RADIATIVE PROCESSE S IN ASTROPHYSICS
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6

SYNCHROTRON RADIATION

Particles accelerated by a magnetic field **B** will radiate. For nonrelativistic velocities the complete nature of the radiation is rather simple and is called *cyclotron radiation*. The frequency of emission is simply the frequency of gyration in the magnetic field.

However, for extreme relativistic particles the frequency spectrum is much more complex and can extend to many times the gyration frequency. This radiation is known as *synchrotron radiation*.

6.1 TOTAL EMITTED POWER

Let us start by finding the motion of a particle of mass m and charge q in a magnetic field using the correct relativistic equations [cf. Eqs. (4.84)].

$$\frac{d}{dt}(\gamma m\mathbf{v}) = \frac{q}{c}\mathbf{v}\times\mathbf{B} \tag{6.1a}$$

$$\frac{d}{dt}(\gamma mc^2) = q\mathbf{v}\cdot\mathbf{E} = 0. \tag{6.1b}$$

This last equation implies that γ = constant or that $|\mathbf{v}|$ = constant. Therefore, it follows that

$$m\gamma\frac{d\mathbf{v}}{dt}=\frac{q}{c}\mathbf{v}\times\mathbf{B}. \tag{6.2}$$

Separating the velocity components along the field $\mathbf{v}_{\|}$ and in a plane normal to the field $\mathbf{v}_{\perp}$ we have

$$\frac{d\mathbf{v}_{\|}}{dt}=0, \qquad \frac{d\mathbf{v}_{\perp}}{dt}=\frac{q}{\gamma mc}\mathbf{v}_{\perp}\times\mathbf{B}. \tag{6.3}$$

It follows that $\mathbf{v}_{\|}$ = constant, and, since the total $|\mathbf{v}|$ = constant, also $|\mathbf{v}_{\perp}|$ = constant. The solution to this equation is clearly uniform circular motion of the projected motion on the normal plane, since the acceleration in this plane is normal to the velocity and of constant magnitude. The combination of this circular motion and the uniform motion along the field is a *helical* motion of the particle (Fig. 6.1). The frequency of the rotation, or gyration, is

$$\omega_B=\frac{qB}{\gamma mc}. \tag{6.4}$$

The acceleration is perpendicular to the velocity, with magnitude $a_{\perp}=\omega_B v_{\perp}$, so that the total emitted radiation is, [cf. Eq. (4.92)].

$$P=\frac{2q^2}{3c^3}\gamma^4\frac{q^2B^2}{\gamma^2m^2c^2}v_{\perp}^2, \tag{6.5a}$$

or

$$P=\frac{2}{3}r_0^2c\beta_{\perp}^2\gamma^2B^2. \tag{6.5b}$$

For an isotropic distribution of velocities it is necessary to average this formula over all angles for a given speed β. Let α be the *pitch angle*, which is the angle between field and velocity. Then we obtain

$$\langle\beta_{\perp}^2\rangle=\frac{\beta^2}{4\pi}\int\sin^2\alpha\,d\Omega=\frac{2\beta^2}{3}, \tag{6.6}$$

and the result

$$P=\left(\tfrac{2}{3}\right)^2r_0^2c\beta^2\gamma^2B^2, \tag{6.7a}$$

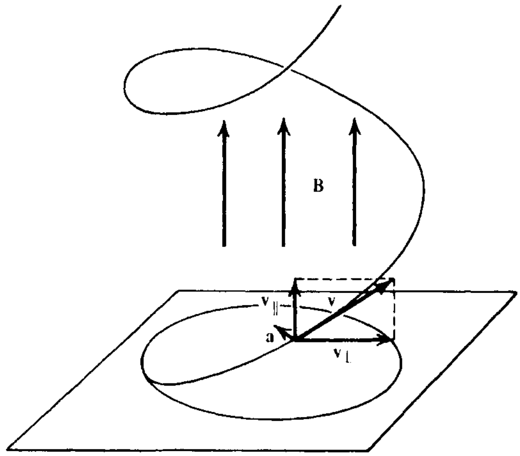

Figure 6.1 Helical motion of a particle in a uniform magnetic field.

which may be written

$$P=\frac{4}{3}\sigma_T c\beta^2\gamma^2 U_B. \tag{6.7b}$$

Here $\sigma_T=8\pi r_0^2/3$ is the Thomson cross section, and U_B is the magnetic energy density, $U_B=B^2/8\pi$.

6.2 SPECTRUM OF SYNCHROTRON RADIATION: A QUALITATIVE DISCUSSION

The spectrum of synchrotron radiation must be related to the detailed variation of the electric field as seen by an observer. Because of beaming effects the emitted radiation fields appear to be concentrated in a narrow set of directions about the particle's velocity. Since the velocity and acceleration are perpendicular, the appropriate diagram is like the one in Fig. 4.11d.

The observer will see a pulse of radiation confined to a time interval much smaller than the gyration period. The spectrum will thus be spread over a much broader region than one of order $\omega_B/2\pi$. This is an essential feature of synchrotron radiation.

We can find orders of magnitude by reference to Fig. 6.2. The observer will see the pulse from points 1 and 2 along the particle's path, where these points are such that the cone of emission of angular width $\sim 1/\gamma$ includes

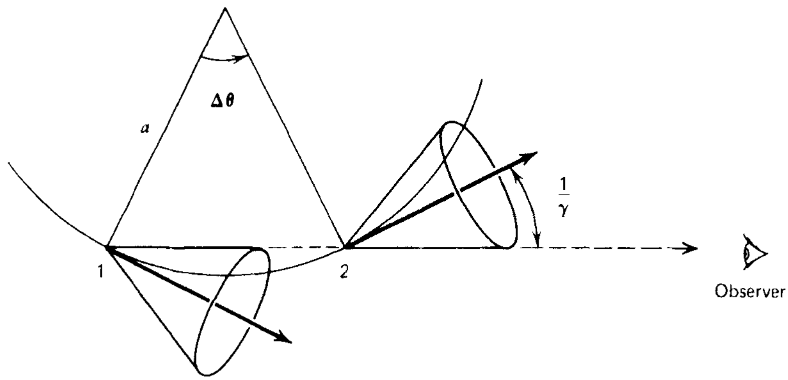

Figure 6.2 Emission cones at various points of an accelerated particle's trajectory.

the direction of observation. The distance Δs along the path can be computed from the radius of curvature of the path, $a=\Delta s/\Delta\theta$.

From the geometry we have $\Delta\theta=2/\gamma$, so that $\Delta s=2a/\gamma$. But the radius of curvature of the path follows from the equation of motion

$$\gamma m\frac{\Delta \mathbf{v}}{\Delta t}=\frac{q}{c}\mathbf{v}\times\mathbf{B},$$

Since $|\Delta\mathbf{v}|=v\,\Delta\theta$ and $\Delta s=v\,\Delta t$, we have

$$\frac{\Delta\theta}{\Delta s}=\frac{qB\sin\alpha}{\gamma mcv}, \tag{6.8a}$$

$$a=\frac{v}{\omega_B\sin\alpha}. \tag{6.8b}$$

Note that this differs by a factor $\sin\alpha$ from the radius of the circle of the projected motion in a plane normal to the field. Thus Δs is given by

$$\Delta s\approx\frac{2v}{\gamma\omega_B\sin\alpha}. \tag{6.8c}$$

The times t_1 and t_2 at which the particle passes points 1 and 2 are such that $\Delta s=v(t_2-t_1)$ so that

$$t_2-t_1\approx\frac{2}{\gamma\omega_B\sin\alpha}. \tag{6.9}$$

Let t_1^A and t_2^A be the *arrival times* of radiation at the point of observation from points 1 and 2. The difference $t_2^A - t_1^A$ is less than $t_2 - t_1$ by an amount $\Delta s/c$, which is the time for the radiation to move a distance Δs. Thus we have

$$\Delta t^A = t_2^A - t_1^A = \frac{2}{\gamma \omega_B \sin\alpha}\left(1 - \frac{v}{c}\right). \tag{6.10a}$$

