Corso di Laurea in Fisica - UNITS

Istituzioni di Fisica per il Sistema Terra

Fluid Kinematics

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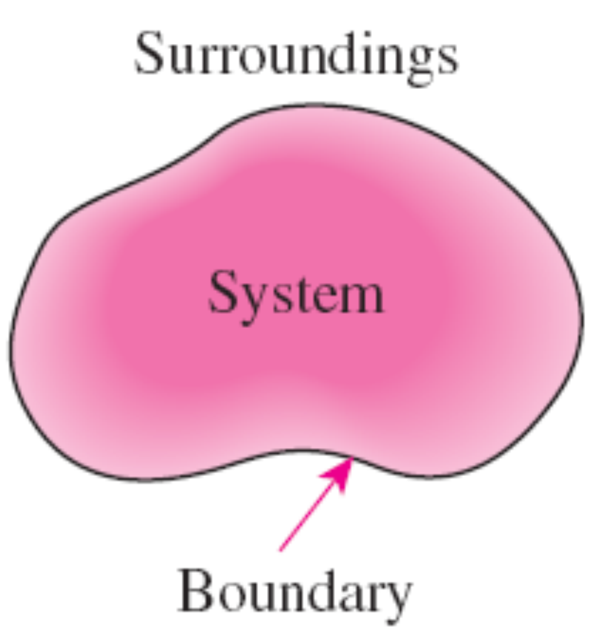
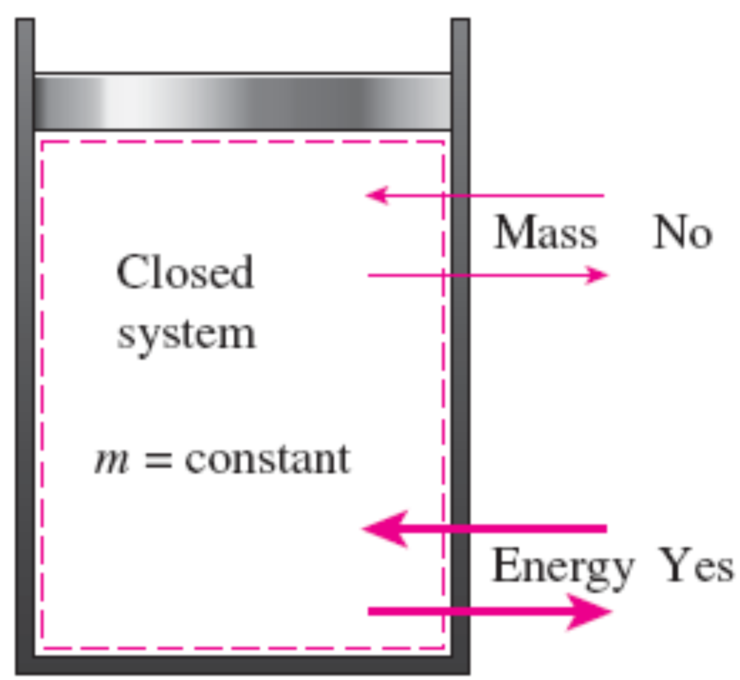
Overview

