen, what is an upper bound for the electron density of the cloud, assuming that the cloud is homogeneous?

3.5—A plane-polarized wave is incident on a sphere of radius a, composed of a solid material. We assume that the wavelength λ is large compared with a. In that case it is known that the electric field at any instant of time is constant throughout the sphere and has the value $E' = E/(1 + 4\pi\alpha/3)$, where E is the external (applied) field and α is the polarizability of the material. The dipole moment per unit volume is simply proportional to the internal electric field $P = \alpha E'$. Show that the total cross section for scattering the radiation is

$$\sigma = \pi a^2 Q_{\text{scatt}}$$

where

$$Q_{\text{scatt}} = \frac{8(ka)^4}{3(1 + 3/4\pi\alpha)^2}.$$

3.6—Consider a medium containing a large number of radiating particles. (For definiteness you may wish to imagine electrons emitting bremsstrahlung.) Each particle emits a pulse of radiation with an electric field $E_0(t)$ as a function of time. An observer will detect a series of such pulses, all with the same shape but with random arrival times $t_1, t_2, t_3, \ldots, t_N$. The measured electric field will be

$$E(t) = \sum_{i=1}^{N} E_0(t - t_i).$$

a. Show that the Fourier transform of $E(t)$ is

$$\hat{E}(\omega) = \hat{E}_0(\omega) \sum_{i=1}^{N} e^{i\omega t_i},$$

where $\hat{E}_0(\omega)$ is the Fourier transform of $E_0(t)$.

b. Argue that

$$\left| \sum_{i=1}^{N} e^{i\omega t_i} \right|^2 = N$$

when averaged over the random arrival times.

c. Thus show that the measured spectrum is simply N times the spectrum of an individual pulse. (Note that this result still holds if the pulses overlap.)

d. By contrast, show that if all the particles are in a region much smaller than a wavelength and they emit their pulses simultaneously, then the measured spectrum will be N^2 times the spectrum of an individual pulse.

3.7—Consider a charge e moving around a circle of radius r_0 at frequency ω_0. By consideration of the current density and its Fourier transform, show that the Fourier transform of the vector potential, $\mathbf{A}_\omega(\mathbf{x})$, is nonzero only at $\omega = \omega_0$ in the dipole approximation, nonzero only at $\omega = 2\omega_0$ in the quadrupole approximation and so on.

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