It should be noted that the factor $(1 - v/c)$ is the same one that enters the Doppler effect [cf. §4.1]. Since $\gamma \gg 1$, we have

$$1 - \frac{v}{c} \approx \frac{1}{2\gamma^2},$$

so that

$$\Delta t^A \approx \left(\gamma^3 \omega_B \sin\alpha\right)^{-1}. \tag{6.10b}$$

Therefore, the width of the observed pulses is smaller than the gyration period by a factor γ^3. The pulse is shown in Fig. 6.3. From our general discussion of spectra associated with particular pulses, §2.3, we expect that the spectrum will be fairly broad, cutting off at frequencies like $1/\Delta t^A$. If we define a critical frequency

$$\omega_c \equiv \frac{3}{2}\gamma^3 \omega_B \sin\alpha \tag{6.11a}$$

or

$$\nu_c = \frac{3}{4\pi}\gamma^3 \omega_B \sin\alpha, \tag{6.11b}$$

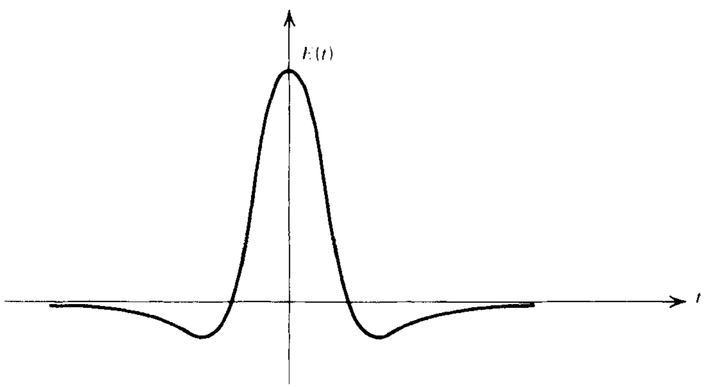

***Figure 6.3** Time-dependence of the electric field in a pulse of synchrotron radiation.*

then we expect the spectrum to extend to something of order ω_c before falling away. We can actually derive quite a lot about the spectrum, simply using the fact that the electric field is a function of θ solely through the combination $\gamma\theta$, (see, e.g., §4.8) where θ is a polar angle about the direction of motion. This is a manifestation of the *beaming effect*. Let us write

$$E(t) \propto F(\gamma\theta), \tag{6.12}$$

where t here refers to time measured in the observer's frame. We set the zero of time and the path length s to be when the pulse is centered on the observer. Using arguments similar to those used to find Δs, we find $\theta \approx s/a$ and $t \approx (s/v)(1-v/c)$. Then the relationship of θ to t is found to be

$$\gamma\theta \approx 2\gamma(\gamma^2\omega_B \sin\alpha)t \propto \omega_c t. \tag{6.13}$$

Therefore, we write the time dependence of the electric field as

$$E(t) \propto g(\omega_c t). \tag{6.14}$$

The proportionality constant here is not yet known, and it may depend on any physical parameters except time t. This is still sufficient for us to derive the general dependence of the spectrum on ω. The Fourier transform of the electric field is

$$\hat{E}(\omega) \propto \int_{-\infty}^{\infty} g(\omega_c t) e^{i\omega t} dt. \tag{6.15a}$$

Changing variables of integration to $\xi \equiv \omega_c t$, we have

$$\hat{E}(\omega) \propto \int_{-\infty}^{\infty} g(\xi) e^{i\omega\xi/\omega_c} d\xi. \tag{6.15b}$$

The spectrum $dW/d\omega d\Omega$ is proportional to the square of $\hat{E}(\omega)$ [cf. Eqs. (2.33) and (3.11a)]. Integrating this over solid angle and dividing by the orbital period, both independent of frequency, then gives for the time-averaged power per unit frequency, [cf. Eq. (2.34)],

$$\frac{dW}{dt\,d\omega} = T^{-1}\frac{dW}{d\omega} \equiv P(\omega) = C_1 F\left(\frac{\omega}{\omega_c}\right), \tag{6.16}$$

where F is a dimensionless function and C_1 is a constant of proportionality. We may now evaluate C_1 by the simple trick of comparing the total power as evaluated by the integral over ω to the previous result in Eq. (6.5):

$$P=\int_0^\infty P(\omega)d\omega=C_1\int_0^\infty F\left(\frac{\omega}{\omega_c}\right)d\omega=\omega_c C_1\int_0^\infty F(x)\,dx, \qquad (6.17a)$$

where we have set $x\equiv\omega/\omega_c$. We do not know what $\int F(x)\,dx$ is until we specify $F(x)$. However, we can regard its nondimensional value as arbitrary, merely setting a convention for the normalization of $F(x)$. We can still find the dependence of the constant C_1 on all the physical parameters. From our previous discussion, we have

$$P=\frac{2q^4B^2\gamma^2\beta^2\sin^2\alpha}{3m^2c^3}, \qquad (6.17b)$$

and

$$\omega_c=\frac{3\gamma^2 qB\sin\alpha}{2mc}. \qquad (6.17c)$$

We thus conclude that for the highly relativistic case ($\beta\approx 1$), the power per unit frequency emitted by each electron is

$$P(\omega)=\frac{\sqrt{3}}{2\pi}\frac{q^3B\sin\alpha}{mc^2}F\left(\frac{\omega}{\omega_c}\right). \qquad (6.18)$$

The choice $\sqrt{3}/2\pi$ for the nondimensional constant has been made to anticipate the conventional choice for the normalization of F, discussed below. If the power per frequency interval $d\nu$ is desired, one can use the relation $P(\nu)=2\pi P(\omega)$.

6.3 SPECTRAL INDEX FOR POWER-LAW ELECTRON DISTRIBUTION

From the formula for $P(\omega)$ given above, it is clear that no factor of γ appears, except for that contained in ω_c. From this fact alone it is possible to derive an extremely important result concerning synchrotron spectra. Often the spectrum can be approximated by a power law over a limited range of frequency. When this is so, one defines the *spectral index* as the constant s in the expression

$$P(\omega) \propto \omega^{-s}. \tag{6.19}$$

This is the negative slope on a $\log P(\omega) - \log \omega$ plot. Often the spectra of astronomical radiation has a spectral index that is constant over a fairly wide range of frequencies: for example, the Rayleigh–Jeans portion of the blackbody law has $s = -2$.