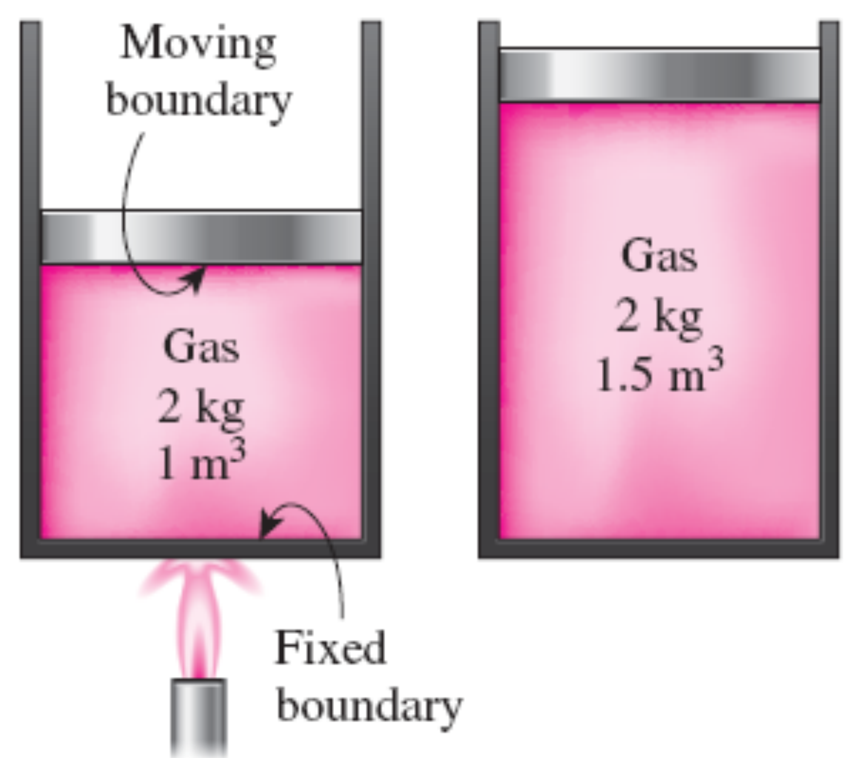
- Fluid Kinematics deals with the motion of fluids without necessarily considering the forces and moments which create the motion.
- Items discussed:
  - Material derivative and its relationship to Lagrangian and Eulerian descriptions of fluid flow.
  - Flow visualization.
  - Plotting flow data.
  - Fundamental kinematic properties of fluid motion and deformation.
  - Reynolds Transport Theorem

Systems and Control Volumes

- System: a quantity of matter or a region in space chosen for study.
- Surroundings: the mass or region outside the system
- Boundary: the real or imaginary surface that separates the system from its surroundings.
  - The boundary of a system can be fixed or movable.
- Systems may be considered to be closed or open.
  - Closed system (Control mass): A fixed amount of mass, and no mass can cross its boundary







- Open system (control volume): A properly selected region in space.
- It usually encloses a device that involves mass flow such as a compressor, turbine, or nozzle.
- Both mass and energy can cross the boundary of a control volume.
- Control surface: The boundaries of a control volume. It can be real or imaginary.

