Corso di Laurea in Fisica - UNITS

ISTITUZIONI DI FISICA PER IL SISTEMA TERRA

LINEAR SYSTEMS

FABIO ROMANELLI

Department of Mathematics & Geosciences

University of Trieste

romanel@units.it

http://moodle2.units.it/course/view.php?id=887

Green's function

Green's function (GF) is a basic solution to a linear differential equation, a building block that can be used to construct many useful solutions.

If one considers a linear differential equation written as:

$$L(x)u(x)=f(x)$$

where L(x) is a linear, self-adjoint differential operator, u(x) is the unknown function, and f(x) is a known non-homogeneous term, the GF is a solution of:

$$L(x)u(x,s)=\delta(x-s)$$

$$G(x,s)$$

Why GF is important?

If such a function G can be found for the operator L, then if we multiply the second equation for the Green's function by f(s), and then perform an integration in the s variable, we obtain:

$$\int L(x)G(x,s)f(s)ds = \int \delta(x-s)f(s)ds = f(x) = Lu(x)$$

$$L\int G(x,s)f(s)ds = Lu(x)$$

$$u(x) = \int G(x,s)f(s)ds$$

Thus, we can obtain the function u(x) through knowledge of the Green's function, and the source term. This process has resulted from the linearity of the operator L.

Linear Systems

$a x_1(t) + b x_2(t) \rightarrow$ **Linear System** $\rightarrow$ $a y_1(t)$, $b y_2(t)$ $= a y_1(t) + b y_2(t)$

$\delta(t) \rightarrow$ **Linear System** $\rightarrow$ $h(t)$ **Impulse Response**

(remember GF definition)

Since any input x(t) can be written as:

$$x(t) = \int x(\tau)\delta(t-\tau)\,d\tau \longrightarrow \int x(\tau)h(t-\tau)\,d\tau = x * h$$

$e^{i\omega t} \rightarrow$ **Linear System** $\rightarrow$ $H(\omega)e^{i\omega t}$ **Transfer Function**

$$\int e^{i\omega\tau}h(t-\tau)\,d\tau = \int e^{i\omega(t-\tau)}h(\tau)\,d\tau = e^{i\omega t}\int e^{-i\omega\tau}h(\tau)\,d\tau$$

$$X(\omega) = \int x(t)e^{-i\omega t}\,dt \longrightarrow X(\omega)\cdot H(\omega)$$

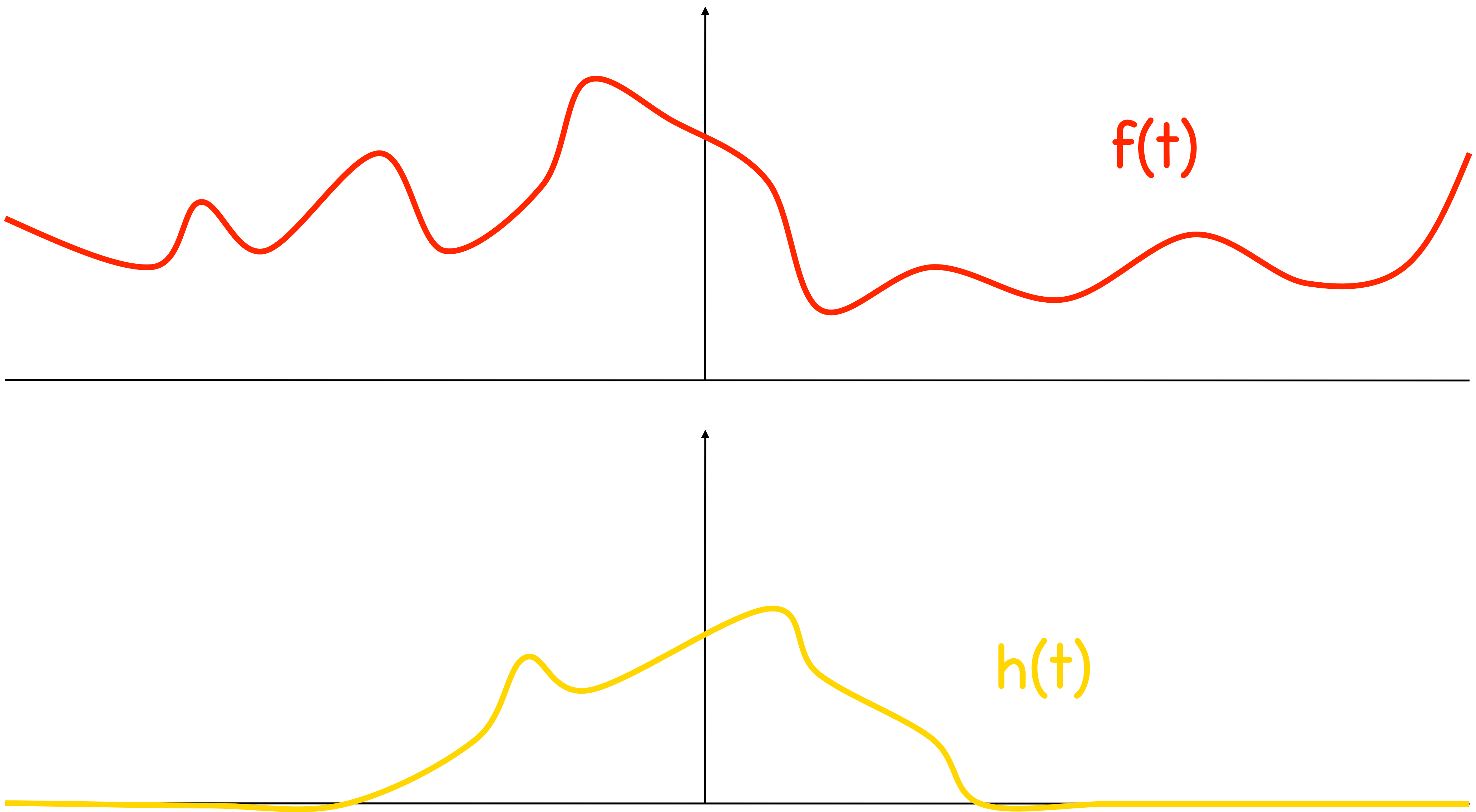
Convolution

- Definition:

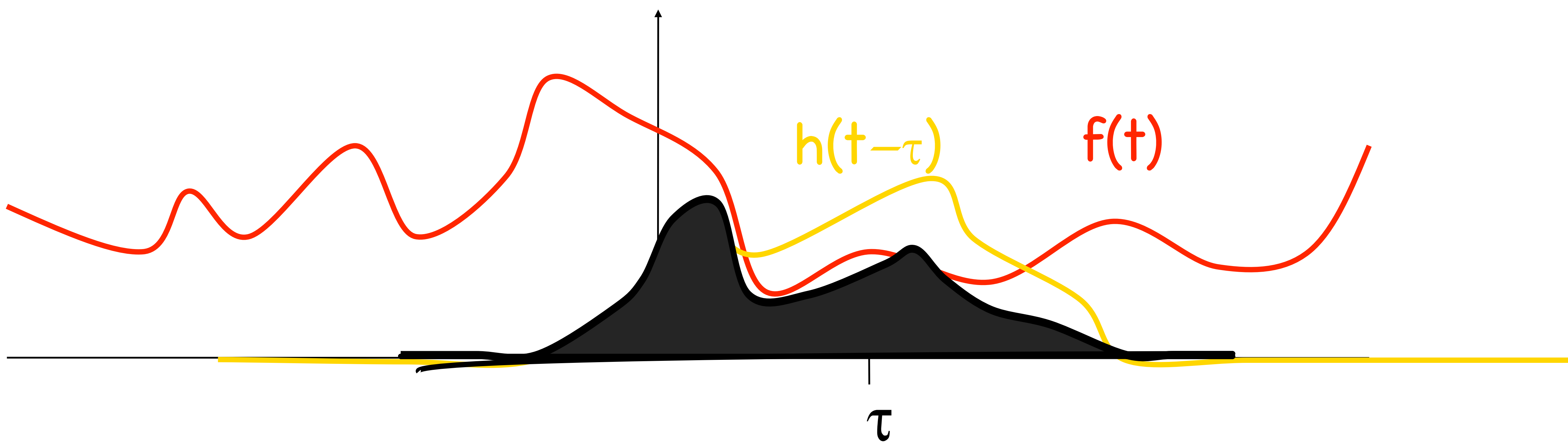
$$f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau$$

Convolution

- Definition: $$f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau$$
- Pictorially



Convolution



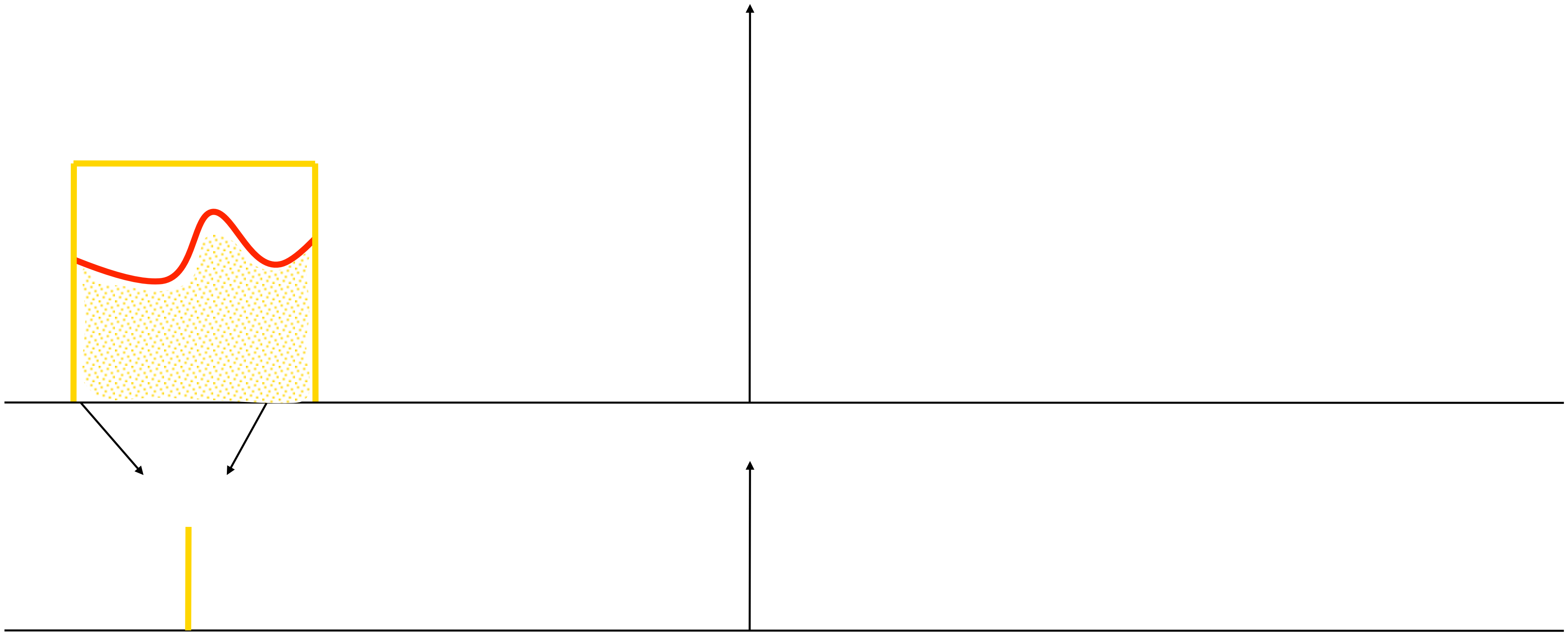
Convolution

- Consider the boxcar function (box filter):

$$h(t) = \begin{cases} 0 & t < -\frac{1}{2} \\ 1 & -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0 & t > \frac{1}{2} \end{cases}$$

Convolution

- This function windows our function f(t)



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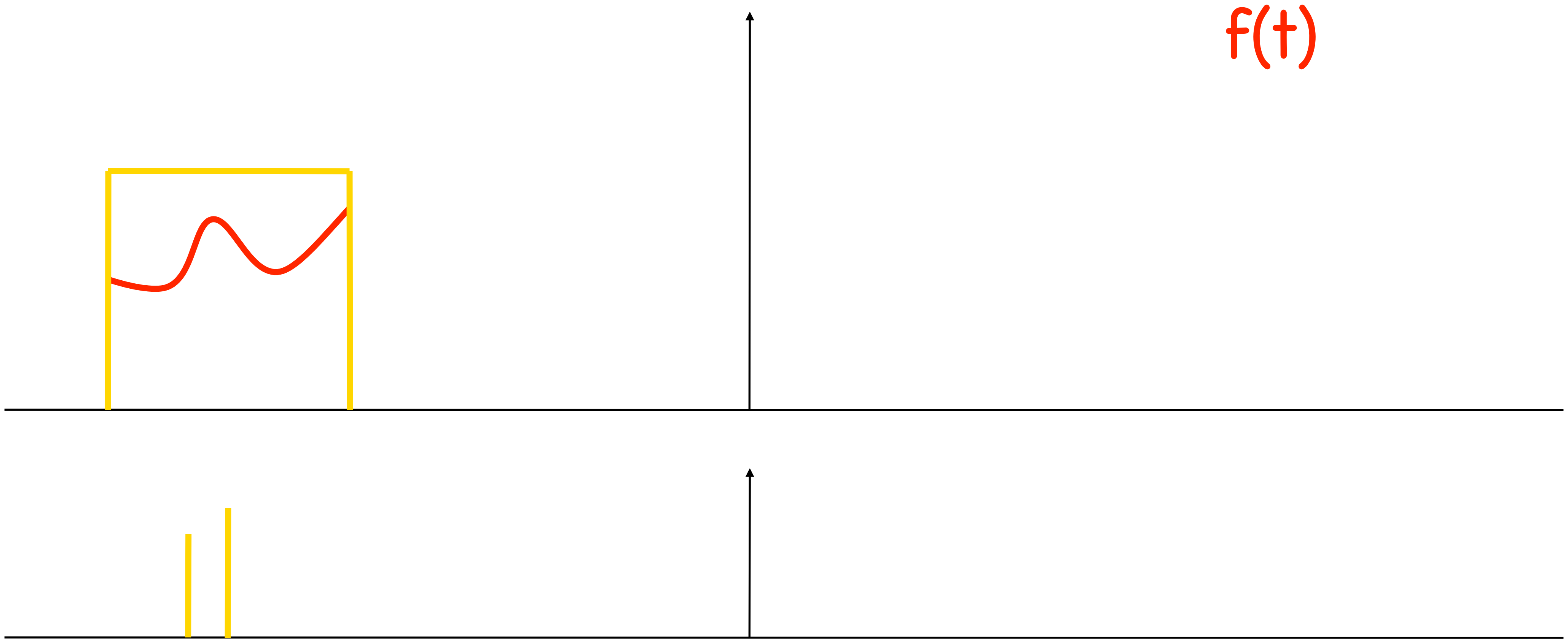