An analogous result sometimes holds for the particle distribution law of relativistic electrons. Often the number density of particles with energies between E and $E + dE$ (or γ and $\gamma + d\gamma$) can be approximately expressed in the form

$$N(E)dE = CE^{-p}dE, \qquad E_1 < E < E_2, \tag{6.20a}$$

or

$$N(\gamma)d\gamma = C\gamma^{-p}d\gamma, \qquad \gamma_1 < \gamma < \gamma_2. \tag{6.20b}$$

The quantity C can vary with pitch angle and the like. The total power radiated per unit volume per unit frequency by such a distribution is given by the integral of $N(\gamma)d\gamma$ times the single particle radiation formula over all energies or γ. Thus, we have

$$P_{\text{tot}}(\omega) = C\int_{\gamma_1}^{\gamma_2} P(\omega)\gamma^{-p}d\gamma \propto \int_{\gamma_1}^{\gamma_2} F\left(\frac{\omega}{\omega_c}\right)\gamma^{-p}d\gamma. \tag{6.21a}$$

Let us change variables of integration to $x \equiv \omega/\omega_c$, noting $\omega_c \propto \gamma^2$;

$$P_{\text{tot}}(\omega) \propto \omega^{-(p-1)/2} \int_{x_1}^{x_2} F(x) x^{(p-3)/2} dx. \tag{6.21b}$$

The limits x_1 and x_2 correspond to the limits γ_1 and γ_2 and depend on ω. However, if the energy limits are sufficiently wide we can approximate $x_1 \approx 0$, $x_2 \approx \infty$, so that the integral is approximately constant. In that case, we have

$$P_{\text{tot}}(\omega) \propto \omega^{-(p-1)/2} \tag{6.22a}$$

so that the spectral index s is related to the particle distribution index p by

$$s = \frac{p-1}{2}. \tag{6.22b}$$

Let us summarize the results of this simplified treatment of synchrotron radiation: We have shown that

1. The angular distribution from a single radiating particle lies close (within $1/\gamma$) to the cone with half-angle equal to the pitch angle.
2. The single-particle spectrum extends up to something of the order of a critical frequency ω_c. More precisely, the spectrum is a function of ω/ω_c alone.
3. For power law distribution of particle energies with index p over a sufficiently broad energy range, the spectral index of the radiation is $s=(p-1)/2$.

6.4 SPECTRUM AND POLARIZATION OF SYNCHROTRON RADIATION: A DETAILED DISCUSSION

Consider the orbital trajectory in Fig. 6.4, where the origin of the coordinates is the location of the particle at the origin of retarded time $t'=0$, and a is the radius of curvature of the trajectory. The coordinate system has been chosen so that the particle has velocity $\mathbf{v}$ along the x axis at time $t'=0$; $\boldsymbol{\epsilon}_{\perp}$ is a unit vector along the y axis in the orbital (x-y) plane, and

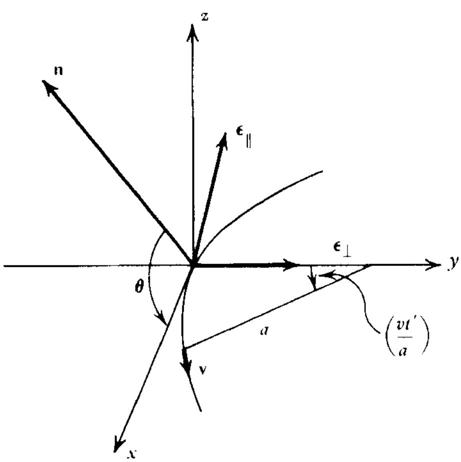

***Figure 6.4** Geometry for polarization of synchrotron radiation. At $t=0$, the particle velocity is along the x axis, and a is the radius of curvature of the trajectory.*

$\epsilon_{\parallel} = \mathbf{n} \times \epsilon_{\perp}$. Using Fig. 6.4, we have

$$\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta}) = -\epsilon_{\perp}\sin\left(\frac{vt'}{a}\right) + \epsilon_{\parallel}\cos\left(\frac{vt'}{a}\right)\sin\theta, \tag{6.23}$$

where we have set $|\boldsymbol{\beta}|=1$. This gives the first factor in Eq. (3.13) for $dW/d\omega\, d\Omega$. For the second factor in that equation, $\exp i\omega[t'-\mathbf{n}\cdot\mathbf{r}(t')/c]$, we note that

$$\begin{aligned} t' - \frac{\mathbf{n}\cdot\mathbf{r}(t')}{c} &= t' - \frac{a}{c}\cos\theta\sin\left(\frac{vt'}{a}\right) \\ &\approx (2\gamma^2)^{-1}\left[(1+\gamma^2\theta^2)t' + \frac{c^2\gamma^2t'^3}{3a^2}\right], \end{aligned} \tag{6.24}$$

where we have expanded the sine and cosine functions for small arguments, used the approximation $(1-v/c)\approx 1/2\gamma^2$, and set $v=c$ elsewhere. Note how the argument of the exponential in Eq. (6.24) is large and the integral is small unless $\gamma\theta \lesssim 1$, $c\gamma t'/a \lesssim 1$, in accordance with our qualitative discussion in 6.2 above.

An expression for the spectrum in the two polarizations states, that is, the intensity along $\epsilon_{\parallel}$ and intensity along $\epsilon_{\perp}$, may now be obtained from Eq. (3.13) and Eqs. (6.23) and (6.24) above. Expanding the sine and cosine functions again in Eq. (6.23), we obtain

$$\frac{dW}{d\omega\, d\Omega} \equiv \frac{dW_{\parallel}}{d\omega\, d\Omega} + \frac{dW_{\perp}}{d\omega\, d\Omega} \tag{6.25a}$$

$$\frac{dW_{\perp}}{d\omega\, d\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int \frac{ct'}{a}\exp\left[\frac{i\omega}{2\gamma^2}\left(\theta_\gamma^2 t' + \frac{c^2\gamma^2 t'^3}{3a^2}\right)\right]dt'\right|^2, \tag{6.25b}$$

$$\frac{dW_{\parallel}}{d\omega\, d\Omega} = \frac{q^2\omega^2\theta^2}{4\pi^2 c}\left|\int \exp\left[\frac{i\omega}{2\gamma^2}\left(\theta_\gamma^2 t' + \frac{c^2\gamma^2 t'^3}{3a^2}\right)\right]dt'\right|^2, \tag{6.25c}$$

where

$$\theta_\gamma^2 \equiv 1 + \gamma^2\theta^2. \tag{6.26a}$$

Now, making the changes of variables

$$y \equiv \gamma\frac{ct'}{a\theta_\gamma}, \tag{6.26b}$$

$$\eta \equiv \frac{\omega a \theta_\gamma^3}{3c\gamma^3}, \tag{6.26c}$$

Eqs. (6.25) become

$$\frac{dW_\perp}{d\omega\, d\Omega}=\frac{q^2\omega^2}{4\pi^2 c}\left(\frac{a\theta_\gamma^2}{\gamma^2 c}\right)^2\left|\int_{-\infty}^{\infty} y\exp\left[\frac{3}{2}i\eta\left(y+\frac{1}{3}y^3\right)\right]dy\right|^2, \tag{6.27a}$$

$$\frac{dW_\parallel}{d\omega\, d\Omega}=\frac{q^2\omega^2\theta^2}{4\pi^2 c}\left(\frac{a\theta_\gamma}{\gamma_c}\right)^2\left|\int_{-\infty}^{\infty}\exp\left[\frac{3}{2}i\eta\left(y+\frac{1}{3}y^3\right)\right]dy\right|^2, \tag{6.27b}$$

where little error is made in extending the limits of integration from $-\infty$ to ∞. The integrals in Eqs. (6.27a) and (6.27b) are functions only of the parameter η. Since most of the radiation occurs at angles $\theta\approx 0$, η can be written as

$$\eta\approx\eta(\theta=0)=\frac{\omega}{2\omega_c}, \tag{6.28}$$

where we have used Eqs. (6.8b) and (6.11a). Thus the frequency dependence of the spectrum depends on ω only through ω/ω_c, as found in our qualitative discussion. It should also be clear that the angular dependence uses θ only through the combination $\gamma\theta$.

