Modeling with ODE

P. Howard

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1 Overview

A wide variety of natural phenomena such as projectile motion, the flow of electric current, and the progression of chemical reactions are well described by equations that relate changing quantities. As the derivative of a function provides the rate at which that function is changing with respect to its independent variable, the equations describing these phenomena often involve one or more derivatives, and we refer to them as differential equations. In these notes we consider two important aspects in the theory of ordinary differential equations: 1. Developing models of physical phenomena; and 2. Determining whether our models are mathematically "well-posed" (do solutions exist? are these solutions unique? do the solutions we find for our equation genuinely correspond with the phenomenon we are modeling).

Solutions to ordinary differential equations cannot be determined uniquely without some outside condition, typically an initial value or a boundary value. In order to understand the nature of this information, consider the general first order equation

$$y' = f(t, y), \tag{1.1}$$

for which $'$ denotes differentiation with respect to t. Assuming $f(t, y)$ is sufficiently differentiable, we can develop a solution to (1.1) for t sufficiently small through the Taylor approximation,

$$y(t) = y(0) + y'(0)t + \frac{1}{2}y''(0)t^2 + ... \tag{1.2}$$

Suppose we know the initial value for $y(t)$, $y(0)$. Observe, then, that we can compute $y'(0)$ directly from (1.1):

$$y'(0) = f(0, y(0)).$$

Similarly, by differentiating (1.1) with respect to t, we have

$$y'' = \frac{\partial}{\partial t}f(t, y) + \frac{\partial}{\partial y}f(t, y)y',$$

and we can compute $y''(0)$ as

$$y''(0) = \frac{\partial}{\partial t}f(0, y(0)) + \frac{\partial}{\partial y}f(0, y(0))y'(0).$$

Proceeding similarly, we can develop the entirety of expansion (1.2).

2 Compartment Analysis

Suppose $y(t)$ denotes the amount of substance in some compartment at time t. For example, $y(t)$ might denote the liters of gasoline in a particular tank or the grams of medicine in a particular organ. We can compute the change in quantity $y(t)$ in terms of the amount of this quantity flowing into the compartment and the amount flowing out, as

$$\frac{dy}{dt} = \text{input rate} - \text{output rate}.$$

Example 2.1. Suppose saltwater is pumped into a tank with constant rate r cm$^3/s$, and is pumpted out at the same rate, and that the concentration of salt in the incoming water is c grams/cm^3. If the volume of water in the tank is V cm^3, find an equation for the amount of salt in the tank at time t.

Let $y(t)$ denote the grams of salt at time t, and notice that

$$\frac{dy}{dt} = \text{input rate} - \text{output rate} = cr - \frac{y(t)}{V}r = r(c - \frac{y(t)}{V}).$$

Example 2.2. (Drug concentration in an organ.) Suppose blood carries a certain drug into an organ at variable rate $r_I(t)$ cm^3/s and out of the organ with variable rate $r_O(t)$ cm^3/s, and that the organ has an initial blood volume V cm^3. If the concentration of drug in the body is $c(t)$ g/cm^3, determine an ODE for the amount of drug in the organ at time t.

Let $y(t)$ denote the amount of drug in the organ at time t, measured in grams. The input rate is then $r_I(t)c(t)$, while the output rate, assuming instantaneous mixing, is $\frac{y(t)}{V(t)}r_O(t)$, where the volume of blood in the organ $V(t)$ can be computed as the initial volume V plus the difference between the blood that flows into the organ over time t and the blood that flows out during the same time:

$$V(t) = V + \int_0^t r_I(s) - r_O(s)ds.$$

We have, then, the ODE

$$\frac{dx}{dt} = c(t)r_I(t) - \frac{y(t)}{V + \int_0^t r_I(s) - r_O(s)ds} r_O(t).$$

$\triangle$

Example 2.3. Suppose M grams of a certain heart medication are injected into a patient at time 0, and that whenever the drug is present in the heart its absorption rate out of the bloodstream (and into the heart tissue) is proportional to the amount in the heart with proportionality constant $r_A\, s^{-1}$. If blood flows into the patient's heart with rate r_I cm$^3/s$ and out with rate r_O cm$^3/s$, and if the volume of blood in the heart at time 0 is V_H and the volume of blood in the patient's body (minus the heart) at time 0 is V_B, develop a model for the amount of drug absorbed into the heart tissue by time t.

Let $y(t)$ denote the amount of drug in the heart at time t, and let $A(t)$ denote the total amount absorbed into the heart tissue by time t. Notice that $\frac{dA}{dt} = r_A y(t)$ (i.e., the rate of absorption $\frac{dA}{dt}$ is proportional to the amount in the heart), and so

$$A(t) = r_A \int_0^t y(s)ds.$$

Now,

$$\frac{dy}{dt} = \frac{M - y(t) - A(t)}{V_B - \int_0^t r_I(s) - r_O(s)ds} r_I(t) - \frac{y(t)}{V_H + \int_0^t r_I(s) - r_O(s)ds} r_O(t) - r_A y.$$

Coupling this with

$$\frac{dA}{dt} = r_A y,$$

we have a system of two equations for the quantities $y(t)$ and $A(t)$.

Example 2.4. (Cleaning the Great Lakes.) The Great Lakes are connected by a network of waterways, as roughly depicted in Figure 2.1. Assume the volume of each lake remains constant, and that water flows into the lake with volume V_k at rate r_k. Suppose pollution

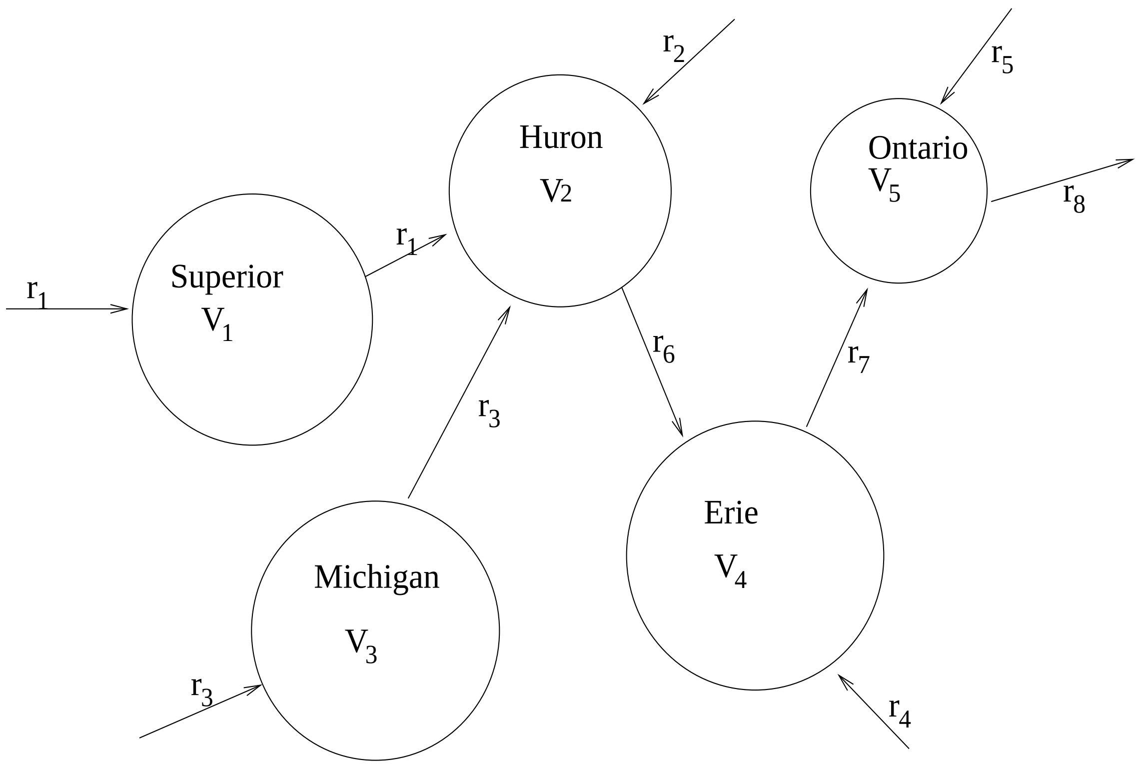

Figure 2.1: Great Lakes waterway network.

stops abruptly (i.e., no more pollution flows into the lakes) and develop a system of ODE that models the progression of pollution as it clears from the lakes.