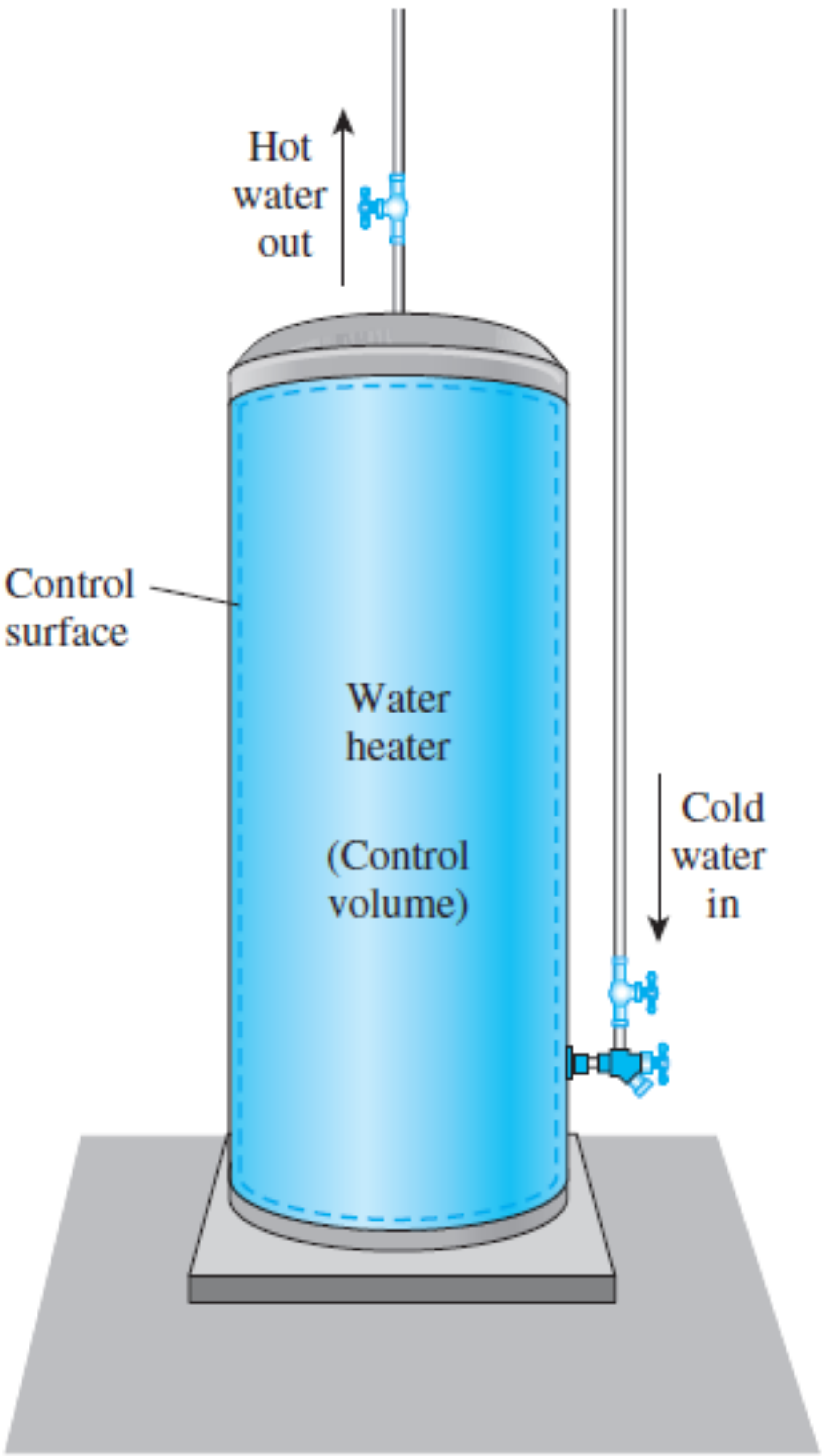
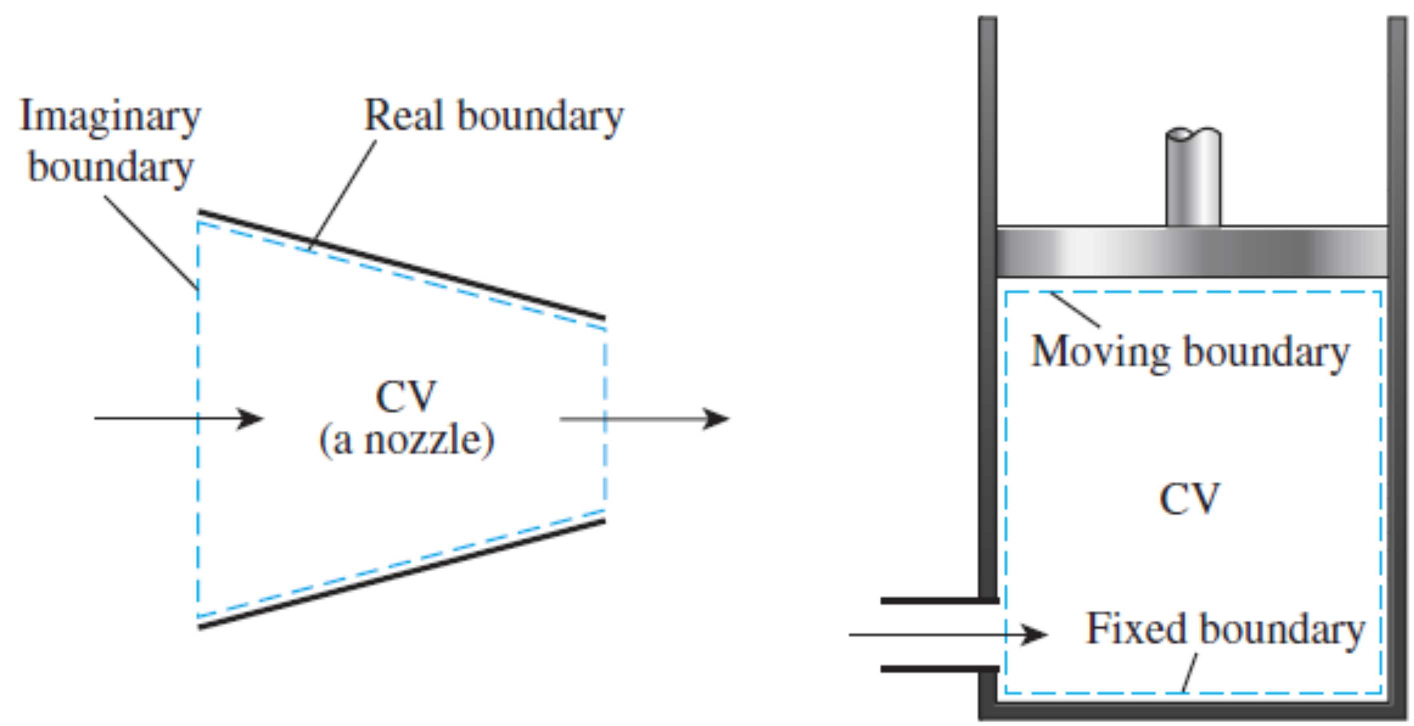