To make further progress, we note that the integrals in Eq. (6.27) may be expressed in terms of the modified Bessel functions of 1/3 and 2/3 order, for example, formulas: 10.4.26, 10.4.31, and 10.4.32 of Abramovitz and Stegun (1965). Therefore we can write

$$\frac{dW_\perp}{d\omega\, d\Omega}=\frac{q^2\omega^2}{3\pi^2 c}\left(\frac{a\theta_\gamma^2}{\gamma^2 c}\right)^2 K_{\frac{2}{3}}^2(\eta), \tag{6.29a}$$

$$\frac{dW_\parallel}{d\omega\, d\Omega}=\frac{q^2\omega^2\theta^2}{3\pi^2 c}\left(\frac{a\theta_\gamma}{\gamma c}\right)^2 K_{\frac{1}{3}}^2(\eta). \tag{6.29b}$$

These formulas can now be integrated over solid angle to give the energy per frequency range radiated by the particle per complete orbit in the projected normal plane. During one such orbit the emitted radiation is almost completely confined to the solid angle shown shaded in Fig. 6.5, which lies within an angle $1/\gamma$ of a cone of half-angle α. Thus it is permissible to take the element of solid angle to be $d\Omega=2\pi\sin\alpha\, d\theta$, and we can write

$$\frac{dW_\perp}{d\omega}=\frac{2q^2\omega^2a^2\sin\alpha}{3\pi c^3\gamma^4}\int_{-\infty}^{\infty}\theta_\gamma^4 K_{\frac{2}{3}}^2(\eta)\,d\theta, \tag{6.30a}$$

$$\frac{dW_\parallel}{d\omega}=\frac{2q^2\omega^2a^2\sin\alpha}{3\pi c^3\gamma^2}\int_{-\infty}^{\infty}\theta_\gamma^2\theta^2 K_{\frac{1}{3}}^2(\eta)\,d\theta. \tag{6.30b}$$

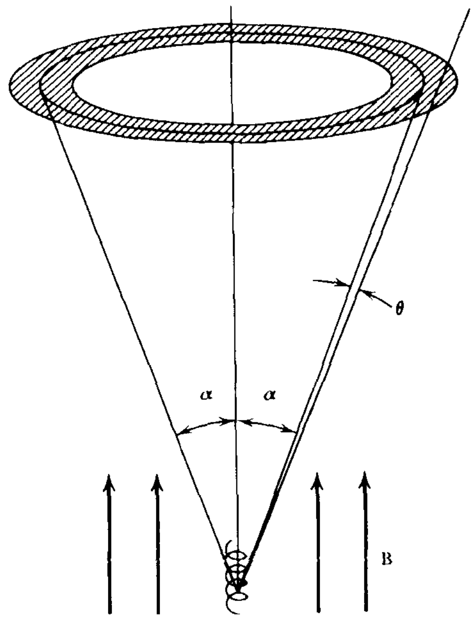

***Figure 6.5** Synchrotron emission from a particle with pitch angle α. Radiation is confined to the shaded solid angle.*

The infinite limits on the integral are convenient and permissible, because the integrand is concentrated to small values of $\Delta\theta$ about α, of order $1/\gamma$. The above integrals can be reduced further (see Westfold, 1959 for details), and we can write

$$\frac{dW_\perp}{d\omega}=\frac{\sqrt{3}\,q^2\gamma\sin\alpha}{2c}\left[F(x)+G(x)\right] \tag{6.31a}$$

$$\frac{dW_\parallel}{d\omega}=\frac{\sqrt{3}\,q^2\gamma\sin\alpha}{2c}\left[F(x)-G(x)\right], \tag{6.31b}$$

where

$$F(x)\equiv x\int_x^\infty K_{\frac{5}{3}}(\xi)\,d\xi, \qquad G(x)\equiv xK_{\frac{2}{3}}(x), \tag{6.31c}$$

and, again $x\equiv\omega/\omega_c$.

To convert this to emitted power per frequency we divide by the orbital period of the charge, $T = 2\pi/\omega_B$,

$$P_\perp(\omega) = \frac{\sqrt{3}\, q^3 B \sin\alpha}{4\pi mc^2}\left[F(x) + G(x)\right], \tag{6.32a}$$

$$P_\parallel(\omega) = \frac{\sqrt{3}\, q^3 B \sin\alpha}{4\pi mc^2}\left[F(x) - G(x)\right]. \tag{6.32b}$$

The total emitted power per frequency is the sum of these:

$$P(\omega) = \frac{\sqrt{3}\, q^3 B \sin\alpha}{2\pi mc^2} F(x), \tag{6.33}$$

in agreement with our previous Eq. (6.18). The function $F(x)$ is plotted in Fig. 6.6. Asymptotic forms for small and large values of x are:

$$F(x) \sim \frac{4\pi}{\sqrt{3}\,\Gamma\left(\frac{1}{3}\right)}\left(\frac{x}{2}\right)^{1/3}, \qquad x \ll 1, \tag{6.34a}$$

$$F(x) \sim \left(\frac{\pi}{2}\right)^{1/2} e^{-x} x^{1/2}, \qquad x \gg 1. \tag{6.34b}$$

To obtain frequency-integrated emission, or emission from a power-law distribution of electrons, it is useful to have expressions for integrals over the F and G functions. From Eq. 11.4.22 of Abramowitz and Stegun (1965)

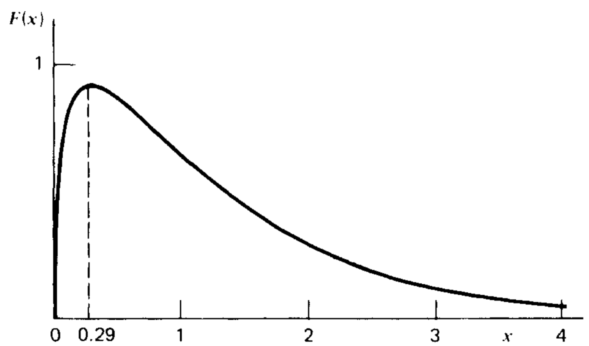

***Figure 6.6** Function describing the total power spectrum of synchrotron emission. Here $x = \omega/\omega_c$. (Taken from Ginzburg, V. and Syrovatskii, S. 1965, Ann. Rev. Astron. Astrophys., 3, 297.)*

one may derive the following relations:

$$\int_0^\infty x^\mu F(x)\,dx = \frac{2^{\mu+1}}{\mu+2}\Gamma\left(\frac{\mu}{2}+\frac{7}{3}\right)\Gamma\left(\frac{\mu}{2}+\frac{2}{3}\right) \tag{6.35a}$$

$$\int_0^\infty x^\mu G(x)\,dx = 2^\mu\Gamma\left(\frac{\mu}{2}+\frac{4}{3}\right)\Gamma\left(\frac{\mu}{2}+\frac{2}{3}\right) \tag{6.35b}$$

where $\Gamma(y)$ is the gamma function of argument y.