We have one differential equation for each lake. Let $x_k(t)$ represent the amount of pollutant in the lake with volume V_k. We obtain the system

$$
\begin{aligned}
\frac{dx_1}{dt} &= -\frac{r_1}{V_1}x_1 \\
\frac{dx_2}{dt} &= +\frac{r_1}{V_1}x_1 + \frac{r_3}{V_3}x_3 - \frac{r_1+r_2+r_3}{V_2}x_2 \\
\frac{dx_3}{dt} &= -\frac{r_3}{V_3}x_3 \\
\frac{dx_4}{dt} &= \frac{r_1+r_2+r_3}{V_2}x_2 - \frac{r_1+r_2+r_3+r_4}{V_4}x_4 \\
\frac{dx_5}{dt} &= \frac{r_1+r_2+r_3+r_4}{V_4}x_4 - \frac{r_1+r_2+r_3+r_4+r_5}{V_5}x_5.
\end{aligned}
$$

△

3 Chemical Reactions

A chemical reaction is a process of molecular rearrangement by which one or more substances may be transformed into one or more new substances. For example, nitrogen dioxide (chemical symbol NO_2) reacts with carbon monoxide (CO) to form nitrogen monoxide and carbon dioxide,

$$NO_2 + CO \longrightarrow NO + CO_2.$$

Energy is released or absorbed in a chemical reaction, but no change in total molecular weight occurs.

3.1 Elementary Reactions

Elementary reactions (sometimes referred to as *simple* reactions) are the building blocks of chemical reactions. They are reactions for which no intermediate steps occur. For example, the reaction of nitrogen oxide with itself to form nitrogen monoxide and nitrogen trioxide is elementary. In this case, the *stoichiometric* equation (i.e., the balanced reaction equation) has the form

$$2NO_2 \longrightarrow NO + NO_3.$$

In particular, this is an example of a second order elementary reaction, because two molecules interact (NO_2 with itself). (A first order reaction, though not elementary, is

$$N_2O \longrightarrow N_2 + O.$$

Higher order reactions are uncommon.[1])

3.2 Complex Reactions

In most cases, several steps occur in the mechanism by which initial reactants form their products. In the reaction above between nitrogen dioxide and carbon monoxide, one proposed mechanism consists of the following elementary steps:

$$\begin{aligned} 2NO_2 &\longrightarrow NO + NO_3 \\ NO_3 + CO &\longrightarrow NO_2 + CO_2. \end{aligned}$$

The designation "proposed" mechanism is a recognition that it is extremely difficult to know with certainty what is happening at the intermollecular level in a given reaction. In general, the best we can do is propose a mechanism that fits all known experimental data.

3.3 Rates of reaction

Recall that for a radioactive substance, the rate of decay is assumed to be proportional to the amount left. For example, the process of carbon dating depends on the decay of carbon-14 (a carbon isotope with six protons and eight neutrons) into nitrogen-14, in which an electron is released,

$$C^{14} \xrightarrow{r} N^{14} + e^-.$$

The assumption that the rate of decay is proportional to the amount left, can be written in the form

$$\frac{d[C^{14}]}{dt} = -r[C^{14}],$$

[1]Which is to say, I couldn't hunt one down to use as an example.

where $[C^{14}]$ represents the concentraction of carbon-14 at time t, typically measured as moles per unit volume. (Recall that 1 mole is approximately 6.024×10^{23} molecules, where 6.024×10^{23} is roughly Avogadro's number, which corresponds with the number of atoms in a 12 gram sample of carbon-12.) According to the conservation of mass, we must have

$$\frac{d[N^{14}]}{dt} = +r[C^{14}],$$

which is simply to say that for each molecule of carbon-14 lost, a molecule of nitrogen-14 is gained.

In general, for elementary reactions, we will assume the *law of mass action.*

Law of mass action. The rate of an elementary chemical reaction is proportional to the product of the concentrations of the reactants.

In the case of our elementary reaction above between nitrogen trioxide and carbon monoxide,

$$NO_3 + CO \xrightarrow{k_2} NO_2 + CO_2,$$

the law of mass action asserts,

$$\frac{d[NO_3]}{dt} = \frac{d[CO]}{dt} = -k_2[NO_3][CO].$$

A good intuitive way to think about this is that since the nitrogen trioxide and the carbon monoxide only react when they come into contact with one another, the chance of reaction is increased if either (or, of course, both) has a high concentration. Again by conservation of mass we have the relations

$$\frac{d[NO_2]}{dt} = \frac{d[CO_2]}{dt} = +k_2[NO_3][CO].$$

Observe that the rate is always determined by the reacting chemicals.

For the reaction in which nitrogen dioxide decomposes into nitrogen monoxide and nitrogen trioxide,

$$2NO_2 \xrightarrow{k_1} NO + NO_3,$$

we regard the left hand side as $NO_2 + NO_2$, so that the decay of nitrogen dioxide can be written,

$$\frac{d[NO_2]}{dt} = -2k_1[NO_2]^2.$$

Observe that the coefficient 2 is critical in this case and indicates that for each reaction that takes place, 2 molecules of nitrogen dioxide are used. The *exponent* 2 is a consequence of the law of mass action. By conservation of mass, we have

$$\frac{d[NO]}{dt} = \frac{d[NO_3]}{dt} = +k_1[NO_2]^2.$$

Finally, notice that the entire reaction $NO_2 + CO \longrightarrow NO + CO_2$ is modeled by a system of ODE,

$$\begin{aligned}
\frac{d[NO_3]}{dt} &= -\,k_2[NO_3][CO] + k_1[NO_2]^2 \\
\frac{d[CO]}{dt} &= -\,k_2[NO_3][CO] \\
\frac{d[NO_2]}{dt} &= -\,2k_1[NO_2]^2 + k_2[NO_3][CO].
\end{aligned}$$

Notice that we have a complete system of ODE and do not need to consider the concentrations of $[NO]$ and $[CO_2]$.

Example 3.1. In certain cases, a reaction can proceed in either direction. For example, in the hydrogenation of ethylene (C_2H_4) to ethane (C_2H_6),

$$C_2H_4 + H_2 \longrightarrow C_2H_6,$$

a proposed mechanism is

$$\begin{gathered}
H_2 \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2H \\
C_2H_4 + H \xrightarrow{k_2} C_2H_5 \\
C_2H_5 + H \xrightarrow{k_3} C_2H_6,
\end{gathered}$$

where the first reaction can proceed in either direction. According to the law of mass action, we can model this mechanism with the following system of ODE,

$$\begin{aligned}
\frac{d[H_2]}{dt} &= -\,k_1[H_2] + k_{-1}[H]^2 \\
\frac{d[H]}{dt} &= 2k_1[H_2] - 2k_{-1}[H]^2 - k_2[C_2H_4][H] - k_3[C_2H_5][H] \\
\frac{d[C_2H_4]}{dt} &= -\,k_2[C_2H_4][H] \\
\frac{d[C_2H_5]}{dt} &= k_2[C_2H_4][H] - k_3[C_2H_5][H].
\end{aligned}$$

△

3.4 Determining Reaction Rates

In the examples above, we have assumed we know which reactions are elementary and which are not. A natural question becomes, how can we determine whether or not a given reaction is elementary?