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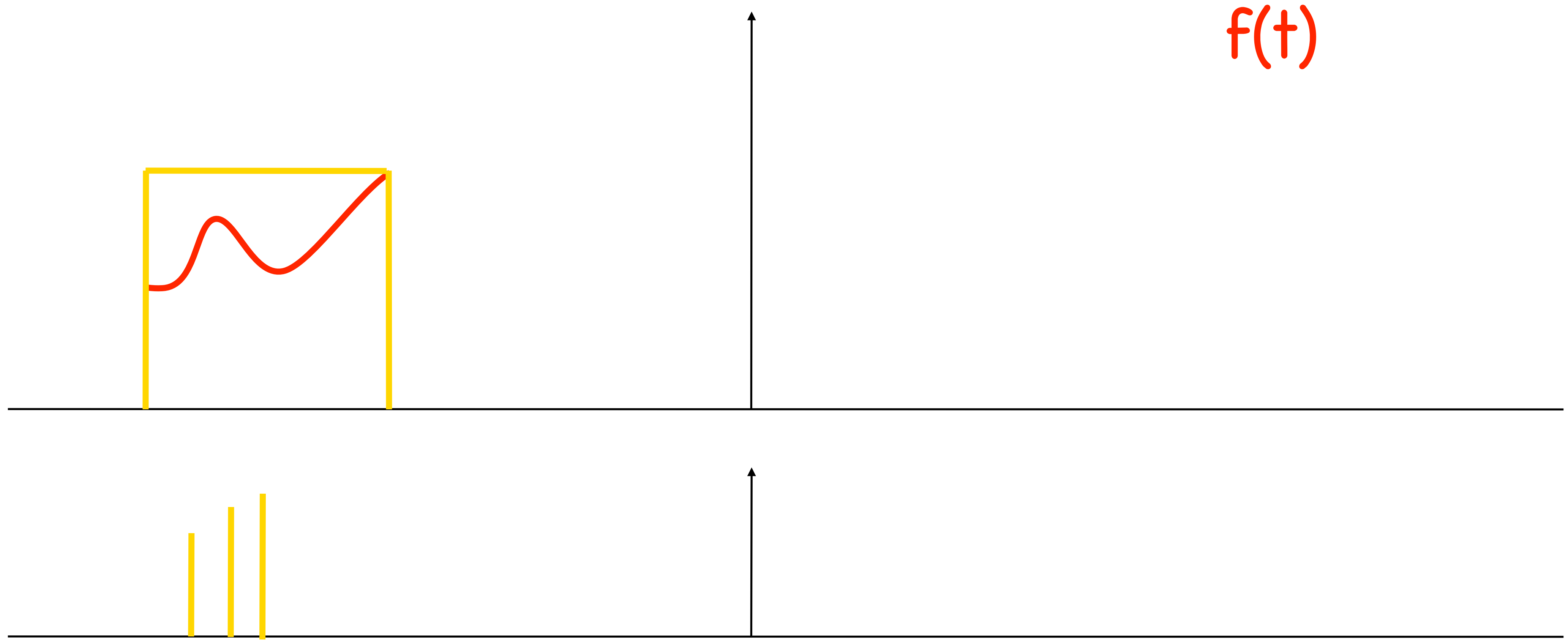

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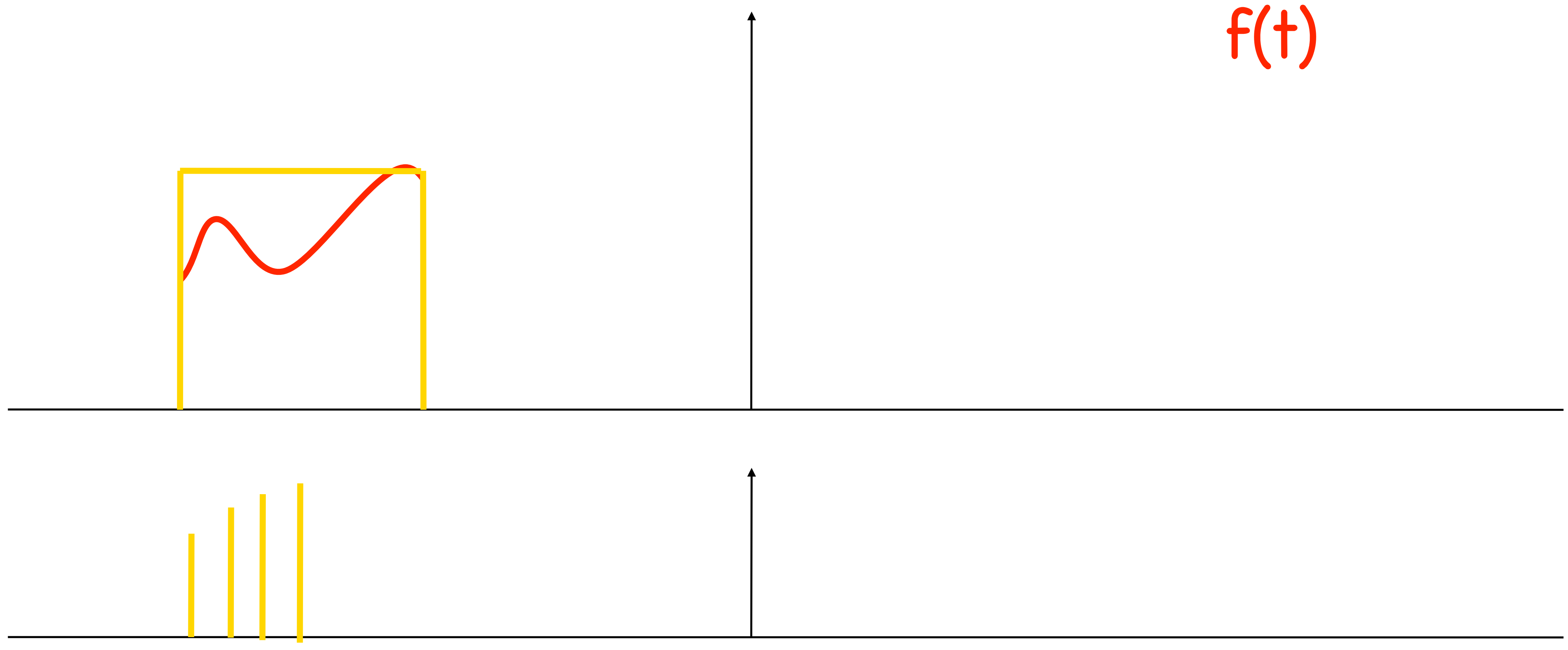

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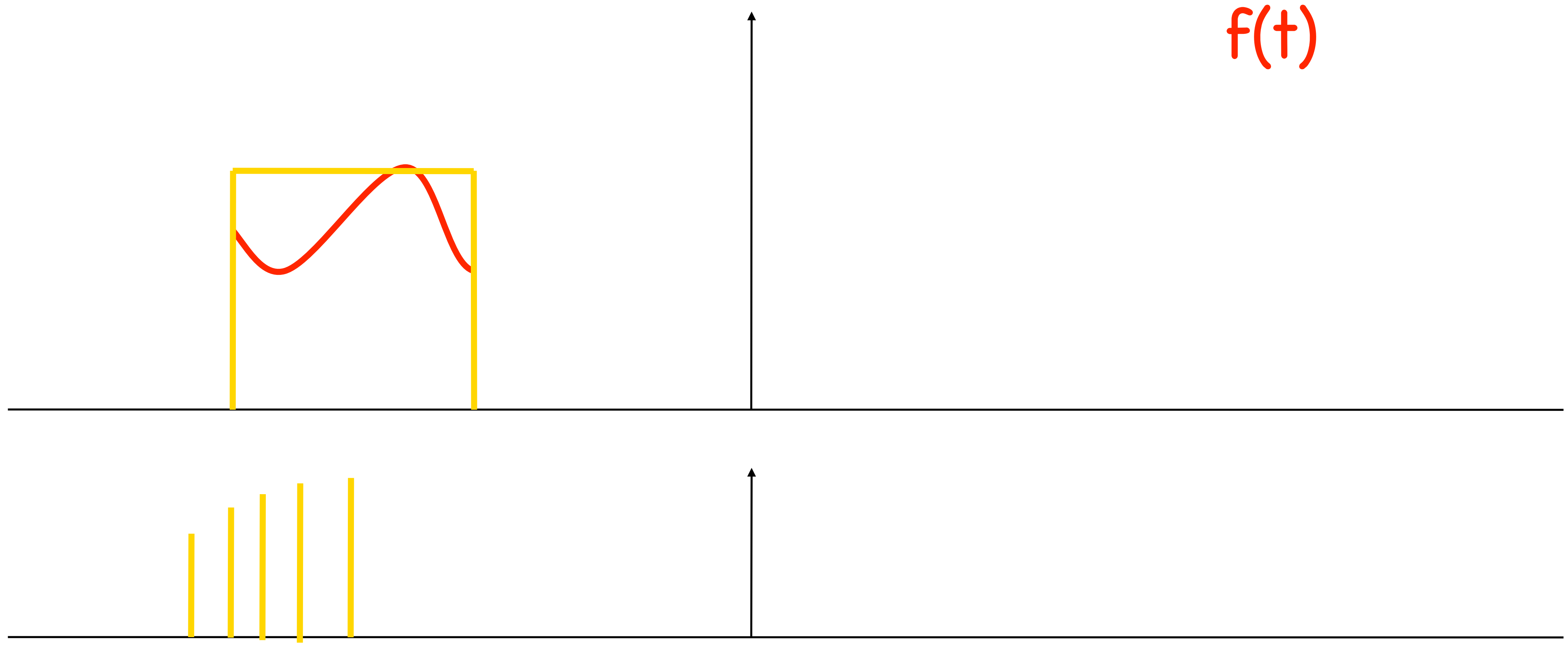

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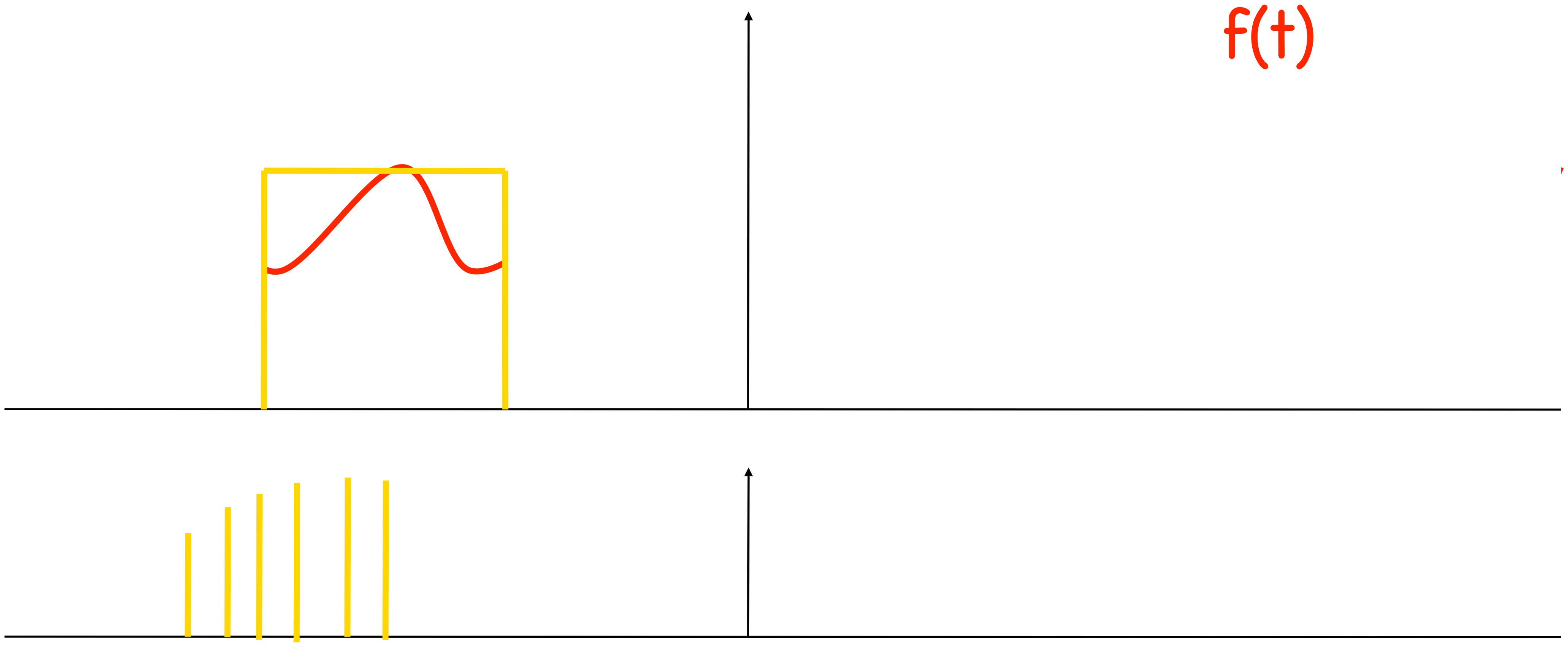