For a *power-law distribution of electrons*, Eq. (6.20b), it can be shown from Eqs. (6.33) and (6.35a) that the total power per unit volume per unit frequency, $P_{\mathrm{tot}}(\omega)$, is

$$P_{\mathrm{tot}}(\omega) = \frac{\sqrt{3}\,q^3 C B \sin\alpha}{2\pi mc^2(p+1)}\Gamma\left(\frac{p}{4}+\frac{19}{12}\right)\Gamma\left(\frac{p}{4}-\frac{1}{12}\right)\left(\frac{mc\omega}{3qB\sin\alpha}\right)^{-(p-1)/2} \tag{6.36}$$

6.5 POLARIZATION OF SYNCHROTRON RADIATION

We can also compute the polarization for synchrotron radiation. The first point to notice is that the radiation from a single charge will be elliptically polarized, the sense of the polarization (right or left handed) being determined by whether the observed line of sight lies just inside or just outside of the cone of maximal radiation (see Fig. 6.5). However, for any reasonable distribution of particles that varies smoothly with pitch angle, the elliptical component will cancel out, as emission cones will contribute equally from both sides of the line of sight. Thus the radiation will be partially linearly polarized, and we can completely characterize the radiation by its powers per unit frequency $P_{\parallel}(\omega)$ and $P_{\perp}(\omega)$, in directions parallel and perpendicular to the projection of the magnetic field on the plane of the sky (see Fig. 6.7). From Eqs. (2.57), (6.32a), and (6.32b) we obtain the degree of linear polarization for particles of a single energy γ:

$$\Pi(\omega) = \frac{P_{\perp}(\omega) - P_{\parallel}(\omega)}{P_{\perp}(\omega) + P_{\parallel}(\omega)} = \frac{G(x)}{F(x)}. \tag{6.37}$$

This polarization is rather high; the polarization of the frequency integrated radiation is 75% (see Problem 6.5b).

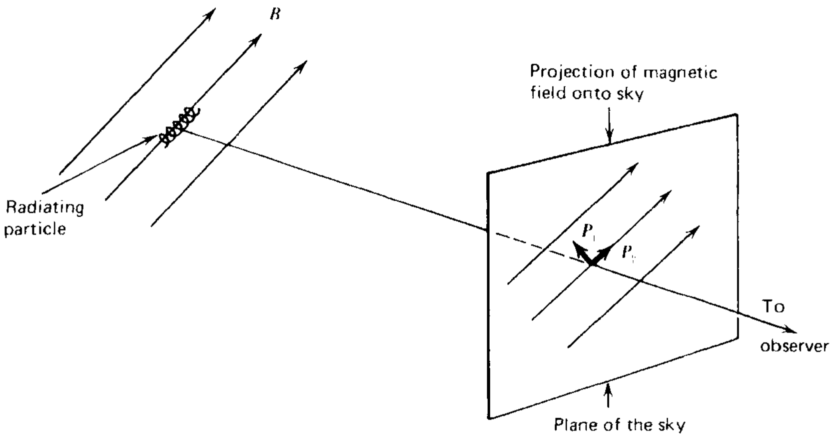

***Figure 6.7** Decomposition of synchrotron polarization vectors on the plane of the sky.*

For particles with a power law distribution of energies, Eq. (6.20), the degree of polarization can be shown to be (see Problem 6.5a)

$$\Pi = \frac{p+1}{p+\frac{7}{3}}. \tag{6.38}$$

6.6 TRANSITION FROM CYCLOTRON TO SYNCHROTRON EMISSION

It is interesting to follow the development of the typical synchrotron spectrum as the electron's energy is varied from the nonrelativistic through the highly relativistic regimes. Let us consider both the electric field at the observation point and the associated spectrum of radiation. For low energies the electric field components vary sinusoidally with the same frequency as the gyration in the magnetic field, and the spectrum consists of a single line, as shown in Figs. 6.8a and 6.8b (see Problem 3.2).

When v/c increases, higher harmonics of the fundamental frequency, ω_B, begin to contribute. It should be clear that the general spectrum, in fact, must be a superposition of contributions at integer multiples of ω_B, since there is periodicity in time intervals $T = 2\pi/\omega_B$. Problem 3.7 demonstrates the general property that a circulating charge produces radiation at harmonics of the fundamental and that increasing harmonics contribute at

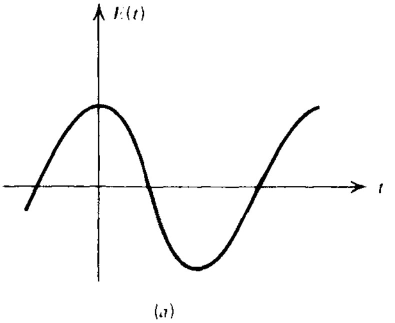

***Figure 6.8a** Time dependence of electric field from slowly moving particle in a magnetic field (cyclotron radiation).*

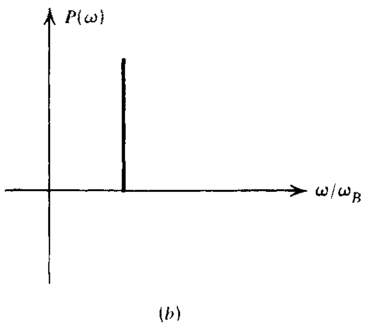

***Figure 6.8b** Power spectrum for a.*

a strength proportional to increasing powers of v/c for $v/c \ll 1$. For example, at slightly relativistic velocities, Fig. 6.8 becomes Fig. 6.9. Here we have adopted the convention that the electric field is positive as the particle approaches the observer. We see that the positive phase of the electric field has become somewhat sharper and more intense relative to the negative phase (Doppler effect). There is now a substantial amount of radiation at the first harmonic of ω_B (i.e., $2\omega_B$).

Finally, for very relativistic velocities, $v \sim c$, we have Fig. 6.10. The originally sinusoidal form of $E(t)$ has now become a series of sharp pulses, which are repeated at time intervals $2\pi/\omega_B$. The spectrum now involves a great number of harmonics, the envelope of which approaches the form of the function $F(x)$. As soon as the frequency resolution becomes large with respect to ω_B, or if other physical broadening mechanisms fill in the spaces between the lines, we approach the results derived earlier. One such

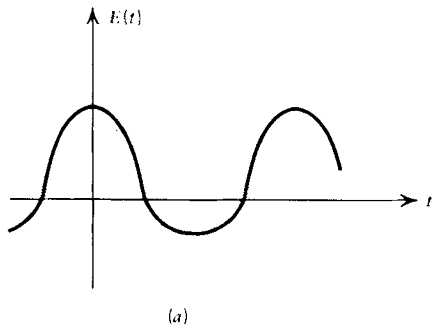

***Figure 6.9a** Time dependence of electric field from a particle of intermediate velocity in a magnetic field.*

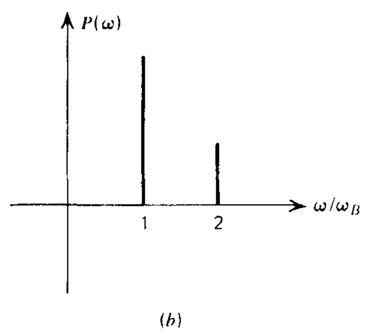

***Figure 6.9b** Power spectrum for a.*

physical broadening mechanism occurs for a distribution of particle energies; then the gyration frequency ω_B is proportional to $1/\gamma$, so that the spectra of the particles do not fall on the same lines. Another effect that will cause the spectrum to become continuous is that emission from different parts of the emitting region may have different values and directions for the magnetic field, so that the harmonics fall at different places in the observed spectrum.

The electric field received by the observer from a distribution of particles consists of a random superposition of many pulses of the kind described here. The net result is a spectrum that is simply the sum of the spectra from the individual pulses (see Problem 3.6).