Example 3.2. Consider the reaction between nitrogen monoxide and hydrogen, given by the stoichiometric equation,

$$2NO + 2H_2 \longrightarrow N_2 + 2H_20.$$

Is this an elementary reaction?

In order to answer this question, we require some experimental data.[2] In Table 3.1, we record concentrations of nitrogen monoxide and hydrogen and initial reaction rates. Keep in mind that the units for concentrations are moles per unit volume, while the units for reaction rates are exactly the units of concentrations divided by time.

Experiment	$[NO]$	$[H_2]$	Initial reaction rate $-\frac{d[NO]}{dt} = -\frac{d[H_2]}{dt}$
1	.1	.1	1.23×10^{-3}
2	.1	.2	2.46×10^{-3}
3	.2	.1	4.92×10^{-3}

Table 3.1: Concentrations of nitrogen monoxide and hydrogen and initial reaction rates.

We now posit the general reaction rate $2k[NO]^a[H_2]^b$ for some three constants a, b, and k. That is, we assume

$$\frac{d[NO]}{dt} = \frac{d[H_2]}{dt} = -2k[NO]^a[H_2]^b,$$

where the 2 is really just for show here since it could be subsumed into k. (Observe that if this reaction is elementary, we regard it as

$$NO + NO + H_2 + H_2 \longrightarrow N_2 + 2H_20,$$

for which a and b will both be 2.) For convenience of notation, we will write the positive rate $-\frac{d[NO]}{dt} = R$, so that we have

$$R = 2k[NO]^a[H_2]^b.$$

Taking now the natural logarithm of both sides, we find

$$\ln R = \ln 2k + a \ln[NO] + b \ln[H_2].$$

Given our data for R, $[NO]$, and $[H_2]$, we can use linear multivariate regression to determine the values of a, b, and k. That is, if we write $X = \ln[NO]$, $Y = \ln[H_2]$, and $Z = \ln R$, we have

$$Z = \ln 2k + aX + bY.$$

(In this case, we have exactly three pieces of data and three unknowns, so the fit will be precise, but in general we would have more data and we would proceed through regression.) In the following MATLAB code, N represents $[NO]$ and H represents H_2.

```
>>N=[.1 .1 .2];
>>H=[.1 .2 .1];
>>R=[1.23e-3 2.46e-3 4.92e-3];
>>M=[ones(size(N))' log(N)' log(H)'];
>>p=M\log(R)'
```

[2]If you've taken first-year chemistry with a lab component you've almost certainly carried out an experiment similar to this one.

p =
0.2070
2.0000
1.0000
>>k=exp(.207)/2
k =
0.6150

In this case we determine that $a = 2$, $b = 1$, and $k = .615$, so that our rate law becomes

$$\frac{d[NO]}{dt} = -2 \cdot (.615)[NO]^2[H_2].$$

We conclude that this reaction is most likely not elementary.

So, what is the mechanism? Well—judging from our analysis, the first reaction might look something like

$$2NO + H_2 \longrightarrow ?$$

At this point, we need a chemist. △

3.5 Carbon Dating

Developed by the American physical chemist Willard F. Libby in 1947, carbon dating is a particular type of radioactive dating, applicable in cases for which the matter to be dated was once living. The radioactive isotope carbon-14 is produced at a relatively constant rate in the atmosphere, and like stable carbon-12, combines with oxygen to form carbon dioxide, which is incorporated into all living things. When an organism dies, its level of carbon-12 remains relatively constant, but its level of carbon-14 begins to decay with rate

$$\frac{d[C^{14}]}{dt} = -r[C^{14}], \quad r = 1.2097 \times 10^{-4} \text{ years}^{-1}$$

(this rate corresponds with the commonly quoted fact that carbon-14 has a half-life of 5730 years, by which we mean the level of carbon-14 in a substance is reduced by half after 5730 years). Since the fraction of carbon-12 to carbon-14 remains relatively constant in living organisms (at the same level as it occurs in the atmosphere, roughly $[C^{14}]/[C^{12}] \cong 1.3 \times 10^{-12}$), we can determine how long an organism has been dead by measuring this ratio and determining how much carbon-14 has radiated away. For example, if we find that half the carbon-14 has radiated away, then we can say the material is roughly 5730 years old. In practice, researchers measure this ratio of carbon-14 to carbon-12 in units called *modern carbons*, in which the living ratio (ratio of carbon-14 to carbon-12 in a living organism) is defined to be 1 modern carbon.

Example 3.3. (Carbon dating the Shroud of Turin)[3] The most famous (and controversial) case of carbon dating was that of the Shroud of Turin, which many believe to have covered

[3]The study we take our data from was originally published in *Nature* vol. 337 (1989), no. 6208 611–615, which is available at www.shroud.com/nature.htm. The data in the form I'm giving it here was given by Remi Van Haelst in his article *Radiocarbon Dating the Shroud of Turin: the Nature Report*, which is available at www.shroud.com/vanhels5.pdf. The results of this study have been widely disputed. One compelling argument against the date we find in this example is that the shroud was patched in medieval times and the examples studied were part of that patch.

Jesus of Nazareth in his tomb. In 1988, samples of the cloth were independently studied by three groups, one at the University of Arizona, one at Oxford University, and one at the Swiss Federal Institute of Technology (ETH) in Zurich. In this example, we will consider the data collected in Zurich. Five measurements were made on the level of modern carbon remaining in the shroud, .8755, .8766, .8811, .8855, and .8855 (two measurements led to the same number). Averaging these, we have a value $M = .8808$. Since the level of carbon-12 remains relatively constant, we can assume that the level of the ratio of carbon-14 to carbon-12 is reduced at the same rate as the level of carbon-14. We have, then

$$\frac{dM}{dt} = -rM; \quad r = 1.2097 \times 10^{-4} \text{ years}^{-1} \Rightarrow M(t) = M(0)e^{-rt}.$$

Setting $M(0) = 1$ as the level of modern carbon when the shroud was made, we need to find t so that

$$.8808 = e^{-1.2097 \times 10^{-4} t}.$$

Solving this relation, we find $t = 1049$ years, which dates the shroud to the year $1988 - 1049 = 939$ A.D. △

4 Population Dynamics

In this section we will regard a population as just about any collection of objects we can count: animals, biological cells, automobiles etc. While the modeling of populations with differential equations is not precise (i.e., there is no equivalent to Newton's second law of motion), it can be extremely useful, and has aided especially in the areas of epidemic control and medical treatment. In this section, we will list and discuss many of the fundamental ODE models used in applications.

Throughout this section, we let $p(t)$ represent the total number of members of a population at time t.

1. Steady production. In the event that some population increases steadily (e.g., 5 members are created per hour), we have simply that the population's first time derivative is constant,

$$\frac{dp}{dt} = c; \quad p(0) = p_0 \Rightarrow p(t) = ct + p_0.$$

Examples include cars coming off an assembly line and T cells being created in bone marrow. In general, constant production is often a reasonable assumption when the population is being produced by an outside source and not reproducing itself.