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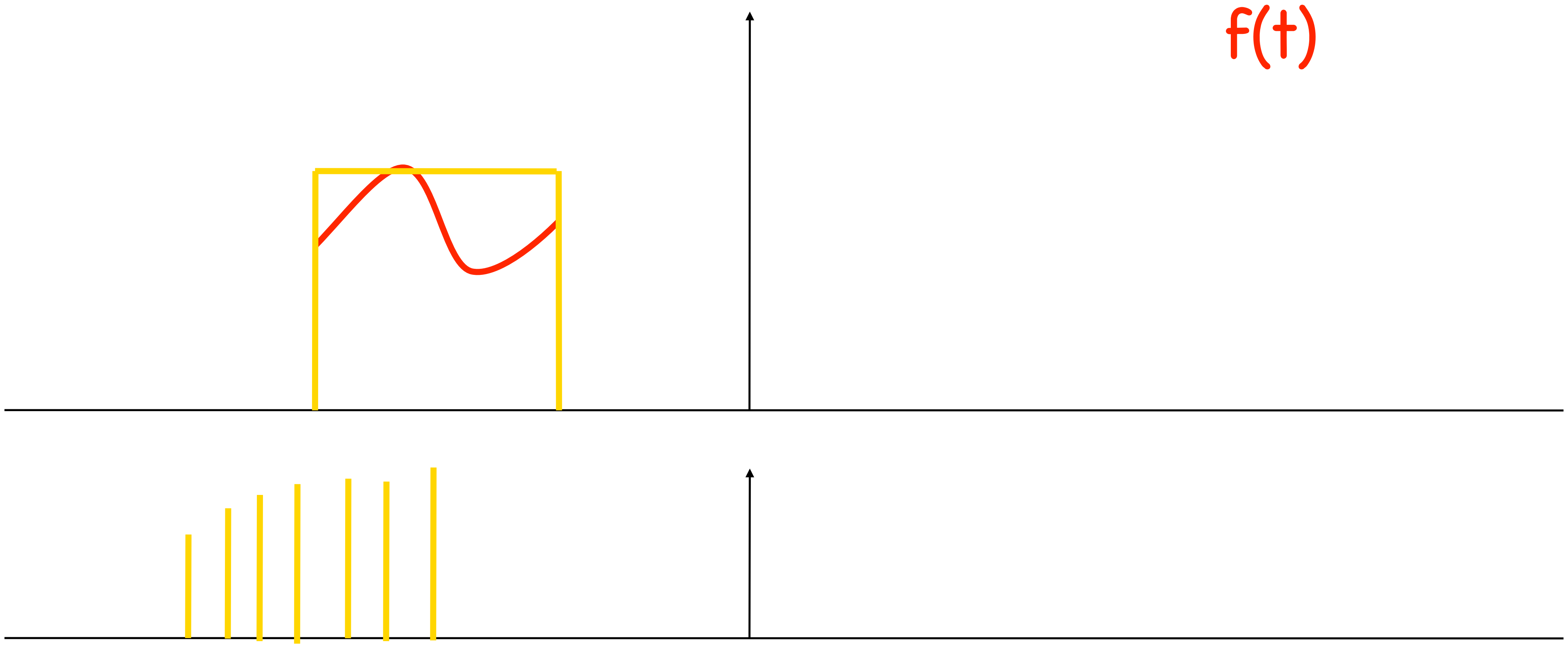

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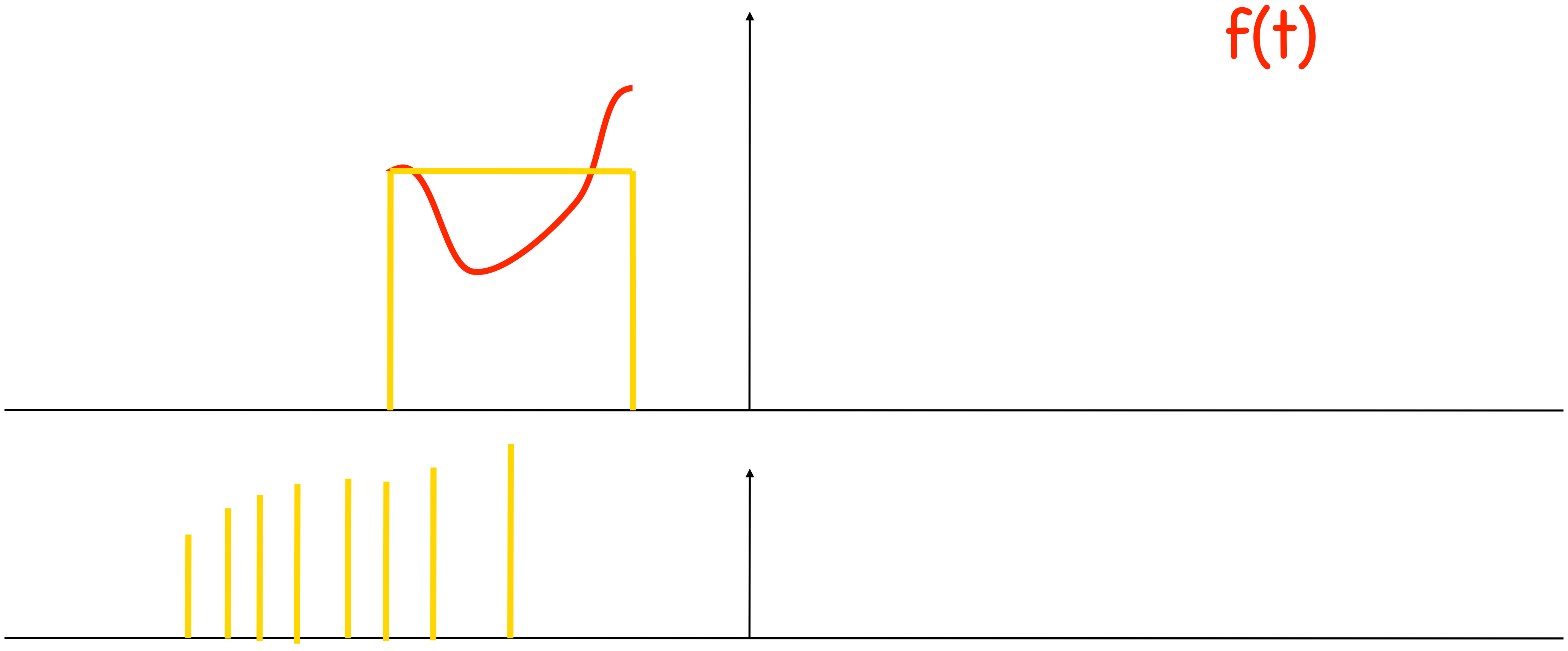

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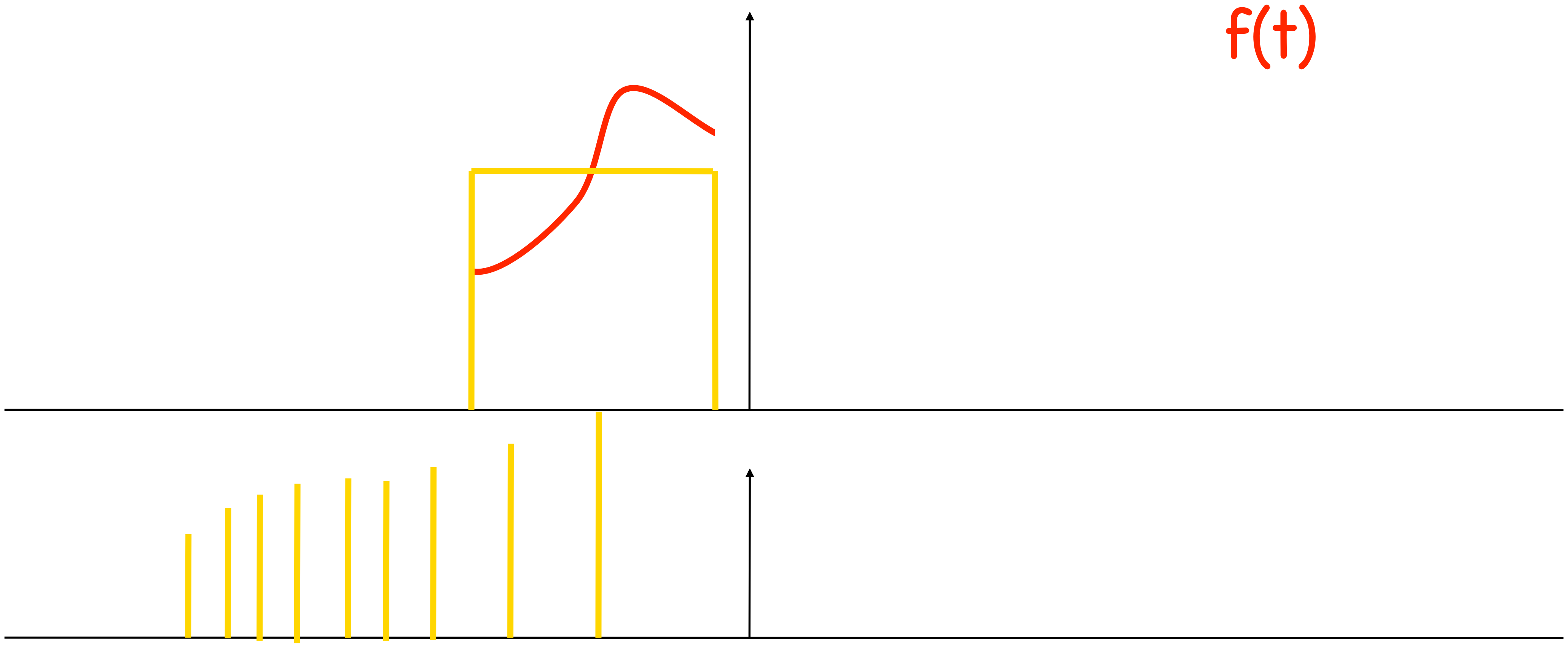

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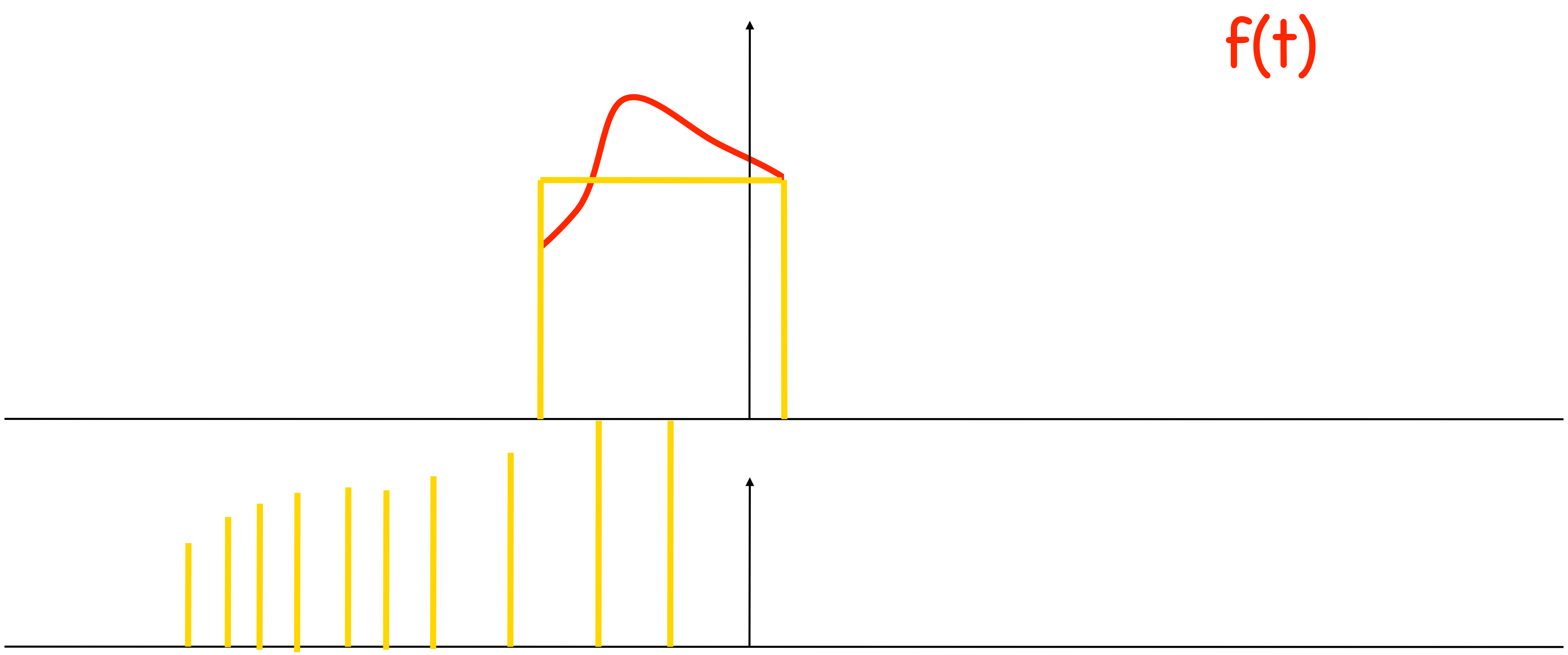