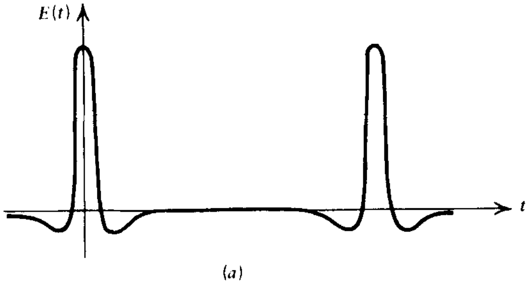

***Figure 6.10a** Time dependence of electric field from a rapidly moving particle in a magnetic field (synchrotron radiation).*

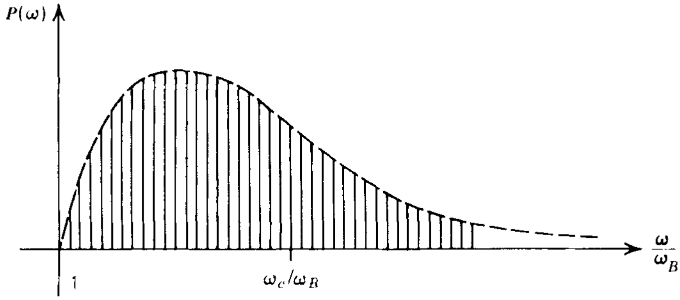

***Figure 6.10b** Power spectrum for a.*

6.7 DISTINCTION BETWEEN RECEIVED AND EMITTED POWER

In about 1968 (e.g., Pacholczyk, 1970; Ginzburg and Syrovatskii, 1969), it was noticed that a proper distinction between received and emitted power had not been made. (In looking at references before then check your formulas carefully.) The problem is that the *received* pulses are not at the frequency ω_B but at an appropriately Doppler-shifted frequency, because of the progressive motion of the particle toward the observer. This can be seen clearly in Fig. 6.11. If $T=2\pi/\omega_B$ is the orbital period of the projected motion, then time-delay effects (cf. §4.1), will give a period between the arrival of pulses T_A satisfying

$$\begin{aligned} T_A &= T\left(1-\frac{v_{\|}}{c}\cos\alpha\right) \\ &= T\left(1-\frac{v}{c}\cos^2\alpha\right)\approx\frac{2\pi}{\omega_B}\sin^2\alpha. \end{aligned} \tag{6.39}$$

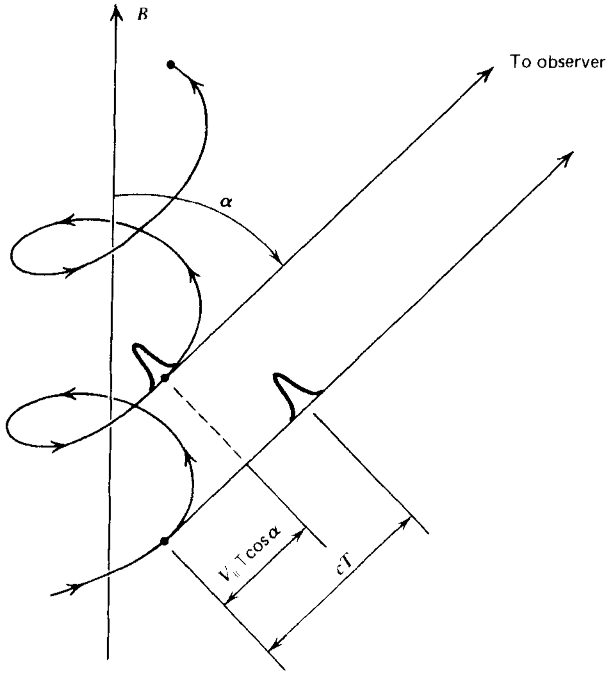

Figure 6.11 Doppler shift of synchrotron radiation emitted by a particle moving toward the observer.

The fundamental *observed* frequency is thus $\omega_B/\sin^2\alpha$ rather than ω_B. This leads to two modifications to the preceding theory, neither of which is serious, fortunately:

1. The first is that the spacing of the harmonics is $\omega_B/\sin^2\alpha$ not ω_B. For extreme relativistic particles this is not important, because one sees a continuum rather than the harmonic structure. In deriving the expression for the pulse width Δt_A and consequently for the critical frequency ω_c, we did take the Doppler compression of the radiation properly into account. Thus the continuum radiation is still a function $F(\omega/\omega_c)$.

2. The second comes from the fact that we found the *emitted power* by dividing the energy by the period T of the gyration. This is correct, but the *received power* must be obtained by dividing the energy by T_A. Thus, we have

$$P_r = \frac{P_e}{\sin^2\alpha}. \tag{6.40}$$

The question arises, should we include the $\sin^2\alpha$ factor in determining the received power? The answer depends on the physical case. Usually one observes a region fairly localized in space with only moderate net velocity toward the receiver. Then any particle that is progressing toward the receiver at one time will at a later time be moving away (and thus not contributing to the power). The average power emitted and received under these circumstances will be the same, because the total number of emitted and received pulses must be the same in the long run (see Problem 6.3). Even over short intervals this will hold when there is a stationary distribution of particles.

We conclude then that for the usual situation encountered in astrophysics one should use the expression for the emitted power to give the proper observed power. Thus the "corrections" due to helical motion are not important for most cases of interest.

6.8 SYNCHROTRON SELF-ABSORPTION

Synchrotron emission is accompanied by absorption, in which a photon interacts with a charge in a magnetic field and is absorbed, giving up its energy to the charge. Another process that can occur is stimulated emission or negative absorption, in which a particle is induced to emit more strongly into a direction and at a frequency where photons already are present. These processes can be interrelated by means of the Einstein coefficients. In our previous discussion of the Einstein coefficients (§1.6) we treated transitions between discrete states, and we must generalize that discussion now to include continuum states. This is easily done by recognizing that the states of an emitting particle are simply the free particle states, defined by its momentum, position, and possibly its internal state. According to the statistical mechanics there is one quantum state associated with the translational degrees of freedom of the particle within a volume of phase space of magnitude h^3. Thus we break up the continuous classical phase space into elements of size h^3, and consider transitions between these states as being between discrete states, for which our previous discussion applies.

A further modification of our previous results is necessary, because for a given energy of a photon $h\nu$ there are many possible transitions possible between states differing in energy by an amount $h\nu$. This means that, in the formula for the absorption coefficient given in Eq. (1.74), we must sum over all upper states 2 and lower states 1:

$$\alpha_\nu = \frac{h\nu}{4\pi} \sum_{E_1} \sum_{E_2} \left[n(E_1) B_{12} - n(E_2) B_{21} \right] \phi_{21}(\nu). \tag{6.41}$$

The profile function $\phi_{21}(\nu)$ is essentially a δ-function that restricts the summations to those states differing by an energy $h\nu = E_2 - E_1$; it will drop out of the final formulas. We have assumed that the emission and absorption are isotropic [as we did for Eq. (1.74)]. For synchrotron emission this requires that the magnetic field be tangled and have no net direction, and that the particle distributions also be isotropic.

It is now our task to reduce Eq. (6.41) to a form depending only on the previously derived formula for synchrotron emission (6.33). It is more convenient here to write the emission in terms of the frequency ν rather than ω, so that we use $P(\nu, E_2) = 2\pi P(\omega)$. We have also explicitly written the argument E_2, the energy of the radiating electron. In terms of the Einstein coefficients we have

$$\begin{aligned} P(\nu, E_2) &= h\nu \sum_{E_1} A_{21}\phi_{21}(\nu) \\ &= (2h\nu^3/c^2)h\nu \sum_{E_1} B_{21}\phi_{21}(\nu) \end{aligned} \tag{6.42}$$

where we have used one of the Einstein relations (1.71b). (Since we are dealing with elementary states, the statistical weights are all unity.)