2. Malthusian model. Named for the British economist Thomas R. Malthus (1766–1834),[4] the Malthusian model assumes that both the birth rate of a population and the death rate

[4] Malthus is perhaps the single most important figure in shaping current socioeconomic views of population dynamics. Prior to the publication of Malthus's *Essay on Population* in 1798, European statesmen and economists largely agreed that rising population was an indication of economic prosperity (this point of view was argued, for example, by the influential Scottish political economist and philosopher Adam Smith (1723–1790) in his *Wealth of Nations* (1776)). According to this point of view, if a king or ruling assemblage wanted to increase its nation's prosperity, it need only increase its pool of taxpayers. In his *Essay on Population*, Malthus pointed out that environments have finite resources, and consequently that rising populations must

of a population are proportional to the current size of the population. For example, in a population of two people, the population will not grow very rapidly, but in a population of 6.2 billion people (roughly the earth's population in 2004) growth is extremely rapid. Letting b represent birth rate and d represent death rate, we write,

$$\frac{dp}{dt} = bp - dp = rp; \quad p(0) = p_0 \Rightarrow p(t) = p_0 e^{rt},$$

where r, which is typically positive, will be referred to as the *growth parameter* of the population.

3. Logistic model. A clear drawback of the Malthusian model is that it assumes there are no inherent limitations on the growth of a population. In practice, most populations have a size beyond which their environment can no longer sustain them. Introduced by the Belgian mathematician Pierre Verhulst in about 1838, the logistic model incorporates this observation through the introduction of a "carrying capacity" K, the greatest population an environment can sustain. We have,

$$\frac{dp}{dt} = rp(1 - \frac{p}{K}); \quad p(0) = p_0 \Rightarrow p(t) = \frac{p_0 K}{(K - p_0)e^{-rt} + p_0}.$$

In order to better understand the role K plays, we recall the idea of equilibrium points or steady states (this will anticipate the general stability discussion of Section 3). An equilibrium point is some point at which a population quits changing: $\frac{dp}{dt} = 0$. In the case of the logistic equation, we can find all equilibrium points by solving the algebraic equation,

$$rp_e(1 - \frac{p_e}{K}) = 0 \Rightarrow p_e = 0, K.$$

We determine whether or not a population moves toward a particular equibrium point by considering the sign of $\frac{dp}{dt}$ on either side of the equilibrium point. For the equilibrium point $p_e = K$, we observe that for $p > K$, $\frac{dp}{dt} < 0$ (that is, the population is decreasing), while for $p < K$, $\frac{dp}{dt} > 0$ (that is, the population is increasing). In this case, the population always approaches K, and we refer to K as a *stable* equilbrium point. *Very generally, stable equilibrium points represent long time behavior of solutions to ODE.*

4. Gompertz model. Named for the British actuary and mathematician Benjamin Gompertz (1779–1865), the Gompertz model is qualitatively similar to the logistic model. We have

$$\frac{dp}{dt} = -rp\ln(\frac{p}{K}); \quad p(0) = p_0.$$

The Gompertz model is often used in the study of tumor growth.

5. General single population model. The logistic and Gompertz models are both special cases of the general population model,

$$\frac{dp}{dt} = \frac{r}{a} p\Big(1 - (\frac{p}{K})^a\Big),$$

eventually lead to famine. Though increasingly clever methods of cultivation have allowed industrialized countries to sustain more people than Malthus would likely have thought possible, his thesis is now widely accepted.

where r and K play qualitatively the same roles as in the logistic and Gompertz models, and a is typically fit to data. We note that $a = 1$ is the logistic model, and the Gompertz model is recovered from a limit as $a \to 0$. More generally, for fixed values of r and K, the parameter a controls how fast the carrying capacity is approached. More precisely, the smaller a is, the faster K is approached.

6. Lotka-Volterra model. Named for the Italian mathematician Vito Volterra (1860–1940) and the Austrian chemist, demographer, ecologist, and mathematician Alfred J. Lotka (1880–1949), the Lotka–Volterra model describes the interaction between a predator (e.g., wildcats) with population $y(t)$ and its prey (e.g., rabbits) with population $x(t)$. We have,

$$\begin{aligned}\frac{dx}{dt} &= ax - bxy; \quad x(0) = x_0\\ \frac{dy}{dt} &= -ry + cxy; \quad y(0) = y_0,\end{aligned}$$

where a, b, c, and r are all taken positive. We observe that in the absence of predators (i.e., in the case $y \equiv 0$) the prey thrive (they have Malthusian growth), while in the absence of prey (i.e., in the case $x \equiv 0$) the predators die off. The interaction or *predation* terms signify that the larger either the predator population or the prey population is, the more often the two populations interact, and that interactions tend to increase the predator population and decrease the prey population. While qualitatively enlightening, the Lotka–Volterra model isn't robust enough to model many real interactions, though see Examples 2.6 and 2.7 in the course notes *Modeling Basics.*

7. Competition models. In addition to predator–prey interactions, we would often like to model two species such as rabbits and deer that compete for the same resources. In the (unlikely) event that each species uses exactly the same amount of the environment, we could model this interaction through the ODE,

$$\begin{aligned}\frac{dx}{dt} &= r_1 x(1 - \frac{x+y}{K})\\ \frac{dy}{dt} &= r_2 y(1 - \frac{x+y}{K}),\end{aligned}$$

where we have simply asserted that if the total population $x+y$ exceeds the carrying capacity, both populations will begin to die off. More generally, we assume that each population has a different carrying capacity and a different interaction with its environment, and only keep this general idea that if either population gets sufficiently large, the other will begin to die off. Under this assumption, a reasonable model, sometimes referred to as the Lotka–Volterra competition model, is,

$$\begin{aligned}\frac{dx}{dt} &= r_1 x(1 - \frac{x + s_1 y}{K_1})\\ \frac{dy}{dt} &= r_2 y(1 - \frac{y + s_2 x}{K_2}),\end{aligned}$$

where K_1 represents the carrying capacity of species x, K_2 represents the carrying capacity of species y, and s_1 represents a scaling for the amount of species x's environment used by species y (and similarly for s_2). For example, suppose species y is larger and eats roughly twice as much per animal as species x. Then we take $s_1 = 2$. It seems fairly natural that if $s_1 = 2$, then $s_2 = 1/2$. That is, if species y uses twice the environment of species x, then species x uses half the environment of species y. While intuitively satisfying, this reciprocity doesn't always fit the data.

8. The SIR epidemic model The most simple model for studying the spread of epidemics involves three populations, the *susceptible* members, $S(t)$, the *infected* members, $I(t)$, and the *removed* members, $R(t)$. (The removed members can either have recovered (in which case they are assumed in this model to be immune) or died.) The SIR model takes the form,

$$\begin{aligned}\frac{dS}{dt} &= -\,aSI\\ \frac{dI}{dt} &= aSI - bI\\ \frac{dR}{dt} &= bI.\end{aligned}$$

In the first equation, we observe that the rate at which susceptible members of the population become infected is proportional to the number of interactions there are between members of the population. The second equation records that each member lost from $S(t)$ moves to population $I(t)$ and that members of $I(t)$ recover or die at some rate b determined by the disease and typically found experimentally.