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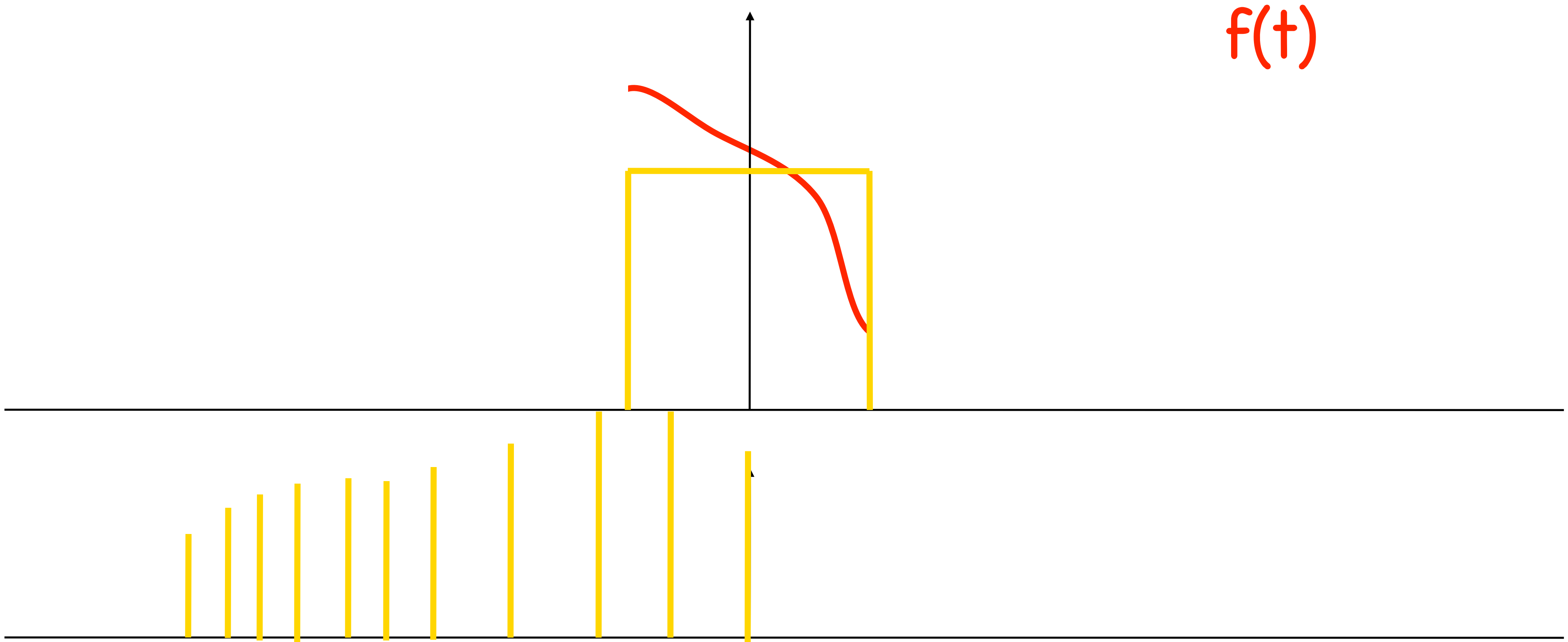

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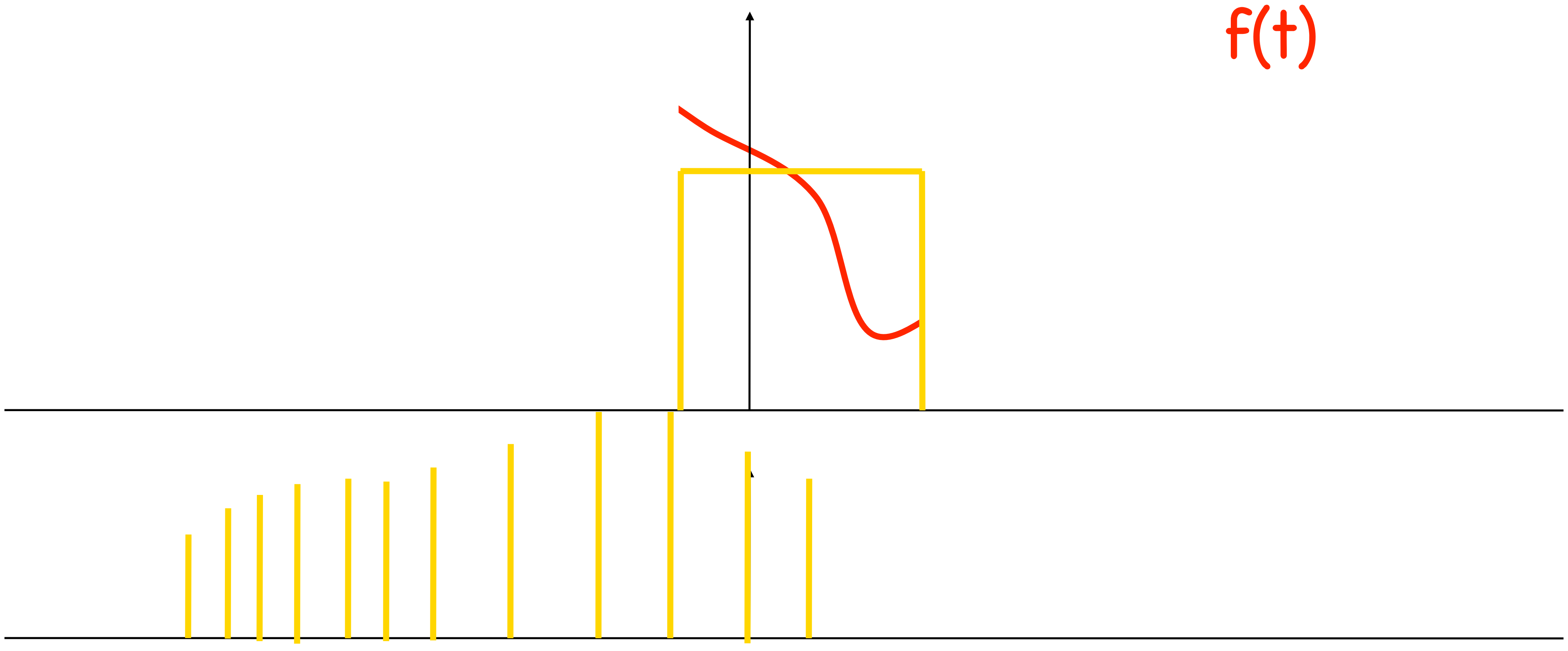

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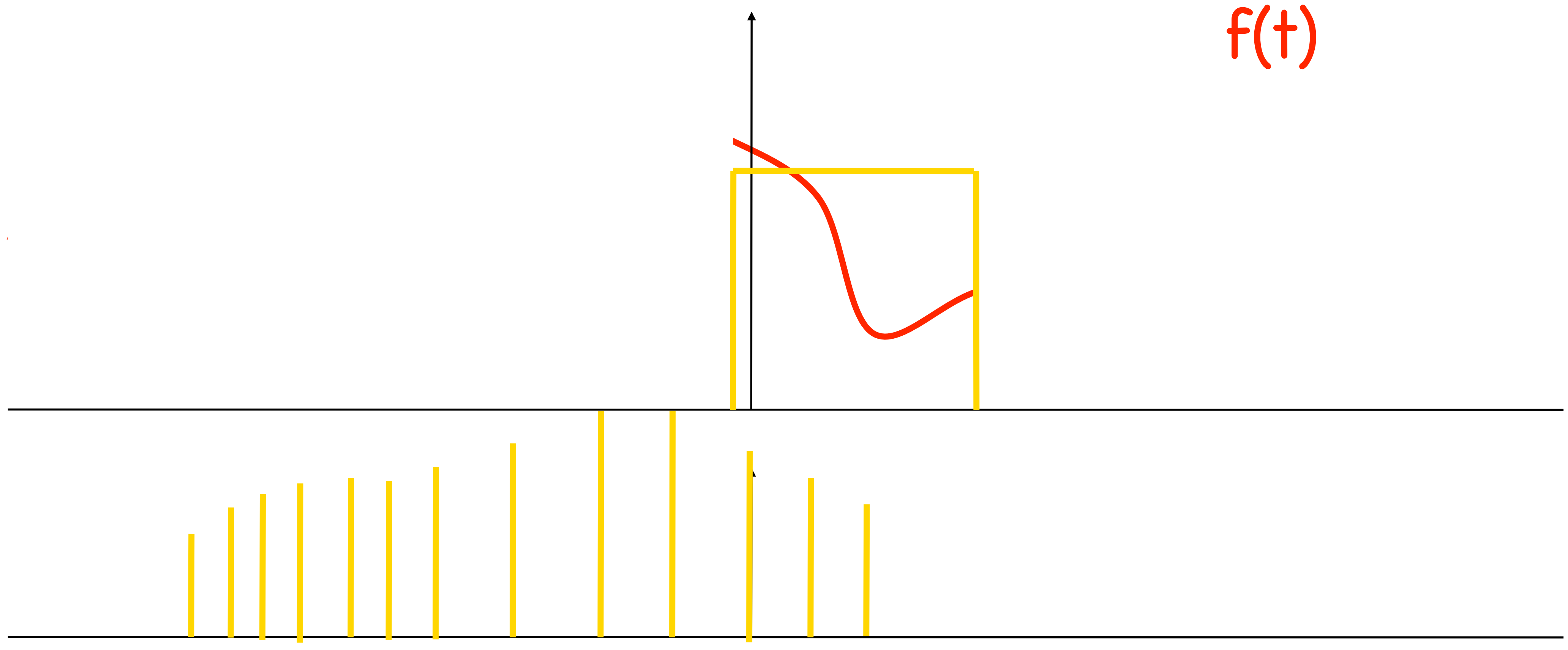

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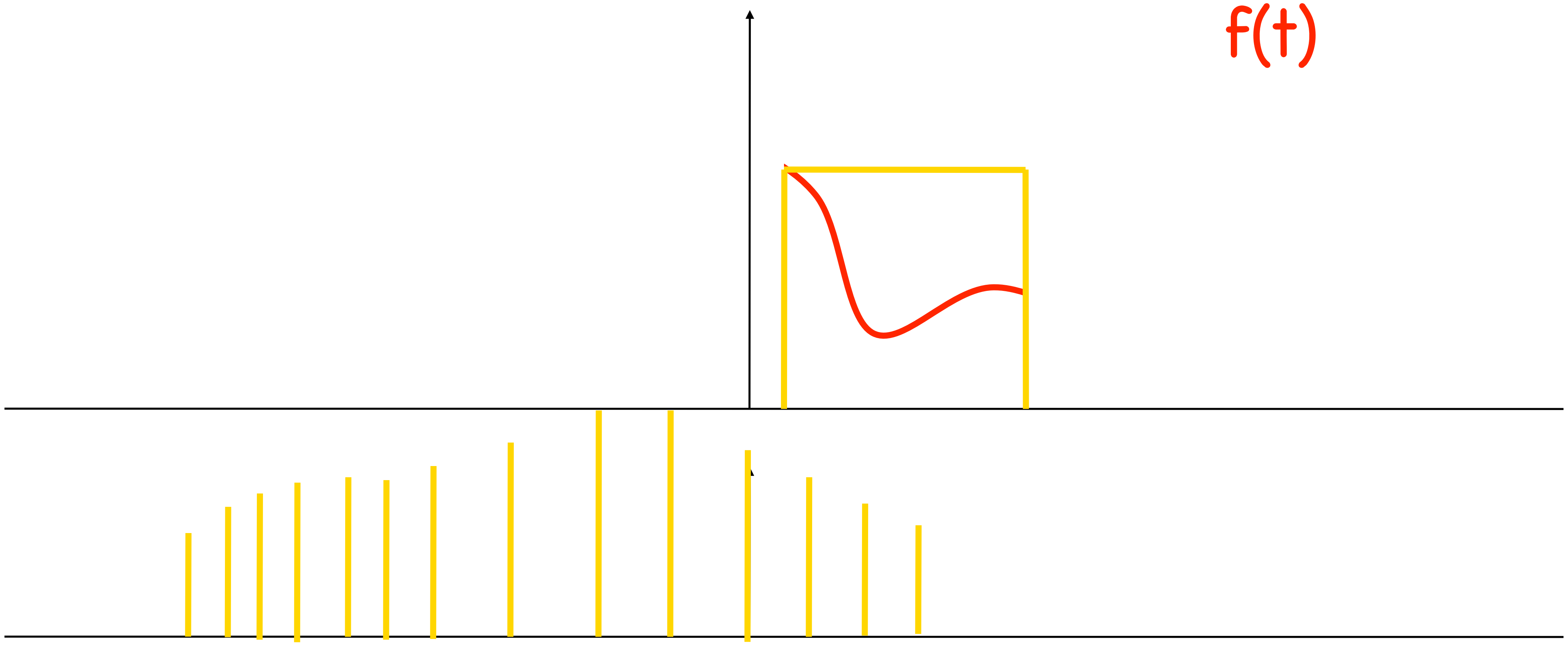

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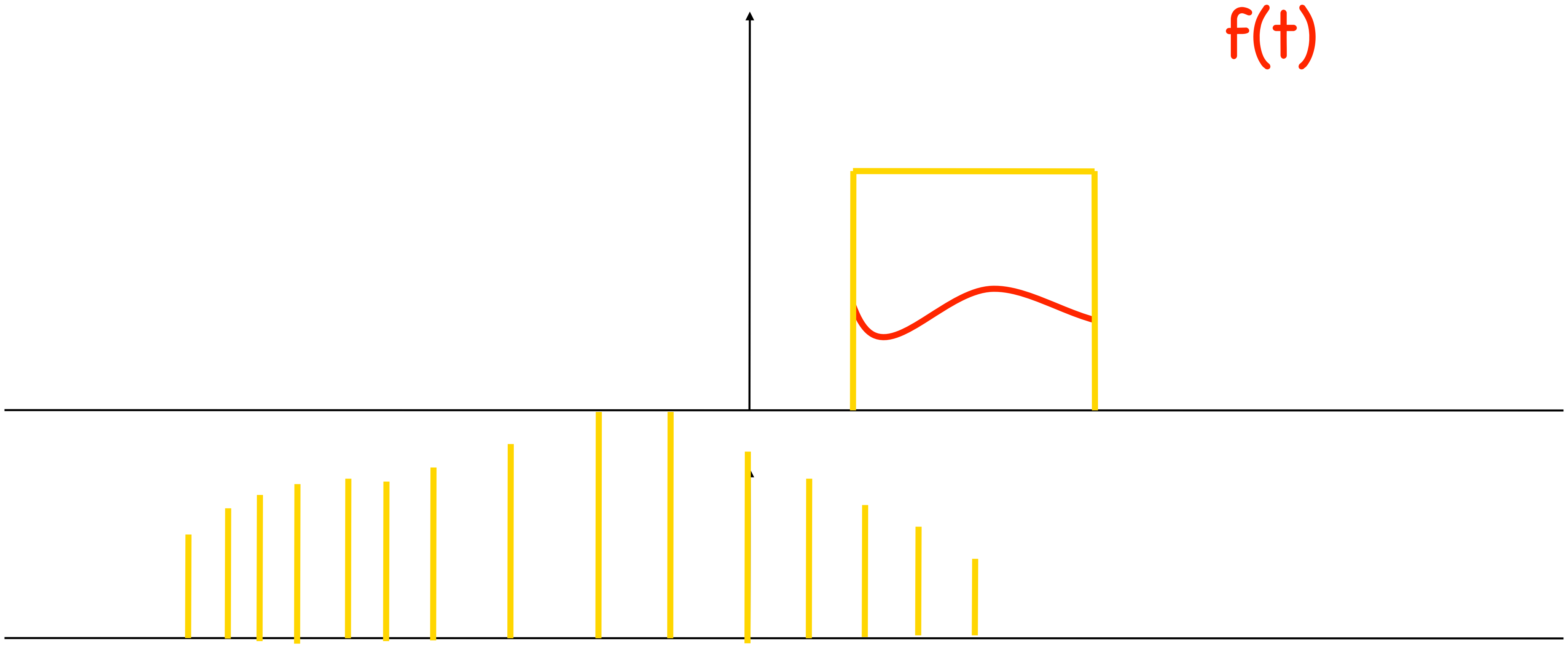