FIGURE 2–5

An open system (a control volume) with one inlet and one exit.



(*a*) A control volume with real and imaginary boundaries

(*b*) A control volume with fixed and moving boundaries

FIGURE 2–4

A control volume can involve fixed, moving, real, and imaginary boundaries.

Properties of a System

- Property: any characteristic of a system.
  - Some familiar properties are pressure P, temperature T, volume V, and mass m.
- Properties are considered to be either intensive or extensive.
  - Intensive properties: those that are independent of the mass of a system, such as temperature, pressure, and density.
  - Extensive properties: Those whose values depend on the size, or extent, of the system.
    - Specific properties: Extensive properties per unit mass.

$(v = V/m)$ $(e = E/m)$

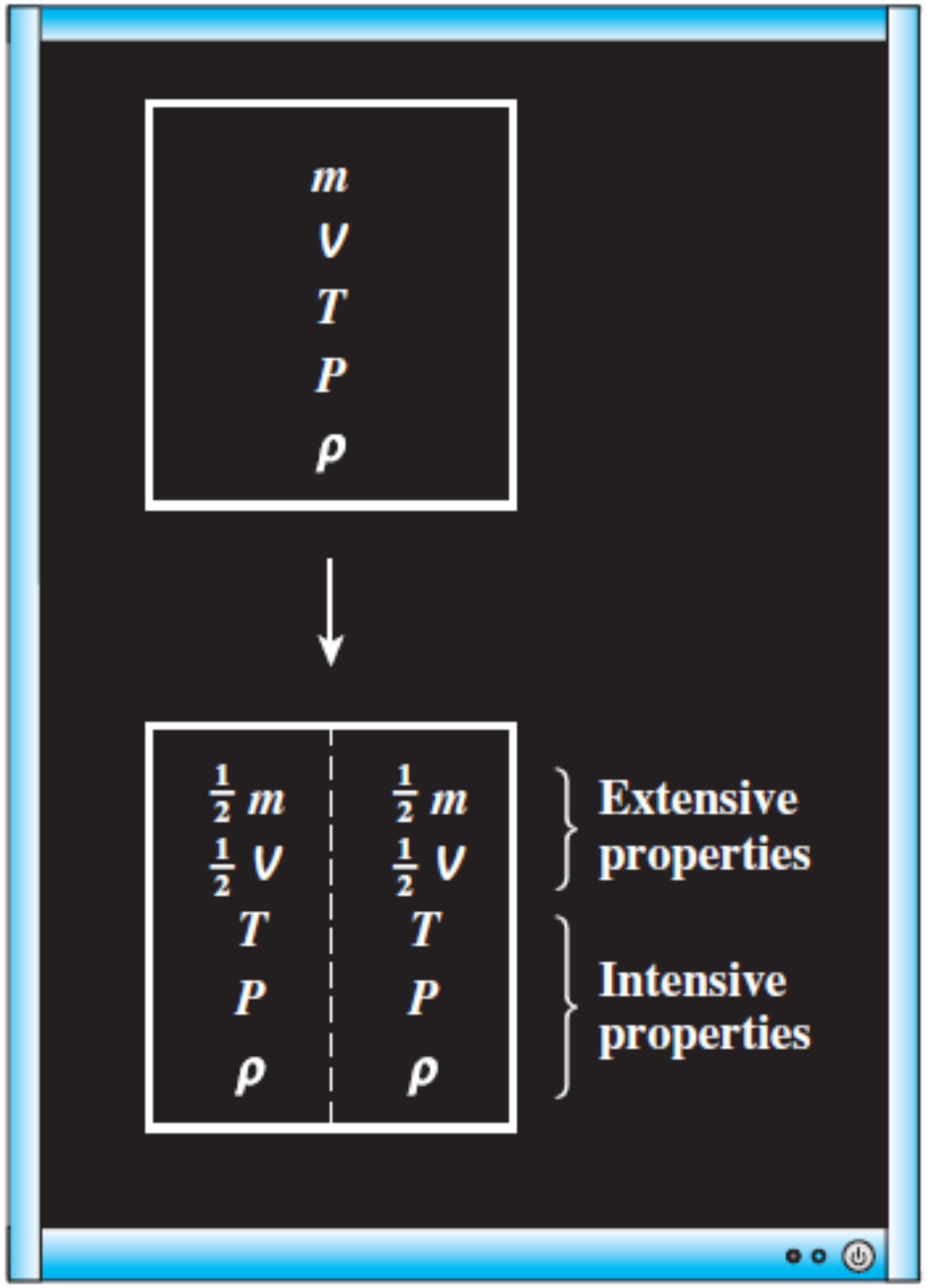


FIGURE 2–6

Criterion to differentiate intensive and extensive properties.

Lagrangian Description

The two ways to describe motion are Lagrangian and Eulerian description.