9. Half saturation constants. In the Lotka–Volterra predator–prey model above, the predator growth due to predation takes the form $+cxy$. Even if there is only one predator left, this claims that if there are enough prey, the predators will continue to grow rapidly. A better expression might be,

$$\frac{cxy}{x+M},$$

for which there is an intrinsic limit on how fast the predator population can grow when saturated with prey. In particular, we mean by full saturation the case in which a limit is taken as the prey population goes to infinity,

$$\lim_{x\to\infty}\frac{cxy}{x+M} = cy.$$

We refer to the constant M as the "half saturation" constant, because when $x = M$, the growth rate is at precisely half saturation,

$$\frac{cxy}{x+x} = \frac{1}{2}cy.$$

10. The Monod growth model. Introduced by the French biologist Jacques Monod (1910–1976), the Monod growth model describes the growth of some organism that is nourished by some nutrient that is available with amount n. If $p(t)$ denotes the population of this organism, then Monod suggested a growth rate with the form

$$R(n) = \frac{rn}{M+n}.$$

Clearly, the growth rate increases as the amount of nutrient increases, but saturation eventually occurs, in which case an increase in the amount of nutrient has little effect on the growth of the organism. If we let δ denote a Malthusian type death rate we obtain the model

$$\frac{dp}{dt} = p(\frac{rn}{M+n} - \delta).$$

Of course since the organism is living off the nutrient the amount of nutrient diminishes in the presence of the organism, and we might describe the evolution of the nutrient population by

$$\frac{dn}{dt} = an - \frac{bnp}{M_2 + n},$$

where we have added a half-saturation constant to the predation expression to ensure that if there is a saturation of nutrient the organism utilizes only a reasonable fixed percentage of it. We now have a system of two equations for the interaction between a population and its nutrient.

11. Size-structured population models. In certain cases, especially when resources are limited, the average size of individuals in a population will begin to shrink as the population grows. This is most notable in vegetation, though one very interesting example was discovered in 2004, when a certain bipedal species was discovered on a remote island called Flores in Indonesia. Quite similar to modern humans, the species was distinguished by its small size, the members standing roughly three feet tall at their adult height. One theory that has been proposed to explain this size is that the species became smaller due to limited resources on the island. Here, we consider a population model in which the average size of individuals in the population, $s(t)$, plays a role in the dynamics. We take

$$\begin{aligned}\frac{dp}{dt} &= rp(1 - \frac{ps}{K})\\ \frac{ds}{dt} &= \kappa(\frac{s}{M_1} - 1)(1 - \frac{s}{M_2})(1 - \frac{ps}{L}).\end{aligned}$$

Here, we have assumed that if the product ps ever exceeds the threshold K, then the population will decline. Also, M_1 is the minimum size of individuals in the population (if $s < M_1$, the population will die out through size going to 0), M_2 is the maximum number of individuals in the population (if $s > M_2$, the population size is reduced), and we assumed that if the product ps ever exceeds some threshold L (possibly equal to K), then the average size will begin to drop.

12. Learning terms. We often want to specify in our model that a species becomes more (or less) adept at some function as time progresses. For example, we might find in a predator–prey situation that prey learn over time to avoid predators. In this case, the growth and decay rates due to predation will depend on the independent variable t. Typically, we assume this change is slow, and logarithmic terms are often employed. In the Lotka–Volterra model, under the assumption that the prey learn to avoid the predators (and the predators do not get more adept at finding the prey), we could write,

$$\begin{aligned}\frac{dx}{dt} &= ax - \frac{b}{[\ln(e+t)]^k}xy\\ \frac{dy}{dt} &= -ry + \frac{c}{[\ln(e+t)]^k}xy,\end{aligned}$$

where we evaluate natural log at $e+t$ so that we get 1 when $t=0$, and k is a new parameter to be fit to data.

13. Delay models. One of the deficiencies with the population models discussed above is that in each case birth rate is assumed to change instantly with a change in population. More generally, we might expect the members of a population to reach some threshold age before giving birth, introducing a time delay into the model. For example, a time-delay Malthusian model would take the form

$$\frac{dp}{dt}(t) = rp(t-T),$$

wherein the growth rate of the population at time t depends on the population at time $t-T$.

14. Difference Equations.[5] In the event that the dependent variable is assumed to take discrete (rather than continuous) values we obtain a difference equation rather than a differential equation. For example, we may want to compute a population value at each time unit $t = 1, 2, \ldots$. One frequently used discrete model, similar to the continuous logistic model, takes the form

$$p_{t+1} = p_t e^{r(1-\frac{p_t}{K})},$$

and is called Ricker's model, or Ricker's logistic equation, in honor of the Canadian biologist William E. Ricker (1908–2001). Notice that we can right this in difference form

$$p_{t+1} - p_t = p_t(e^{r(1-\frac{p_t}{K})} - 1).$$

15. Difference Equation Systems. As when modeling with differential equations, when we model with difference equations we often obtain a system of difference equations. As an example, we consider a 2001 paper in the journal *Conservation Biology* [P. C. Cross and S. R. Beissinger, *Using logistic regression to analyze the sensitivity of PVA models: a comparison of models based on African wild dog model*, Conservation Biology **15** (2001), no. 5, 1335–1346.] in which the authors consider African wild dogs, dividing the population into three categories: pups, yearlings and adults. We assume pups survive to become yearlings with probability S_p, that yearlings reproduce pups with probability R_y, that yearlings survive to adulthood with probability S_y, that adults reproduce pups with probability R_a, and finally that adults have an annual probability of survival S_a (see Figure 4.1).

Suppose we know initial populations P_0, Y_0, and A_0, and would like to determine the corresponding populations a year later. For pups, we lose the entirety of P_0 (either because

[5] The final two models in this section are not ODE models, but are well worth mentioning.

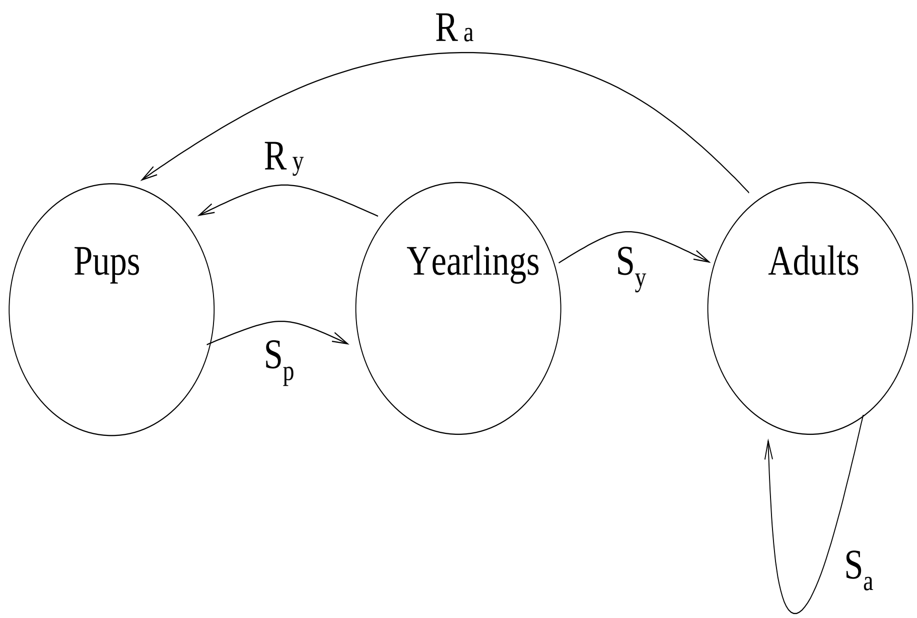

Figure 4.1: Transitions in African wild dog population categories.

the pups become yearlings or because they don't survive), and we gain pups according to the reproduction rates of yearlings and adults. Arguing similarly for yearlings and adults, we arrive at the model,

$$\begin{aligned} P_1 &= R_y Y_0 + R_a A_0 \\ Y_1 &= S_p P_0 \\ A_1 &= S Y_0 + S_a A_0. \end{aligned}$$

(It might be tempting to think we should have a loss term in the pups equation of the form $-S_p P_0$, but keep in mind that after one year *all* pups are lost (if they've survived, they've become yearlings), and we have only gotten new pups from reproduction.) More generally, we obtain the system

$$\begin{aligned} P_{t+1} &= R_y Y_t + R_a A_t \\ Y_{t+1} &= S_p P_t \\ A_{t+1} &= S_y Y_t + S_a A_t. \end{aligned}$$