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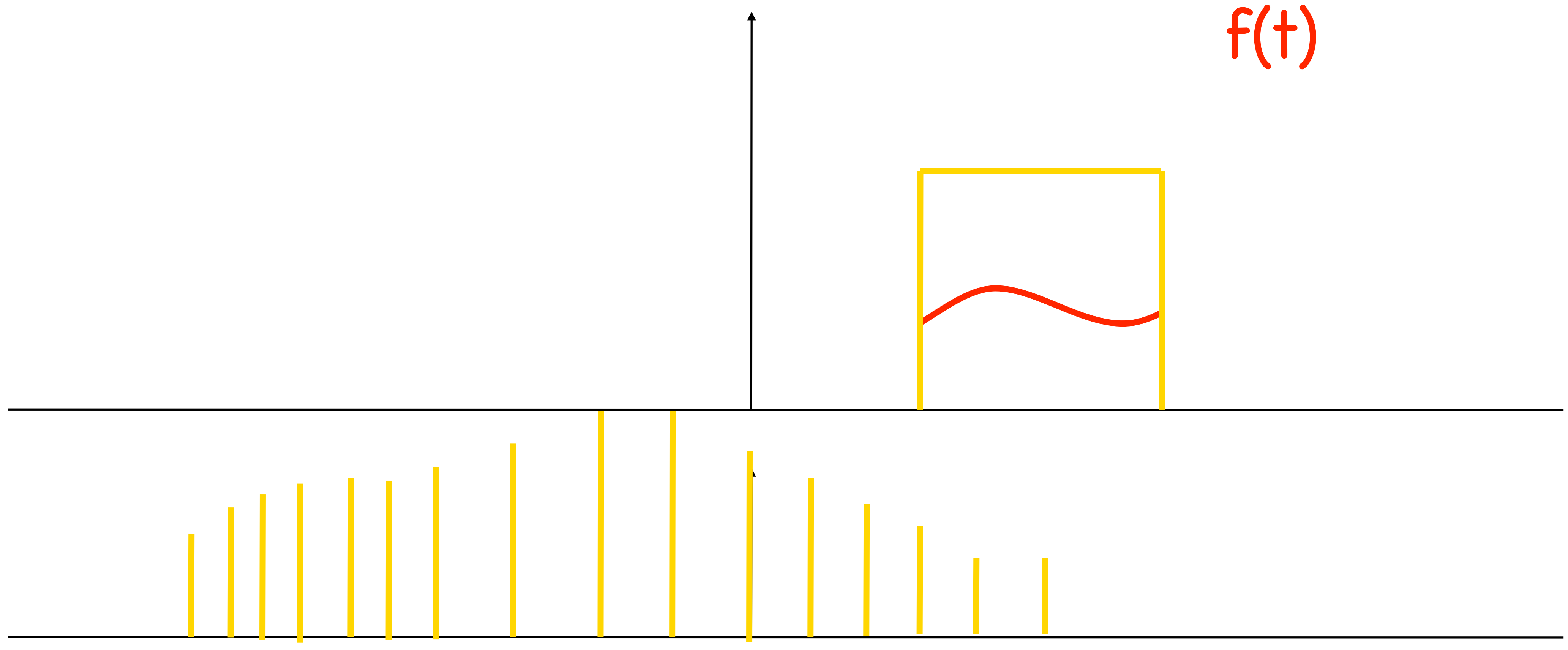

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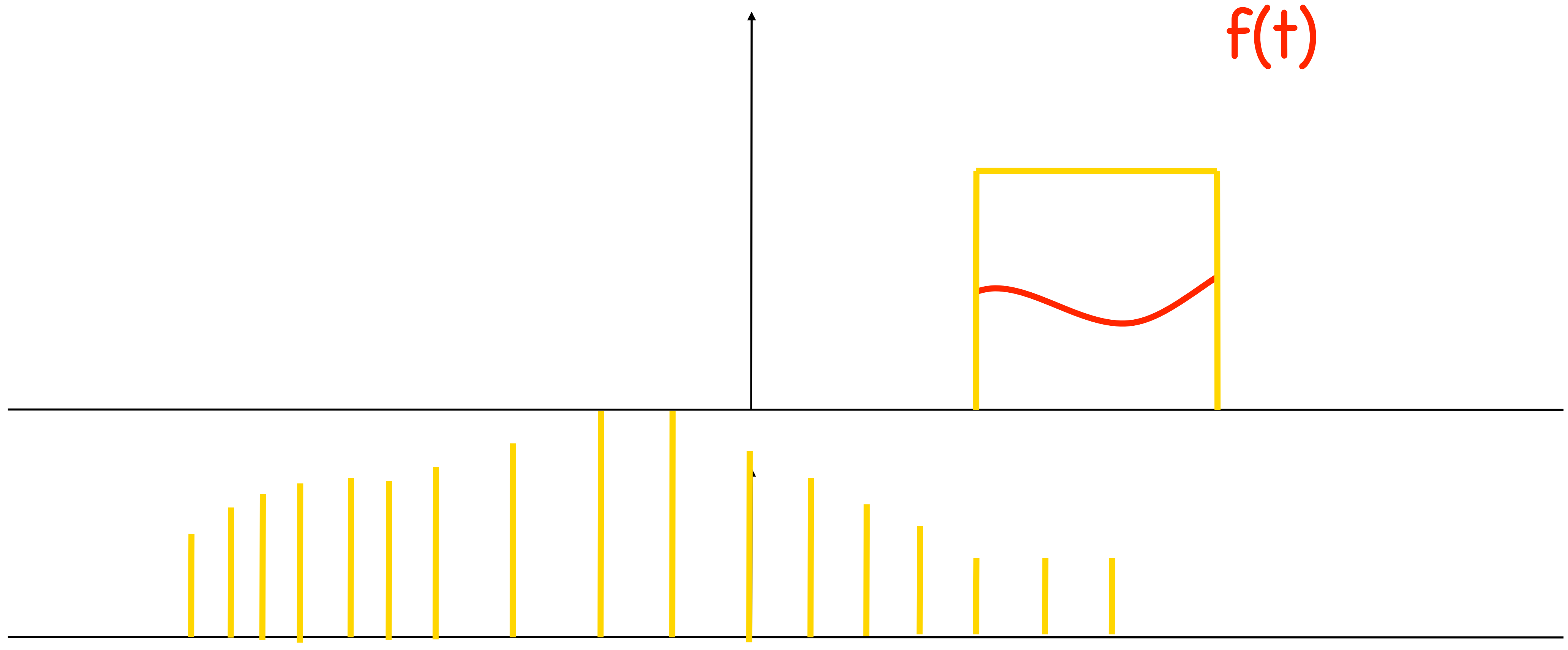

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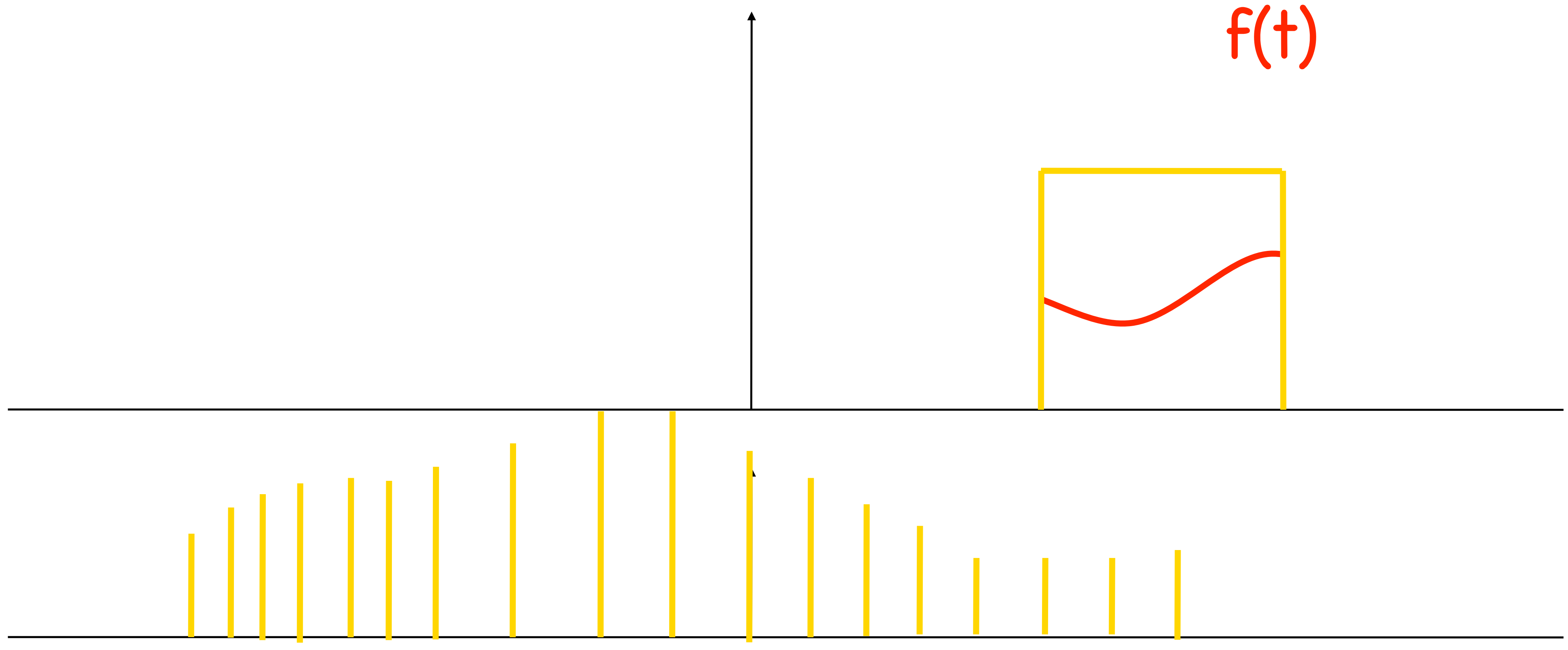

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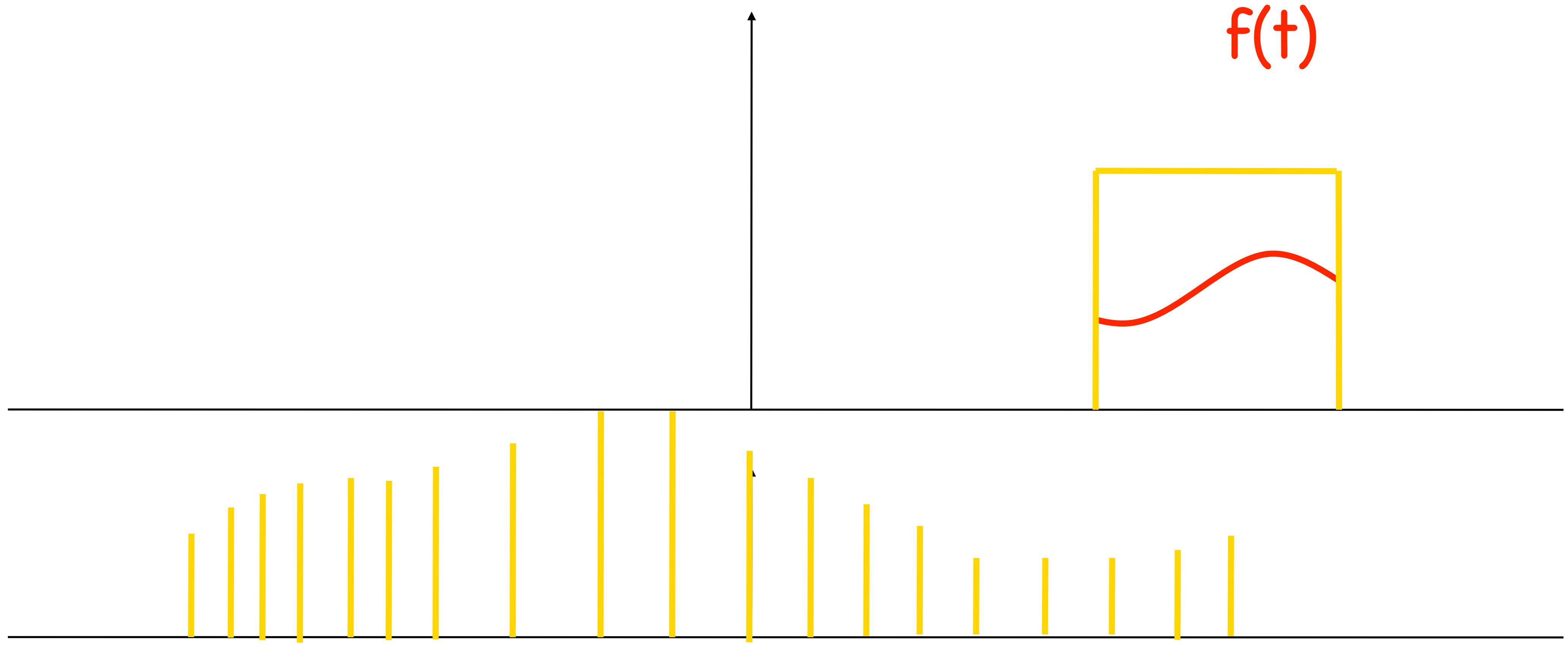