- Lagrangian description of fluid flow tracks the position and velocity of individual particles (eg. Billiard ball on a pooltable).
  - Motion is described based upon Newton's laws.
  - Difficult to use for practical flow analysis.
    - Fluids are composed of *billions* of molecules.
    - Interaction between molecules hard to describe/model.
  - However, useful for specialized applications
    - Sprays, particles, bubble dynamics, rarefied gases.
    - Coupled Eulerian-Lagrangian methods.
  - Named after Italian mathematician Joseph Louis Lagrange (1736-1813).

Eulerian Description

- Eulerian description of fluid flow: a flow domain or control volume is defined by which fluid flows in and out.
- We define field variables which are functions of space and time.
  - Pressure field, P=P(x,y,z,t)
  - Velocity field, $\vec{V} = \vec{V}(x, y, z, t)$

$$\vec{V} = u(x, y, z, t)\vec{i} + v(x, y, z, t)\vec{j} + w(x, y, z, t)\vec{k}$$

  - Acceleration field, $\vec{a} = \vec{a}(x, y, z, t)$

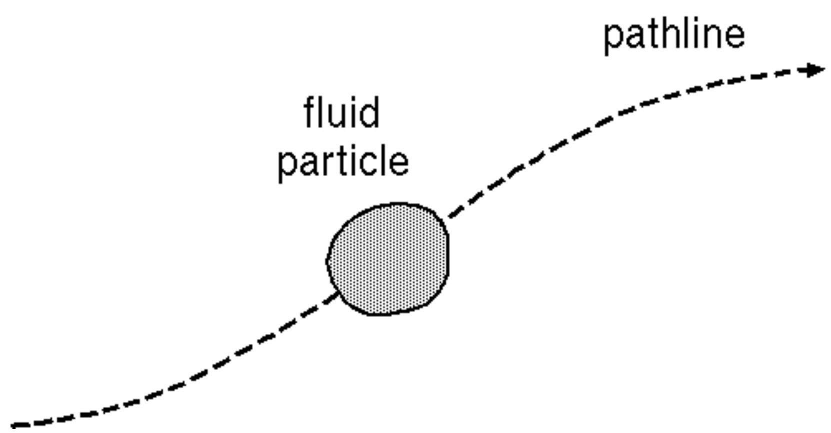
$$\vec{a} = a_x(x, y, z, t)\vec{i} + a_y(x, y, z, t)\vec{j} + a_z(x, y, z, t)\vec{k}$$