In matrix form

$$\begin{pmatrix} P_{t+1} \\ Y_{t+1} \\ A_{t+1} \end{pmatrix} = \begin{pmatrix} 0 & R_y & R_a \\ S_p & 0 & 0 \\ 0 & S_y & S_a \end{pmatrix} \begin{pmatrix} P_t \\ Y_t \\ A_t \end{pmatrix}.$$

In order to solve such an equation, we first observe

$$\begin{pmatrix} P_2 \\ Y_2 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 & R_y & R_a \\ S_p & 0 & 0 \\ 0 & S_y & S_a \end{pmatrix} \begin{pmatrix} P_1 \\ Y_1 \\ A_1 \end{pmatrix} = \begin{pmatrix} 0 & R_y & R_a \\ S_p & 0 & 0 \\ 0 & S_y & S_a \end{pmatrix}^2 \begin{pmatrix} P_0 \\ Y_0 \\ A_0 \end{pmatrix}.$$

More generally, the year k population can now be determined as

$$\begin{pmatrix} P_k \\ Y_k \\ A_k \end{pmatrix} = \begin{pmatrix} 0 & R_y & R_a \\ S_p & 0 & 0 \\ 0 & S_y & S_a \end{pmatrix}^k \begin{pmatrix} P_0 \\ Y_0 \\ A_0 \end{pmatrix}.$$

The matrix in these calculations is typically referred to as the transition matrix.

5 Newtonian mechanics.

Given a point particle with position $\vec{x}(t)$ with constant mass, Newton's second law of motion, $F = m\vec{a}$, can be written as an ODE,

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = m\frac{d^2\vec{x}}{dt^2}.$$

In the event that the mass of our particle is changing, we have the generalized form,

$$\vec{F} = \frac{d\vec{p}}{dt},$$

where $\vec{p} = m\vec{v}$ is momentum.

Example 5.1. (Drag racing) Recall that for an object on a flat surface, the force due to friction is proportional to the object's weight, with proportionality constant the coefficient of static or dynamic friction,

$$F = -\mu mg.$$

Observe in this relationship that we can reasonably regard $\mu \leq 1$. If not, then it would take less force to lift the object and carry it than it would to push it. Since the entire force pushing a dragster forward is due to friction (between the tires and the road), we expect the maximum force propelling the dragster forward to be $F = mg$. Under this assumption, we can determine the minimum time it will take a dragster to complete a standard quarter-mile course (402.34 meters). If $x(t)$ represents position at time t along the course (with initial position and initial velocity assumed 0), then we have, according to Newton's second law,

$$\frac{d^2x}{dt^2} = g \Rightarrow x(t) = \frac{1}{2}gt^2.$$

We compute the minimum track time as,

$$t = \sqrt{\frac{2(402.34)}{9.81}} = 9.06 \text{ seconds}.$$

Let's put this to the test. On June 2, 2001, Kenny Bernstein set the world record for a quarter mile track with a time $t = 4.477$ seconds.[6] △

Example 5.2. Consider the motion of a mass, m, swinging at the end of a rigid rod, as depicted in Figure 5.1. Assume air resistance is negligible.

[6] Race car drivers "burn out" their tires at the beginning of a race, and this makes the tires adhere to the racing surface, so that they can "push off." Viewed another way, the cars get more difficult to lift.

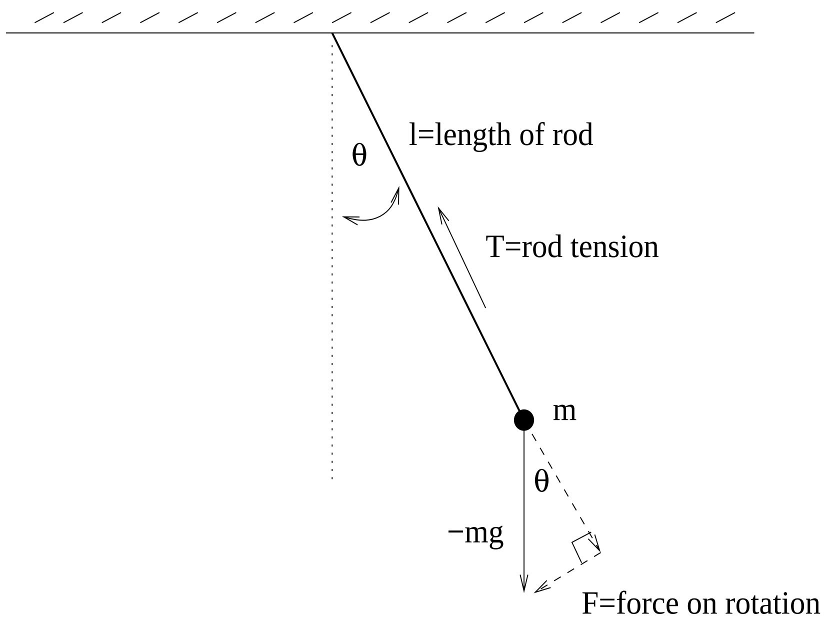

Figure 5.1: Pendulum motion under the influence of gravity alone.

The force due to gravity on m acts vertically downward, and must be decomposed into a force $-T$, which is exactly balanced by the rod, and a force F, directed tangentially to the arc of motion. Observing the right triangle, with hypotenuse of length $-mg$, we have

$$\cos\theta = \frac{T}{mg} \Rightarrow T = mg\cos\theta,$$
$$\sin\theta = -\frac{F}{mg} \Rightarrow F = -mg\sin\theta.$$

Measuring distance as arclength, $d = l\theta$, Newton's second law of motion ($F = ma$) determines

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta$$
$$\theta(0) = \theta_0, \quad \frac{d}{dt}\theta(0) = \omega_0. \tag{5.1}$$

Example 5.3. (Planetary motion) Consider the earth–sun system in two space dimensions. We choose some arbitrary origin $(0, 0)$ and let $\vec{r}_1 = (x_1, y_1)$ represent the position of the sun (mass M) relative to the origin and $\vec{r}_2 = (x_2, y_2)$ represent the position of the earth (mass m) relative to this origin. (See Figure 5.2.)

According to Newton's law of gravitation, the magnitude of the force exerted by one (point) mass on another is proportional to the product of the masses and inversely proportional to the distance between the masses squared, with constant of proportionality G. Ignoring direction, we have

$$F = \frac{GMm}{d^2}.$$

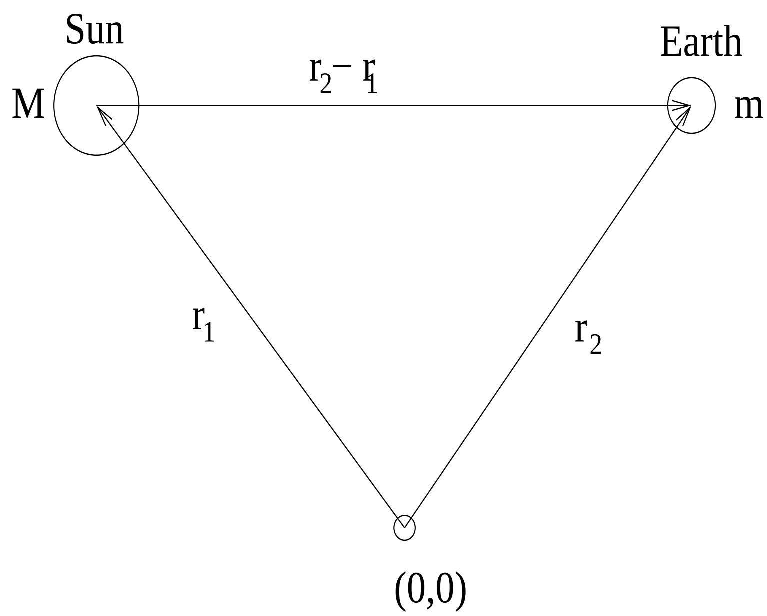

Figure 5.2: The earth–sun system in two dimensions.