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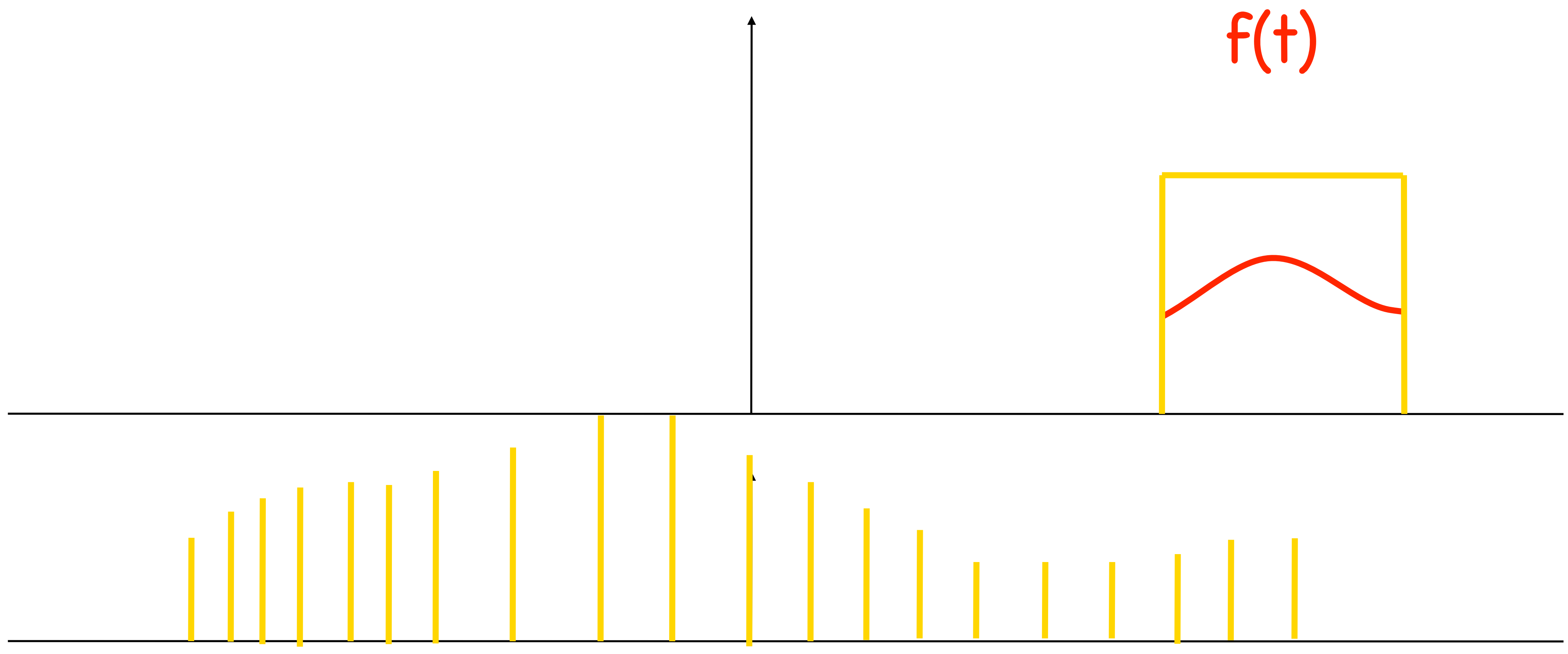

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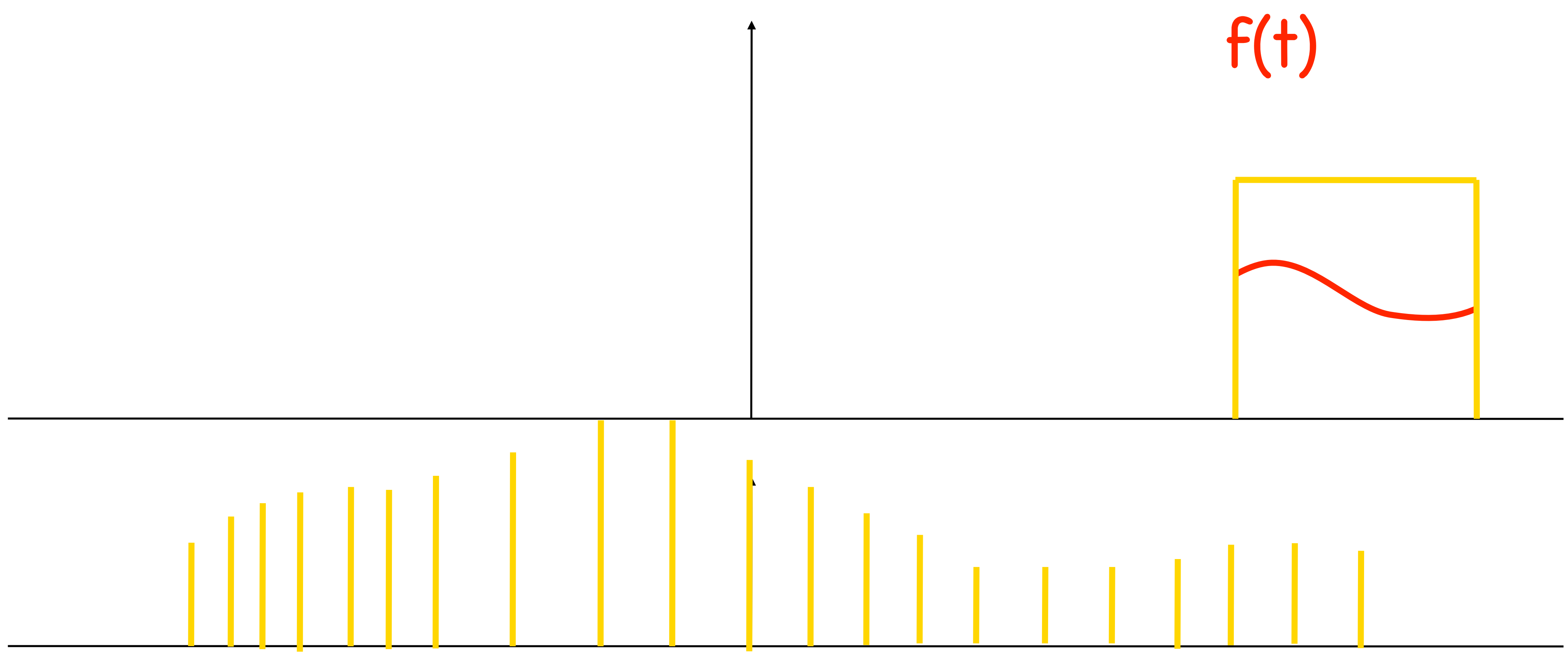

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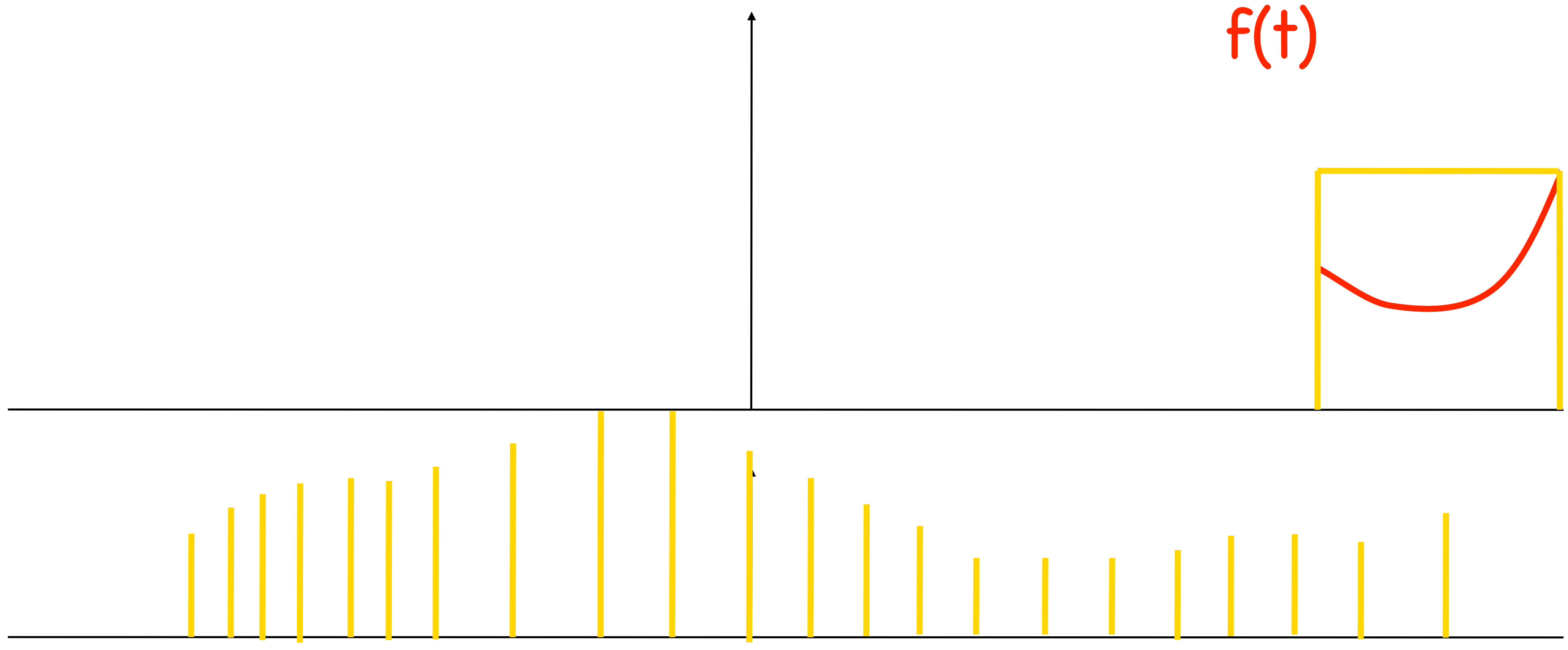
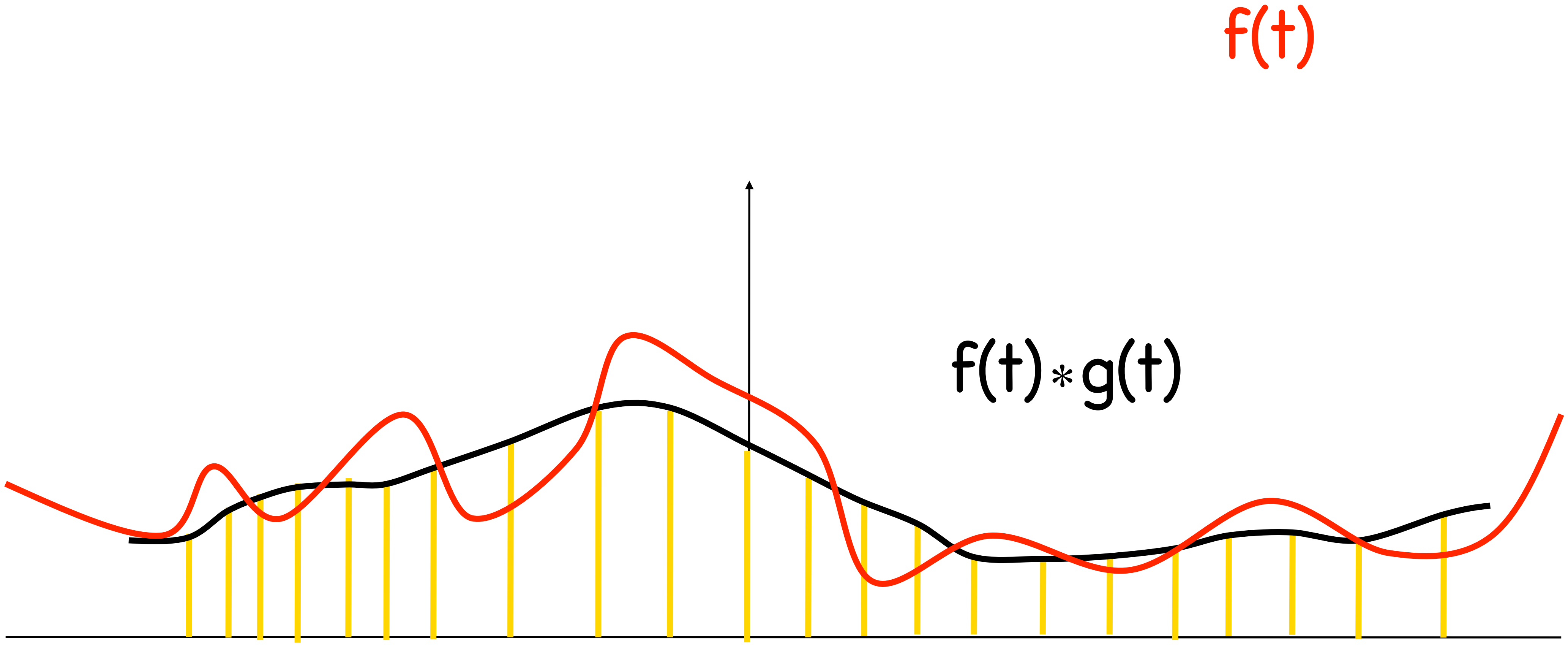


Convolution

- This particular convolution smooths out some of the high frequencies in f(t).