The parts of the absorption coefficient (6.41) due to stimulated emission can be now written in terms of $P(\nu, E_2)$:

$$\frac{-h\nu}{4\pi} \sum_{E_1} \sum_{E_2} n(E_2) B_{21}\phi_{21} = \frac{-c^2}{8\pi h\nu^3} \sum_{E_2} n(E_2) P(\nu, E_2). \tag{6.43}$$

The true absorption part can be written

$$\frac{h\nu}{4\pi} \sum_{E_1} \sum_{E_2} n(E_1) B_{12}\phi_{21} = \frac{c^2}{8\pi h\nu^3} \sum_{E_2} n(E_2 - h\nu) P(\nu, E_2). \tag{6.44}$$

Here we have used the Einstein relation $B_{12} = B_{21}$. Also we have made use of the *continuous* nature of the problem by moving $n(E_1)$ from under the summation sign and replacing it by $n(E_2 - h\nu)$. This is permissible because $\phi_{21}(\nu)$ acts essentially like a δ function, enforcing the energy relation $E_1 = E_2 - h\nu$. Therefore, we have

$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3} \sum_{E_2} \left[n(E_2 - h\nu) - n(E_2) \right] P(\nu, E_2). \tag{6.45}$$

Let us introduce the isotropic electron distribution function $f(p)$ by $f(p)d^3p$ = number of electrons/volume with momenta in d^3p about p.

According to statistical mechanics, the number of quantum states/volume range d^3p is simply $\tilde{\omega}h^{-3}d^3p$, where $\tilde{\omega}$ is the internal statistical weight of the electron ($=2$ for spin$=1/2$ particles). The electron density per quantum state is thus $(h^3/\tilde{\omega})f(p)$. Therefore, we can make the replacements

$$\sum_2 \to \frac{\tilde{\omega}}{h^3}\int d^3p_2, \qquad n(E_2)\to\frac{h^3}{\tilde{\omega}}f(p_2).$$

Then Eq. (6.45) becomes

$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3}\int d^3p_2\left[f(p_2^*)-f(p_2)\right]P(\nu,E_2) \tag{6.46}$$

where p_2^* is the momentum corresponding to energy $E_2-h\nu$. Before specializing this formula further, let us check that it yields the correct result for a *thermal distribution* of particles, that is

$$f(p)=K\exp\left[-\frac{E(p)}{kT}\right].$$

We note that

$$\begin{aligned} f(p_2^*)-f(p_2) &= K\exp\left(-\frac{E_2-h\nu}{kT}\right)-K\exp\left(-\frac{E_2}{kT}\right)\\ &= f(p_2)(e^{h\nu/kT}-1). \end{aligned}$$

Thus the absorption coefficient is

$$(\alpha_\nu)_{\text{thermal}} = \frac{c^2}{8\pi h\nu^3}(e^{h\nu/kT}-1)\int d^3p_2 f(p_2)P(\nu,E_2). \tag{6.47}$$

But the integral here simply represents the total power per volume per frequency range, which is $4\pi j_\nu$ for isotropic emission. Recognizing the formula for $B_\nu(T)$ this can be written

$$(\alpha_\nu)_{\text{thermal}} = \frac{j_\nu}{B_\nu(T)},$$

which is the correct result for thermal emission (Kirchhoff's Law).

Because the electron distribution is isotropic it is convenient to use the energy rather than the momentum to describe the distribution function, that is, $N(E)$, as in Eq. (6.20). We shall also assume the extreme relativistic relation $E=pc$. Then from the relation

$$N(E)dE=f(p)4\pi p^2dp \tag{6.48}$$

we obtain

$$\alpha_\nu=\frac{c^2}{8\pi h\nu^3}\int dE\,P(\nu,E)E^2\left[\frac{N(E-h\nu)}{(E-h\nu)^2}-\frac{N(E)}{E^2}\right], \tag{6.49}$$

where we now have simply written E instead of E_2.

We now assume that $h\nu \ll E$. This is, in fact, a necessary condition for the application of classical electrodynamics, so is already an implicit restriction on our formula for $P(\nu,E)$. Expanding to first order in $h\nu$ we obtain

$$\alpha_\nu=-\frac{c^2}{8\pi\nu^2}\int dE\,P(\nu,E)E^2\frac{\partial}{\partial E}\left[\frac{N(E)}{E^2}\right]. \tag{6.50}$$

Let us again look at the case of a thermal distribution, which for ultrarelativistic particles is

$$N(E)=KE^2e^{-E/kT}. \tag{6.51}$$

This leads to the result

$$(\alpha_\nu)_{\text{thermal}}=\frac{c^2}{8\pi\nu^2kT}\int N(E)P(\nu,E)dE=\frac{j_\nu c^2}{2\nu^2kT},$$

which is Kirchhoff's law in the Rayleigh–Jeans regime. This is to be expected, because of the assumption $h\nu \ll E$.

For a *power law distribution of particles*, Eq. (6.20), we have

$$-E^2\frac{d}{dE}\left[\frac{N(E)}{E^2}\right]=(p+2)CE^{-(p+1)}=\frac{(p+2)N(E)}{E},$$

and the absorption coefficient (6.50) can be written

$$\alpha_\nu=\frac{(p+2)c^2}{8\pi\nu^2}\int dE\,P(\nu,E)\frac{N(E)}{E}. \tag{6.52}$$

It is straightforward to show, using Eqs. (6.33) and (6.35a), that the integral gives

$$\alpha_\nu = \frac{\sqrt{3}\,q^3}{8\pi m}\left(\frac{3q}{2\pi m^3 c^5}\right)^{p/2} C(B\sin\alpha)^{(p+2)/2}\Gamma\left(\frac{3p+2}{12}\right)\Gamma\left(\frac{3p+22}{12}\right)\nu^{-(p+4)/2}. \tag{6.53}$$

The source function can be found from

$$S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{P(\nu)}{4\pi\alpha_\nu} \propto \nu^{5/2}, \tag{6.54}$$

using Eq. (6.53). A simple way of deriving this latter result is to note that S_ν can be written as $S_\nu \propto \nu^2 \bar{E}$ where $\bar{E}$ is a mean particle energy [cf. Eqs. (6.52) and (6.54). The appropriate value for $\bar{E}$ is the energy of those electrons whose critical frequency equals ν, that is, $\bar{E}^2 \propto \nu_c = \nu$, so that one obtains the proportionality given in Eq. (6.54). It is of some interest that the source function is a power law with an index $-\frac{5}{2}$, independent of the value of p. It should be particularly noted that this index is not equal to -2, the Rayleigh–Jeans value, because the emission is nonthermal.

For optically thin synchrotron emission, the observed intensity is proportional to the emission function, while for optically thick emission it is proportional to the source function. Since the emission and source functions for a nonthermal power law electron distribution are proportional to $\nu^{-(p-1)/2}$ and $\nu^{5/2}$, respectively, [cf. eqs. (6.22a) and (6.54)] we see that the optically thick region occurs at low frequencies and produces a low-frequency cutoff of the spectrum (see Fig. 6.12).

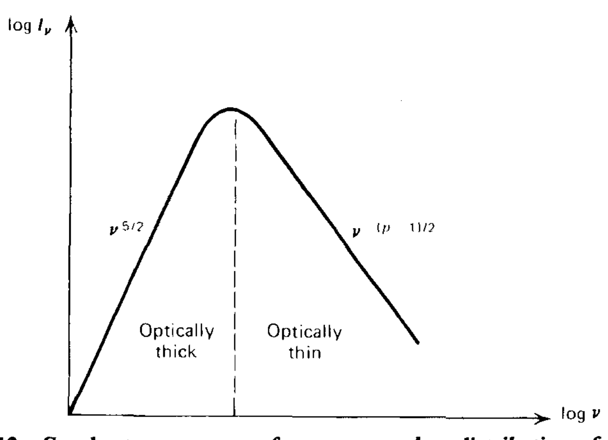

Figure 6.12 Synchrotron spectrum from a power-law distribution of electrons.