In order to incorporate direction, we assume the force on either mass is directed radially toward the other mass. The force on the sun due to the earth is given by,

$$\vec{F}_{\text{sun}} = \frac{GMm}{|\vec{r}_2 - \vec{r}_1|^3}(\vec{r}_2 - \vec{r}_1),$$

while the force on the earth due to the sun is given by,

$$\vec{F}_{\text{earth}} = -\frac{GMm}{|\vec{r}_2 - \vec{r}_1|^3}(\vec{r}_2 - \vec{r}_1).$$

Finally, according to Newton's second law of motion, we can set $\vec{F} = m\vec{a}$, for which we obtain the vector ODE

$$\begin{aligned} M\frac{d^2\vec{r}_1}{dt^2} &= \frac{GMm}{|\vec{r}_2 - \vec{r}_1|^3}(\vec{r}_2 - \vec{r}_1) \\ m\frac{d^2\vec{r}_2}{dt^2} &= -\frac{GMm}{|\vec{r}_2 - \vec{r}_1|^3}(\vec{r}_2 - \vec{r}_1), \end{aligned}$$

or component-wise

$$\begin{aligned} x_1'' &= \frac{Gm(x_2 - x_1)}{((x_2 - x_1)^2 + (y_2 - y_1)^2)^{3/2}}, \\ y_1'' &= \frac{Gm(y_2 - y_1)}{((x_2 - x_1)^2 + (y_2 - y_1)^2)^{3/2}}, \\ x_2'' &= -\frac{GM(x_2 - x_1)}{((x_2 - x_1)^2 + (y_2 - y_1)^2)^{3/2}}, \\ y_2'' &= -\frac{GM(y_2 - y_1)}{((x_2 - x_1)^2 + (y_2 - y_1)^2)^{3/2}}. \end{aligned}$$

$\triangle$

5.1 Newtonian Mechanics in Polar Coordinates

In Example 2.7, we described planetary motion in the standard cartesian coordinate system, for which our unit vectors are $\hat{e}_x = (1, 0)$ and $\hat{e}_y = (0, 1)$. For many physical systems, we find it convenient to describe motion in polar coordinates, for which a point P is described by its distance from the origin r and its angle from the horizontal axis θ. In this case, the unit vectors are $\hat{e}_r$ and $\hat{e}_\theta$, where $\hat{e}_r$ is a vector of unit length directed from the origin to the point P and $\hat{e}_\theta$ is a vector of unit length orthogonal to $\hat{e}_r$ (see Figure 5.3). In particular, notice that while $\hat{e}_x$ and $\hat{e}_y$ are independent of x and y, $\hat{e}_r$ and $\hat{e}_\theta$ depend on the point (r, θ).

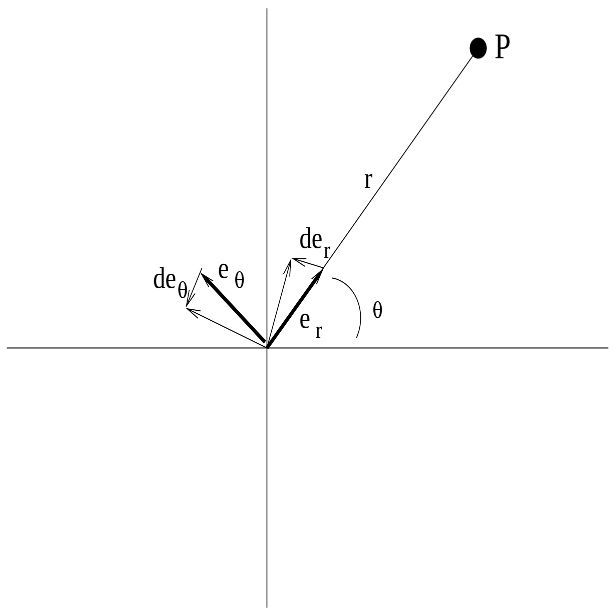

Figure 5.3: Unit vectors in polar coordinates.

In the event of an incremental change in angle θ, $d\theta$, we have incremental changes in $\hat{e}_r$ and $\hat{e}_\theta$, described by the law of cosines[7]

$$|d\hat{e}_r|^2 = 1^2 + 1^2 - 2\cos(d\theta),$$

from which we have

$$\frac{|d\hat{e}_r|}{d\theta} = \frac{\sqrt{2 - 2\cos(d\theta)}}{d\theta},$$

which is

$$\frac{|\hat{e}_r(\theta + d\theta) - \hat{e}_r(\theta)|}{d\theta} = \frac{\sqrt{4\sin^2\frac{d\theta}{2}}}{d\theta}.$$

Taking a limit as $d\theta \to 0^+$, we obtain

$$\frac{|d\hat{e}_r|}{d\theta} = 1,$$

[7]The law of cosines states that if a triangle has sidelengths a, b, and c, and the angle between side a and b is θ, then $c^2 = a^2 + b^2 - 2ab\cos\theta$.

and finally we observe that the direction of $d\hat{e}_r$ is that of $\hat{e}_\theta$ so that

$$\frac{d\hat{e}_r}{d\theta} = \hat{e}_\theta.$$

Proceeding similarly with $d\hat{e}_\theta$, we find additionally

$$\frac{d\hat{e}_\theta}{d\theta} = -\hat{e}_r.$$

We are now in a position to develop an expression for acceleration in polar coordinates. Let $\vec{x}(t)$ be the position vector of some point particle at time t. Then $\vec{x}$ can be described in polar coordinates as $\vec{x}(t) = r(t)\hat{e}_r(t)$, where we keep in mind that $\hat{e}_r$ is a vector. That is, $\hat{e}_r(t)$ will always be a unit vector pointing in the direction of $\vec{x}(t)$ and r will be a scalar equal in value to the length of $\vec{x}(t)$. We compute

$$\frac{d\vec{x}}{dt} = r_t\hat{e}_r + r\frac{d\hat{e}_r}{dt} = r_t\hat{e}_r + r\hat{e}_\theta\frac{d\theta}{dt},$$

and similarly

$$\frac{d^2\vec{x}}{dt^2} = (r_{tt} - r(\theta_t)^2)\hat{e}_r + (2r_t\theta_t + r\theta_{tt})\hat{e}_\theta.$$

That is, acceleration in the radial direction is given by $r_{tt} - r(\theta_t)^2$, while acceleration in the angular direction is given by $2r_t\theta_t + r\theta_{tt}$. In the case of a *central* force such as gravity, all force is in the radial direction, and Newton's second law of motion becomes the system

$$\begin{aligned} m(r_{tt} - r(\theta_t)^2) &= -F \\ m(2r_t\theta_t + r\theta_{tt}) &= 0. \end{aligned}$$

In particular, if we assume the sun's position is fixed in space, the earth's motion around it can be described by the system

$$\begin{aligned} r_{tt} - r(\theta_t)^2 &= -\frac{GM}{r^2} \\ 2r_t\theta_t + r\theta_{tt} &= 0. \end{aligned}$$

6 Hamilton's Method[8]

In the case of conservative systems, equations of motion can often be obtained from the conservation of energy. We will refer to this general method as Hamilton's method.