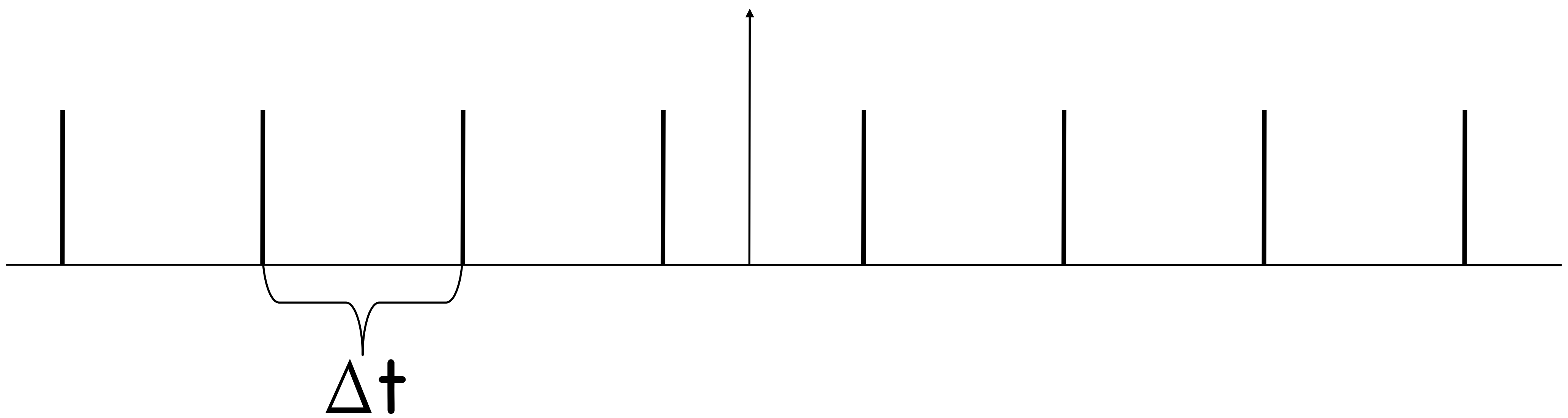


Sampling Function

- A Sampling Function or Impulse Train is defined by:

$$S_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - k\Delta t)$$

where Δt is the sample spacing.



Sampling Function

- The Fourier Transform of the Sampling Function is itself a sampling function.
- The sample spacing is the inverse.

$$S_{\Delta t}(t) \Leftrightarrow S_{\frac{1}{\Delta t}}(\omega)$$

Convolution Theorem

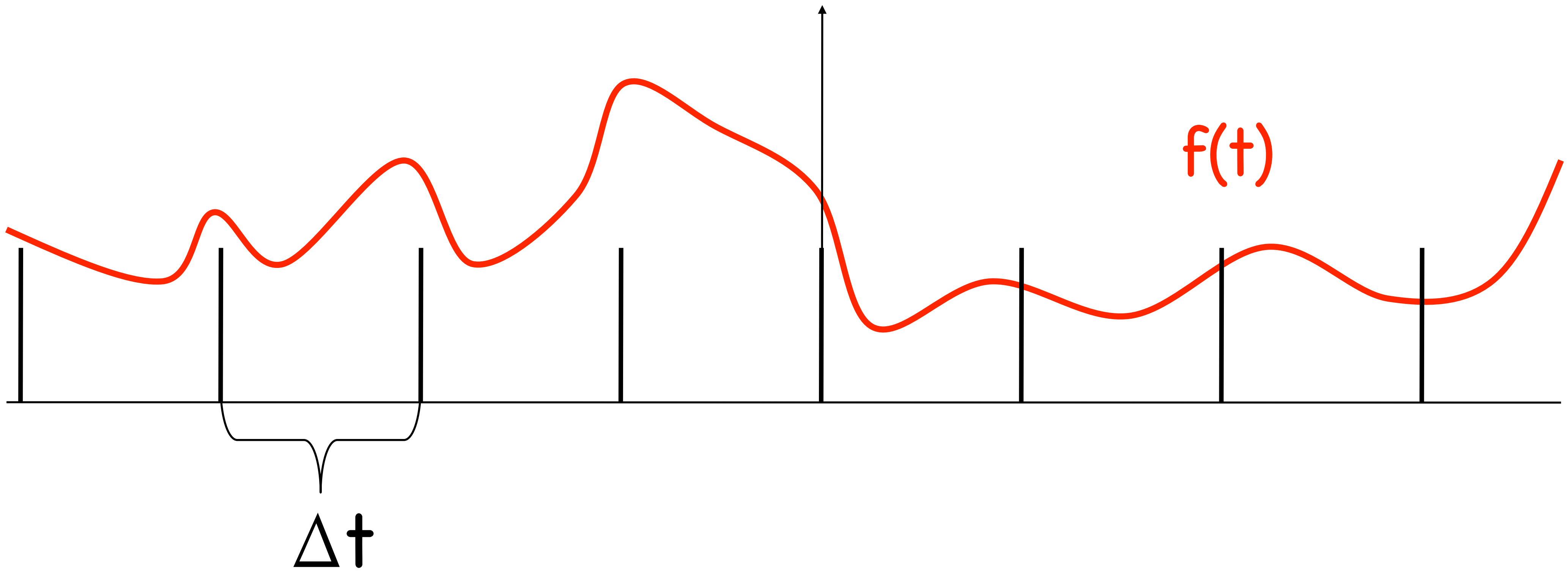
- The convolution theorem states that convolution in the spatial domain is equivalent to multiplication in the frequency domain, and viceversa.

$$f(t) * g(t) \Leftrightarrow F(\omega) \cdot G(\omega)$$

$$f(t) \cdot g(t) \Leftrightarrow F(\omega) * G(\omega)$$

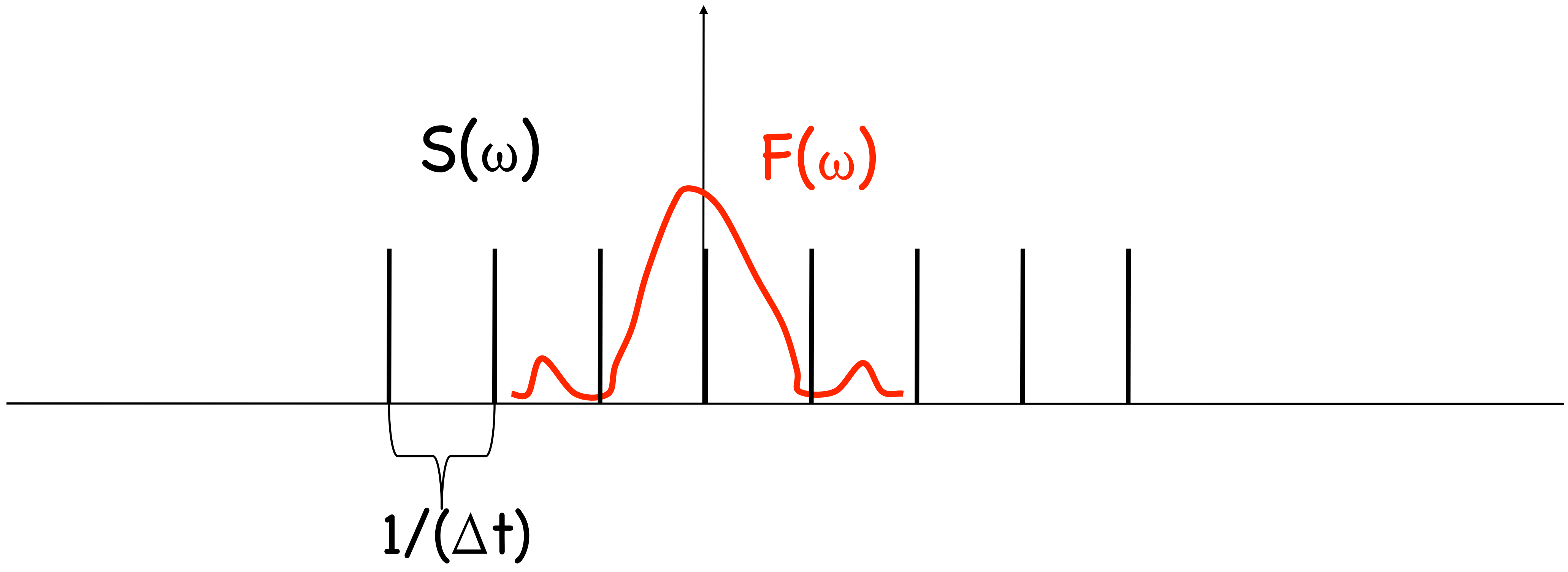
- This powerful theorem can illustrate the problems with our point sampling and provide guidance on avoiding aliasing.

- Consider: $f(t) \cdot S_{\Delta t}(t)$



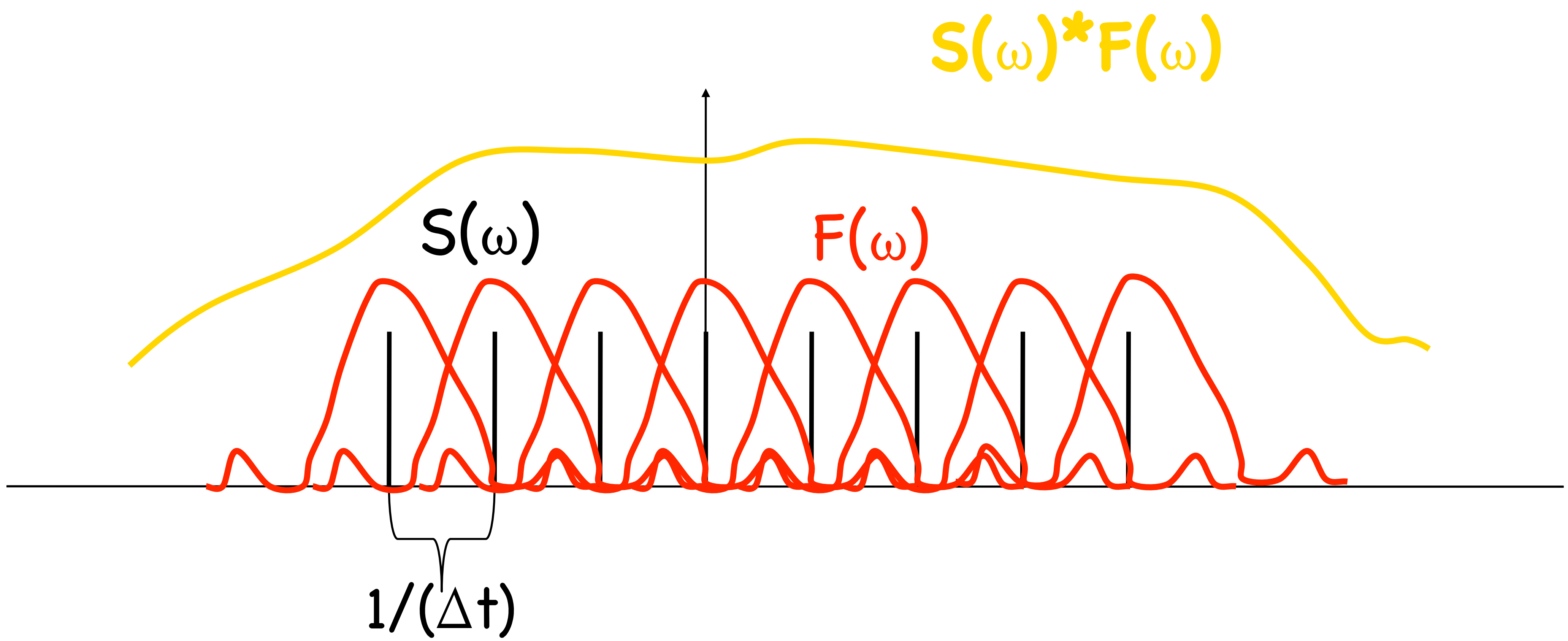
Convolution Theorem

- What does this look like in the Fourier domain?



Convolution Theorem

- In Fourier domain we would convolve



Aliasing

- What this says, is that any frequencies greater than a certain amount will appear intermixed with other frequencies.
- In particular, the higher frequencies for the copy at $1/\Delta t$ intermix with the low frequencies centered at the origin.

Aliasing and Sampling

- Note, that the sampling process introduces frequencies out to infinity.
- We have also lost the function f(t), and now have only the discrete samples.
- This brings us to our next powerful theory.

Sampling Theorem

- The Shannon Sampling Theorem:

  A band-limited signal f(t), with a cutoff frequency of λ, that is sampled with a sampling spacing of Δt may be perfectly reconstructed from the discrete values $f[n\Delta t]$ by convolution with the sinc(t) function, provided the Nyquist limit: $\lambda < 1/(2\Delta t)$

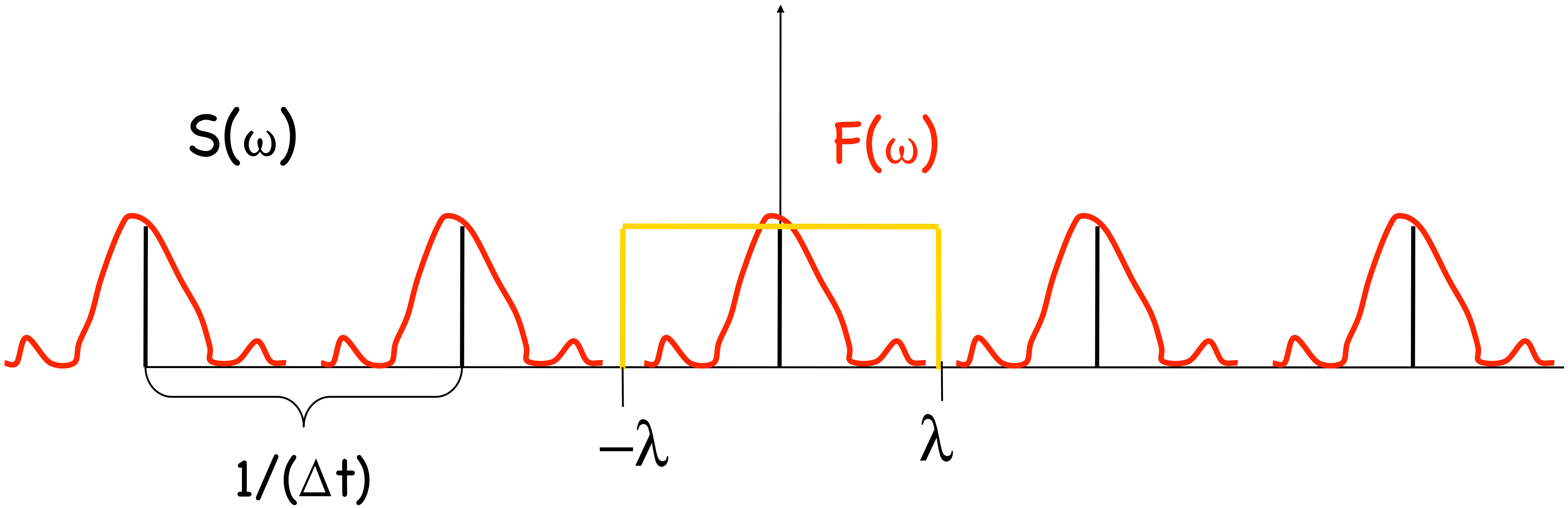
- Why is this?