6.9 THE IMPOSSIBILITY OF A SYNCHROTRON MASER IN VACUUM

It is possible to prove that the absorption coefficient is positive for an arbitrary distribution of particle energies $N(E)$. That is, if we attempted to cause a population inversion by increasing $N(E)$ at a certain energy E_0 so that emission from E_0 to $E_0 - h\nu$ was a maser transition, we would inevitably be making a still stronger positive absorption somewhere else that would more than compensate. To show this analytically we can integrate Eq. (6.50) by parts, noting that $N(E)P(\nu,E)$ vanishes for low and high energies:

$$\alpha_\nu = \frac{c^2}{8\pi\nu^2}\int \frac{N(E)}{E^2}\frac{d}{dE}\left[E^2P(\nu,E)\right]dE.$$

For any fixed ν,

$$E^2P(\nu,E) \propto x^{-1}F(x) = \int_x^\infty K_{\frac{5}{3}}(\eta)\,d\eta.$$

This is clearly a monotonically decreasing function of x, since $K_{\frac{5}{3}}(\eta)$ is positive. Therefore, $E^2P(\nu,E)$ is a monotonically increasing function of E, and α_ν is positive.

We actually should also look at the absorption coefficients for specific polarization states to complete the proof of impossibility of masers. For the two states of polarization

$$P(\nu,E) \propto F(x) \pm G(x).$$

Since $x^{-1}G(x) = K_{\frac{2}{3}}(x)$, which decreases monotonically with x, we need only consider the polarization state in the parallel direction. By use of Eq. 10.1.22 of Abramowitz and Stegun (1965), we obtain the identity

$$\frac{1}{x}\left[F(x) - G(x)\right] = \frac{2}{3}\int_x^\infty K_{\frac{2}{3}}(\eta)\eta^{-1}\,d\eta,$$

which again is clearly monotonically decreasing with x.

Although synchrotron masers cannot exist in vacuum, it is possible to show that in a plasma, where the index of refraction is not unity, such synchrotron maser emission is possible.

PROBLEMS

6.1—An ultrarelativistic electron emits synchrotron radiation. Show that its energy decreases with time according to

$$\gamma=\gamma_0(1+A\gamma_0 t)^{-1}, \qquad A=\frac{2e^4B_\perp^2}{3m^3c^5}.$$

Here γ_0 is the initial value of γ and $B_\perp = B\sin\alpha$. Show that the time for the electron to lose half its energy is

$$t_{\frac{1}{2}}=(A\gamma_0)^{-1}=\frac{5.1\times10^8}{\gamma_0 B_\perp^2}.$$

How does one reconcile the decrease of γ here with the result of constant γ implied by Eqs. (6.1)?

6.2—A region of space contains relativistic electrons and magnetic fields. Let a typical linear scale of this region be l. Suppose the region is compressed (by passage of a shock wave, perhaps). Assume that the compression is the same in all directions. We want to see what effect this compression has on various properties of the electrons and magnetic field.

a. Show that the magnetic field satisfies $B\propto l^{-2}$.

b. If the compression is slow, show that the momentum of an electron satisfies $p\propto l^{-1}$, and that magnetic flux through electron orbits is approximately conserved.

c. Show that the synchrotron emission $P\propto l^{-6}$, that the critical frequency $\nu_c\propto l^{-4}$ and that the half-life for the electron $t_{\frac{1}{2}}\propto l^5$. (This shows that moderate compression can profoundly effect observed emission.)

6.3—Ultrarelativistic electrons are emitting synchrotron radiation in a fairly uniform magnetic field. The observer's line of sight makes an angle α with respect to B. (See Fig. 6.13).

The electrons are confined to the region between points 1 and 2 by constrictions in the magnetic field, which reflect the electrons back and forth along the field lines while maintaining their pitch angles. Show that a given electron, while radiating continually in its own frame, produces observable radiation only for a fraction $\frac{1}{2}\sin^2\alpha$ of the time.

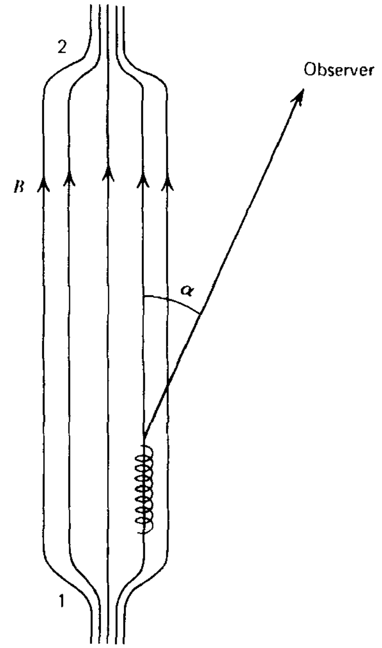

***Figure 6.13** Synchrotron emission from electrons confined between positions 1 and 2.*

6.4—The spectrum shown in Fig. 6.14 is observed from a point source of unknown distance d. A model for this source is a spherical mass of radius R that is emitting synchrotron radiation in a magnetic field of strength B. The space between us and the source is uniformly filled with a thermal bath of hydrogen that emits and absorbs mainly by bound-free transitions, and it is believed that the hydrogen bath is unimportant compared to the synchrotron source at frequencies where the former is optically thin. The synchrotron source function can be written as

$$S_\nu = A(\mathrm{erg\,cm^{-2}\,s^{-1}\,Hz^{-1}})\left(\frac{B}{B_0}\right)^{-1/2}\left(\frac{\nu}{\nu_0}\right)^{5/2}.$$

The absorption coefficient for synchrotron radiation is

$$\alpha_\nu^s = C(\mathrm{cm^{-1}})\left(\frac{B}{B_0}\right)^{(p+2)/2}\left(\frac{\nu}{\nu_0}\right)^{-(p+4)/2},$$

and that for bound-free transitions is

$$\alpha_\nu^{bf} = D(\mathrm{cm^{-1}})\left(\frac{\nu}{\nu_0}\right)^{-3},$$

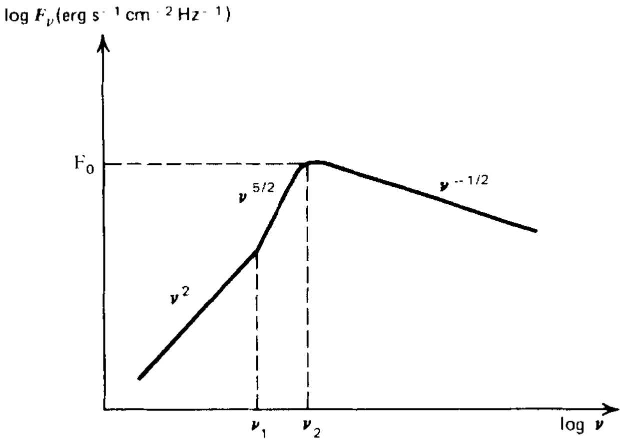

Figure 6.14 Observed spectrum from a point source.

where A, B_0, ν_0, C and D are constants and p is the power law index for the assumed power law distribution of relativistic electrons in the synchrotron source.

a. Find the size of the source R and the magnetic field strength B in terms of the solid angle $\Omega = \pi(R^2/d^2)$ subtended by the source and the constants A, B_0, ν_0, C, D.

b. Now using D and ν_1, in addition to the previous constants, find the solid angle of the source and its distance.

6.5

a. Derive the linear polarization for a power-law distribution of electrons, $N(\gamma) = C\gamma^{-p}$, emitting synchrotron radiation, Eq. (6.38).

b. Show that the linear polarization for the frequency-integrated synchrotron emission of particles of the same γ is 75%.

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