Example 5.4. Consider an object of mass m with height $y(t)$ falling under the influence of gravity only. We have

$$\begin{aligned} \text{Kinetic Energy } &= \frac{1}{2}mv^2 \\ \text{Potential Energy } &= mgy, \end{aligned}$$

[8]Developed in 1833 by the Irish mathematician William Rowan Hamilton (1805–1865).

where *potential energy* is described through the function $U(y)$ such that

$$\text{Force } = -\frac{\partial U}{\partial y}.$$

The total energy for this system is

$$H = \text{Kinetic Energy} + \text{Potential Energy } = \frac{1}{2}mv^2 + mgy,$$

where we designate this quantity by H because it correponds with the *Hamiltonian* of the system. (The Hamiltonian is not always the total energy of the system, but it is in this case.) For a conservative system (in the event that energy is conserved) we have $\frac{dH}{dt} = 0$, and we can compute

$$\frac{dH}{dt} = \frac{1}{2}m2v\frac{dv}{dt} + mg\frac{dy}{dt} = 0 \Rightarrow \frac{dv}{dt} = -g \Rightarrow \frac{d^2y}{dt^2} = -g.$$

△

Example 5.5. Consider again the pendulum of Example 2.7. In this case, we would like to begin by writing down the total energy associated with the pendulum's motion. The kinetic energy is

$$\text{K.E.} = \frac{1}{2}mv^2 = \frac{1}{2}ml^2(\frac{d\theta}{dt})^2,$$

where we have observed that the distance in this problem is arclength $s = l\theta$, and thus

$$v = \frac{ds}{dt} = l\frac{d\theta}{dt}.$$

For the potential energy, we have

$$\text{P.E.} = mgL,$$

where in this case L is the vertical distance between the pendulum's position and its height when hanging straight down. That is,

$$L = l - l\cos\theta.$$

We have, then

$$H = \frac{1}{2}ml^2(\frac{d\theta}{dt})^2 + mgl(1 - \cos\theta).$$

We now obtain the pendulum equation by setting

$$\frac{dH}{dt} = 0.$$

We compute

$$\frac{dH}{dt} = ml^2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + mgl\sin\theta\frac{d\theta}{dt} = 0.$$

Upon division by $ml\frac{d\theta}{dt}$, we obtain

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta.$$

This method leads naturally to an analysis of the period of a pendulum, so we also consider that. Our goal, then, is to start the pendulum at an initial angle θ_0 and with 0 initial velocity, and to determine the time it will take for the pendulum to return to its starting position. At $t = 0$, we have $\theta'(0) = 0$ and consequently

$$H = mgl(1 - \cos\theta_0).$$

Since H is constant, we know that H continues to have this value for all time so that

$$\frac{1}{2}ml^2(\frac{d\theta}{dt})^2 + mgl(1 - \cos\theta) = mgl(1 - \cos\theta_0).$$

Subtracting mgl from both sides and solving for $\frac{d\theta}{dt}$, we find

$$\frac{d\theta}{dt} = \pm\sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)}.$$

Here, $+$ indicates motion from the left to the right and $-$ indicates the return motion. Since the period can be computed by considering only one of these and doubling the obtained value, we can focus our attention on the equation

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)}.$$

This equation corresponds with the pendulum's swinging from an angle $-\theta_0$ to an angle θ_0. In particular, $\theta(t)$ is a strictly increasing function of t on this interval and thus is invertible. This means we are justified in inverting our equation as

$$\frac{dt}{d\theta} = \frac{1}{\sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)}}.$$

We now integrate t as a function of θ from $-\theta_0$ up to θ_0. We obtain

$$t(\theta_0) - t(-\theta_0) = \int_{-\theta_0}^{\theta_0} \frac{d\theta}{\sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)}}.$$

The left-hand side of this last equality is exactly one-half of the period, and so we finally have

$$P = \sqrt{\frac{2l}{g}} \int_{-\theta_0}^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}.$$

Unfortunately, this integral does not have an integrand that can be written down in closed form. By defining ϕ by the relation

$$\sin(\frac{1}{2}\theta) = \sin(\frac{1}{2}\theta_0)\sin\phi,$$

it can be shown that

$$P(\theta_0) = 4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-\sin^2\frac{\theta_0}{2}\sin^2\phi}} = 4\sqrt{\frac{l}{g}}K(\sin\frac{\theta_0}{2}),$$

where $K(\cdot)$ is a *complete elliptic integral.* All we have really done here is replace one nasty integral with another, but complete elliptic integrals have been well studied. In particular, we can now compute the period of our pendulum in MATLAB with the command *ellipke,* though to be precise MATLAB's terminology is slightly different from standard, and the input to *ellipke* should be of the form $\sin^2\frac{\theta_0}{2}$. △

7 Variational Methods

Consider a point (x_1, y_1) on a curve $y = f(x)$ emerging from the origin. The arclength of such a curve is given by the *functional*

$$F[y] = \int_0^{x_1} \sqrt{1 + y'(x)^2}dx,$$

where by *functional* we mean a mapping that takes functions as its input and returns numbers as output. Suppose we want to determine the shortest possible path between $(0, 0)$ and (x_1, y_1). (Intuitively, of course, we know this is a line.) The question becomes: What curve $y(x)$ minimizes the functional F. What we would like to do is take a derivative of F and set it to 0. The difficulty lies in taking a derivative with respect to a function (the input for F). The study of such a theory is referred to as the *calculus of variations.*

In order to set some notation, we will designate the minimizing function as $y_e(x)$ and define a family of *variations* of this minimzer as the functions

$$y_s(x) = y_e(x) + sy(x),$$

where the functions $y(x)$ can be *any* continuously differentiable functions for which $y(0) = y(x_1) = 0$ (we assume $y_s(x)$ and $y_e(x)$ agree at the endpoints). We say that $y(x)$ belongs to the *function class* $C_0^1[0, x_1]$: the collection of all continuously differentiable functions on $x \in [0, x_1]$ that vanish at the endpoints.

According to our assumption, $F[y_e] =$ minimum, and consequently

$$F[y_s]\Big|_{s=0} = \min \Rightarrow \frac{\partial}{\partial s}F[y_s]\Big|_{s=0} = 0.$$

That is, since $\phi(s) := F[y_s]$ is minimized at $s = 0$ (for *any* $y(x)$), its s-derivative at $s = 0$ must be 0. We have

$$F[y_s] = F[y_e + sy] = \int_0^{x_1} \sqrt{1 + (y_e' + sy')^2}dx,$$

from which

$$\frac{\partial}{\partial s}F[y_s] = \int_0^{x_1} \frac{(y_e' + sy')y'}{\sqrt{1 + (y_e' + sy')^2}}dx.$$

Upon setting $\frac{\partial}{\partial s} F[y_s]\Big|_{s=0} = 0$, we have

$$\int_0^{x_1} \frac{y_e' y'}{\sqrt{1 + (y_e')^2}} dx = 0.$$

Integrating by parts, we find that this integral equation can be re-written as

$$\int_0^{x_1} \frac{y_e'' y}{(1 + (y_e')^2)^{3/2}} dx = 0.$$

At this point, we argue as follows: since $y(x)$ can be almost any function we choose (it only needs to be continuously differentiable and to vanish on the boundary), we can choose it to always have the same sign as y_e''. In this case, the numerator and denominator of our integrand will both necessarily be positive, and consequently the only chance our integral has of being 0 is if $y_e'' = 0$. In that case, we have the boundary value problem

$$\begin{aligned} y_e'' &= 0 \\ y(0) &= 0 \\ y(x_1) &= y_1. \end{aligned}$$

In this way we have converted the problem of minimizing a functional into the problem of solving an ODE.

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