  The Nyquist limit will ensure that the copies of $F(\omega)$ do not overlap in the frequency domain.

  We can completely reconstruct or determine f(t) from $F(\omega)$ using the Inverse Fourier Transform.

Sampling Theory

- In order to do this, we need to remove all of the shifted copies of $F(\omega)$ first.
- This is done by simply multiplying $F(\omega)$ by a box function of width 2λ.



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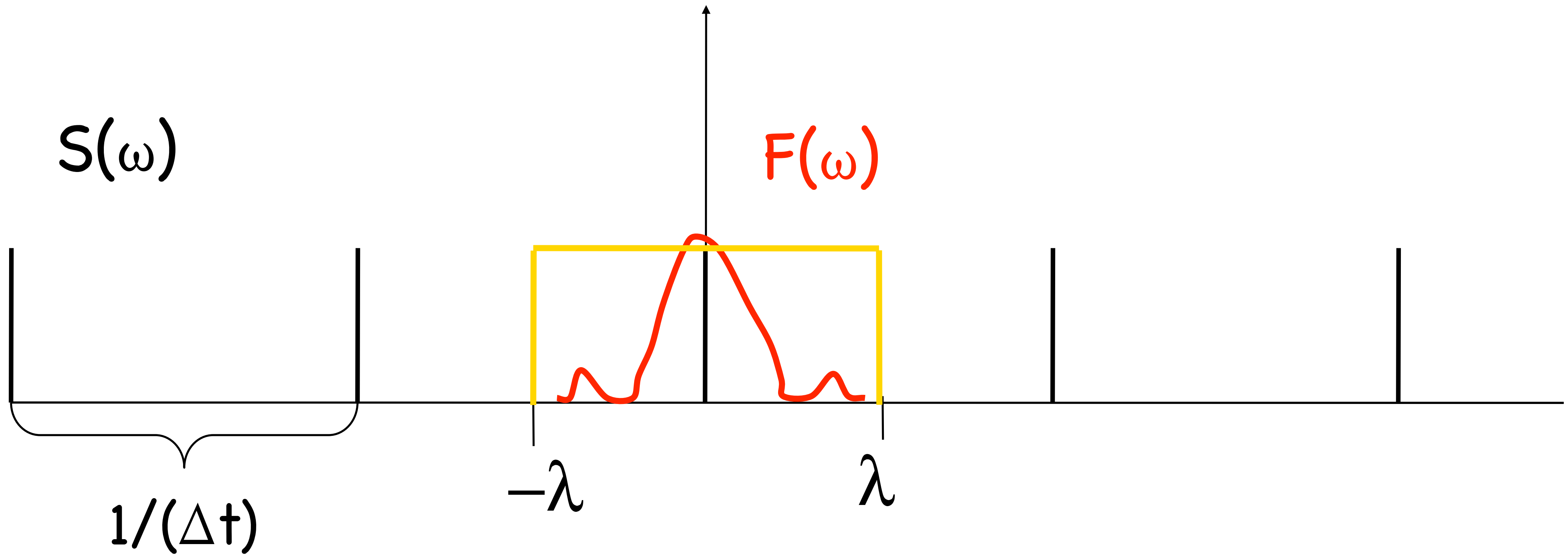


Sampling Theory

- So, given $f[n\Delta t]$ and an assumption that $f(t)$ does not have frequencies greater than $1/(2\Delta t)$, we can write the formula:

$$f[n\Delta t] = f(t) \cdot S_{\Delta t}(t) \Leftrightarrow F(\omega) * S_{\Delta t}(\omega)$$

$$F(\omega) = (F(\omega) * S_{\Delta t}(\omega)) \cdot Box_{1/(2\Delta t)}(\omega)$$

therefore,

$$f(t) = f[n\Delta t] * sinc(t)$$

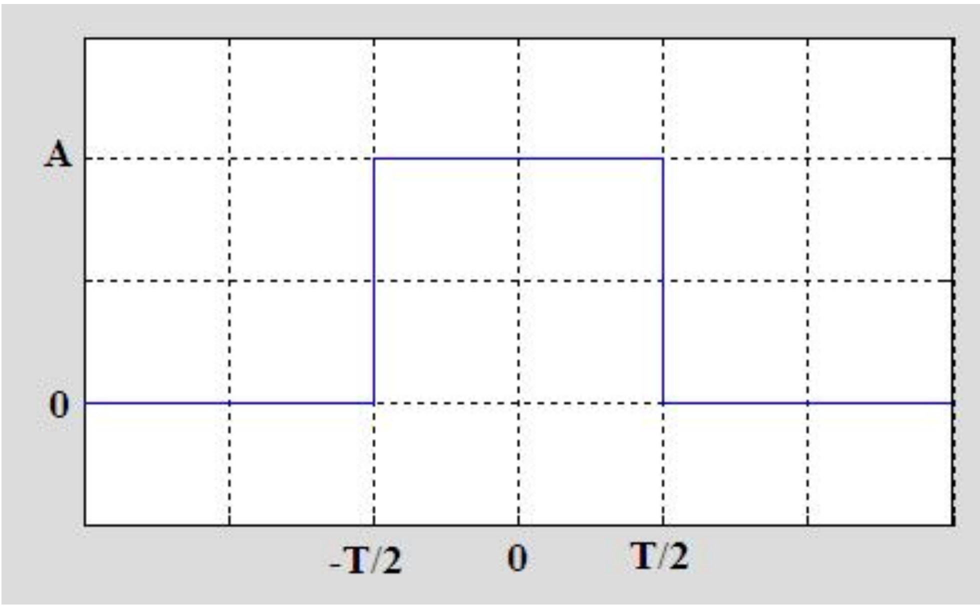
http://www.thefouriertransform.com/pairs/box.php

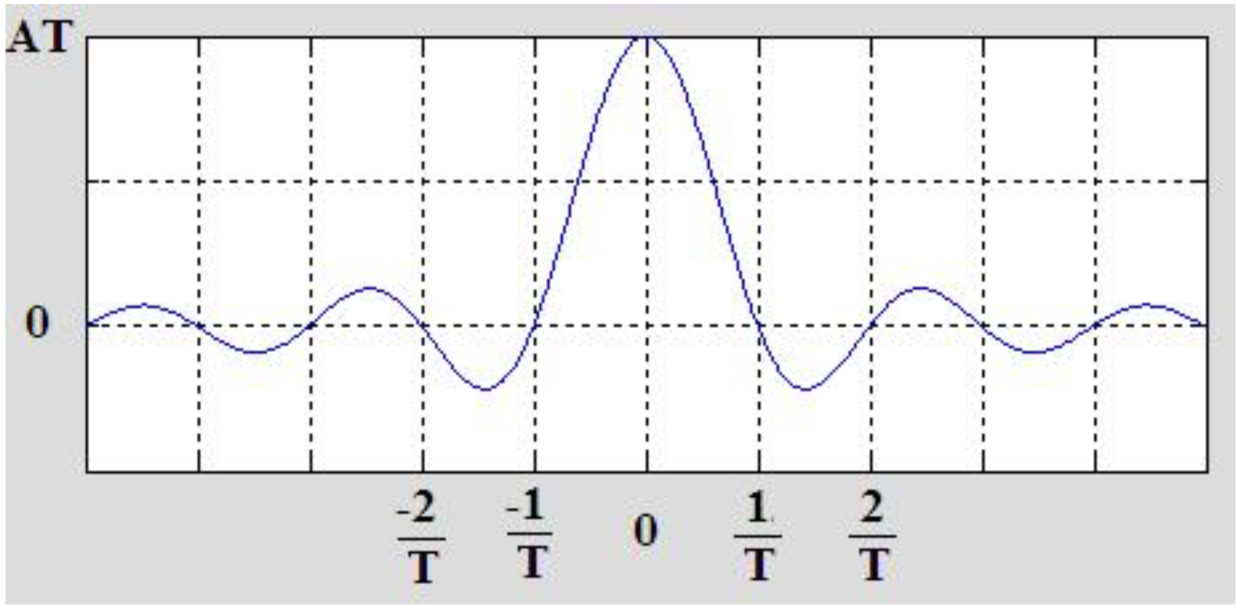
http://195.134.76.37/applets/AppletNyquist/Appl_Nyquist2.html

FT of Boxcar

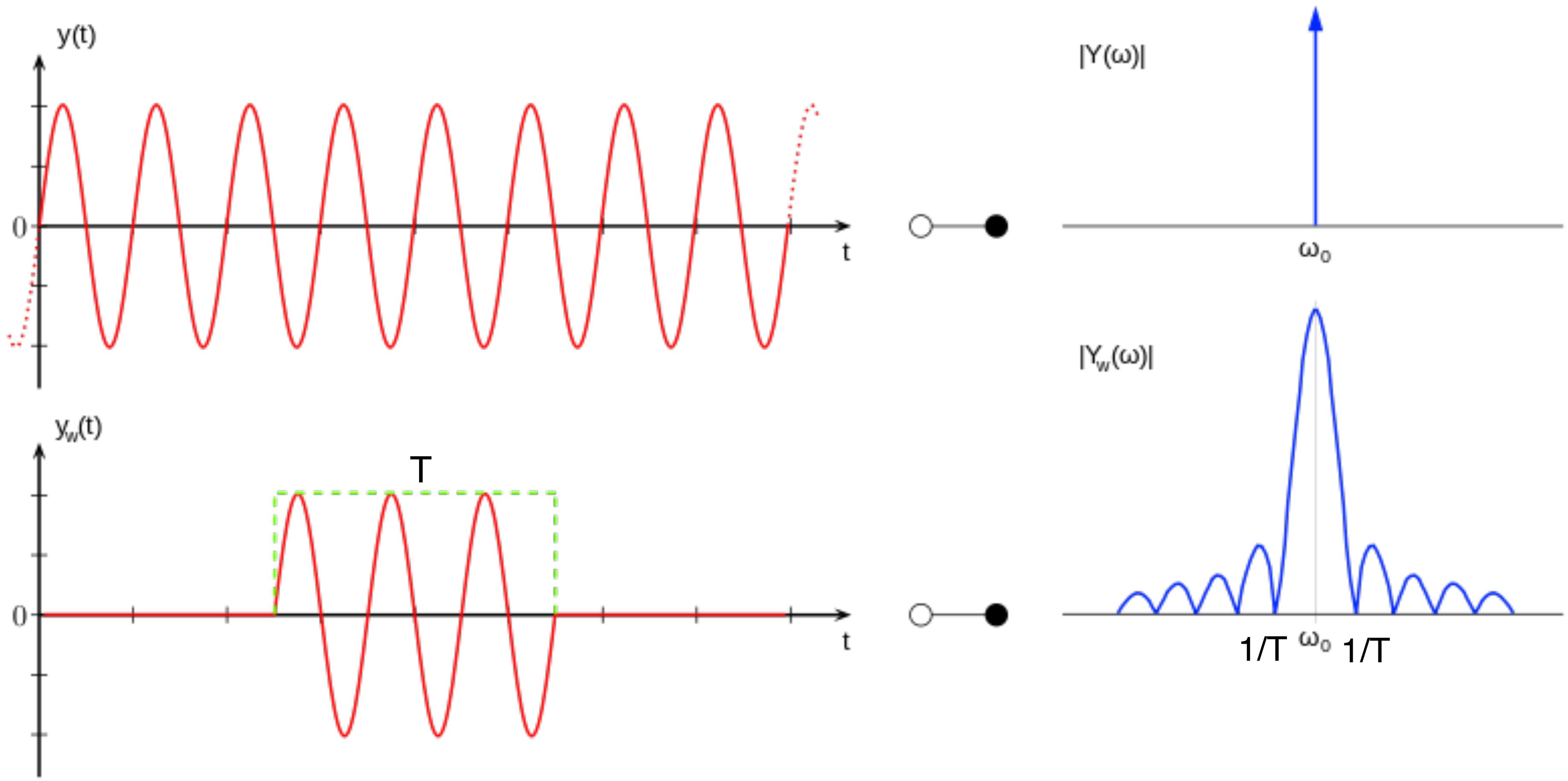
$$\int_{-\infty}^{+\infty} B_T(t) e^{-i\omega t} dt = \int_{-T/2}^{+T/2} e^{-i\omega t}\, dt = \frac{\sin(\pi f T)}{\pi f T}$$

$$\propto \operatorname{sinc}(\pi f T) = \operatorname{sinc}(\omega T / 2)$$





Spectral leakage



- Resolving power in frequency domain is related to maximum duration in time domain:

$$\Delta f \geq \frac{1}{2\pi T} \left(= \frac{1}{2\pi N \Delta t} \right)$$

- Resolving power in time domain decides maximum resolvable frequency:

$$\Delta f \geq \frac{1}{2\Delta t}$$

$$\Delta f \Delta t \geq \frac{1}{2\sqrt{\pi N}}$$

https://www.youtube.com/watch?v=MBnnXbOM5S4