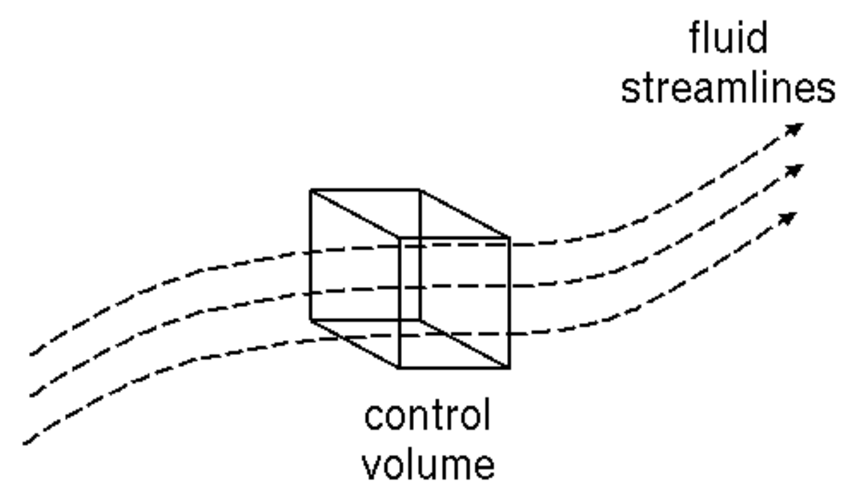
  - These (and other) field variables define the flow field.
  - Well suited for formulation of initial boundary-value problems (PDE's).
  - Named after Swiss mathematician Leonhard Euler (1707-1783).

Lagrangian vs. Eulerian description

A fluid flow field can be thought of as being comprised of a large number of finite sized fluid particles which have mass, momentum, internal energy, and other properties. Mathematical laws can then be written for each fluid particle. This is the Lagrangian description of fluid motion.

Another view of fluid motion is the Eulerian description. In the Eulerian description of fluid motion, we consider how flow properties change at a fluid element that is fixed in space and time (x,y,z,t), rather than following individual fluid particles.





Governing equations can be derived using each method and converted to the other form.

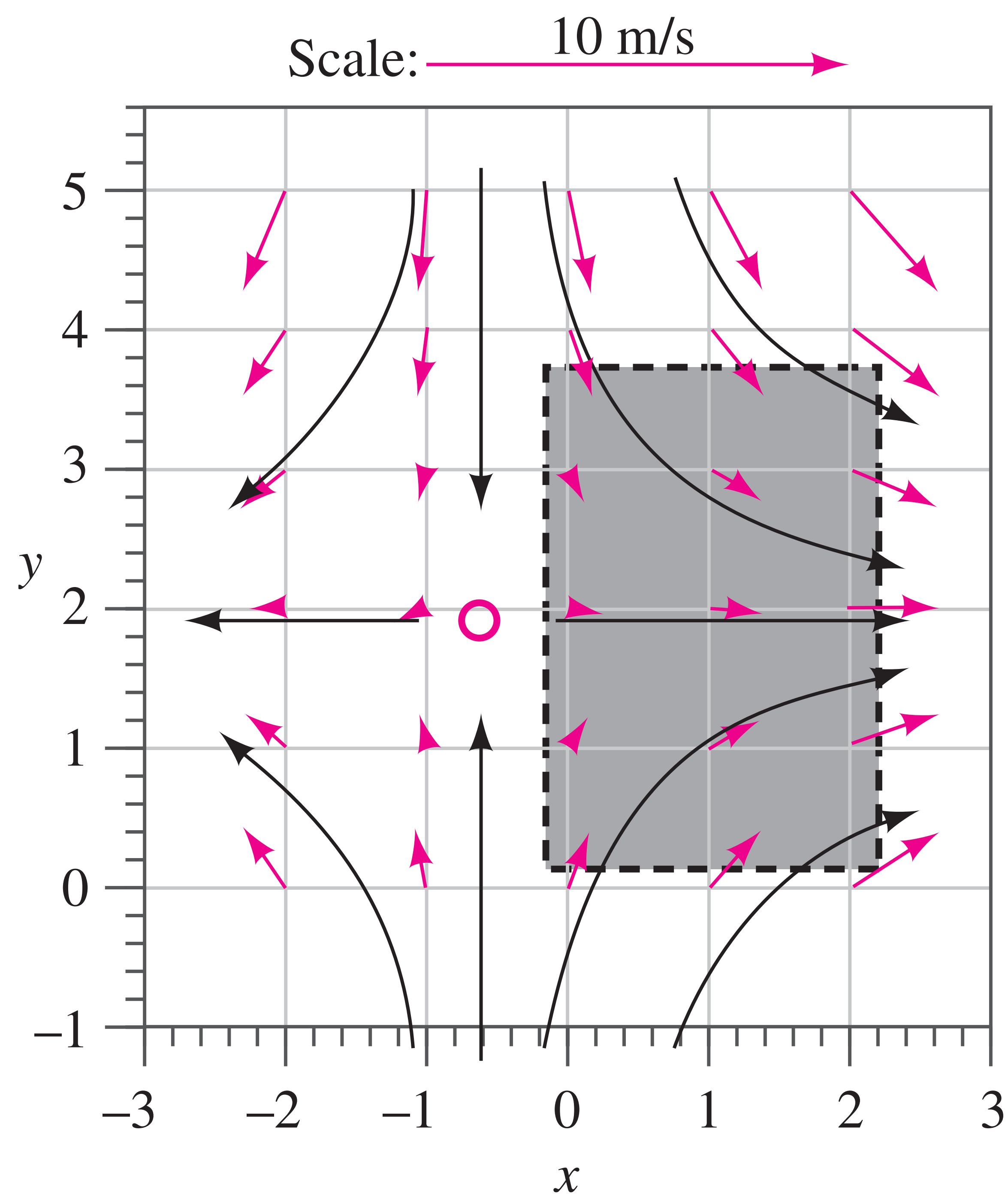
A Steady Two-Dimensional Velocity Field

- A steady, incompressible, two-dimensional velocity field is given by:

$$\vec{V} = (u, v) = (0.5 + 0.8x)\,\vec{i} + (1.5 - 0.8y)\,\vec{j}$$

A *stagnation point* is defined as *a point in the flow field where the velocity is identically zero.*

(*a*) Determine if there are any stagnation points in this flow field and, if so, where?
(*b*) Sketch velocity vectors at several locations in the domain between x = - 2 m to 2 m and y = 0 m to 5 m; qualitatively describe the flow field.



Acceleration Field

- Consider a fluid particle and Newton's second law,

$$\vec{F}_{particle} = m_{particle}\vec{a}_{particle}$$

- The acceleration of the particle is the time derivative of the particle's velocity.

$$\vec{a}_{particle} = \frac{d\vec{V}_{particle}}{dt}$$

- However, particle velocity at a point at any instant in time t is the same as the fluid velocity,

$$\vec{V}_{particle} = \vec{V}\left(x_{particle}(t), y_{particle}(t), z_{particle}(t), t\right)$$

- To take the time derivative of, chain rule must be used.

$$\vec{a}_{particle} = \frac{\partial \vec{V}}{\partial t}\frac{dt}{dt} + \frac{\partial \vec{V}}{\partial x}\frac{dx_{particle}}{dt} + \frac{\partial \vec{V}}{\partial y}\frac{dy_{particle}}{dt} + \frac{\partial \vec{V}}{\partial z}\frac{dz_{particle}}{dt}$$

Where ∂ is the partial derivative operator and
d is the total derivative operator.

Acceleration Field

- Since $$\frac{dx_{particle}}{dt} = u, \frac{dy_{particle}}{dt} = v, \frac{dz_{particle}}{dt} = w$$

$$\vec{a}_{particle} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

- In vector form, the acceleration can be written as

$$\vec{a}(x,y,z,t) = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \bullet \vec{\nabla}\right)\vec{V}$$

- First term is called the local acceleration and is nonzero only for unsteady flows.
- Second term is called the advective acceleration and accounts for the effect of the fluid particle moving to a new location in the flow, where the velocity is different.

If we move a parcel in time Δt

Using Taylor series expansion, assuming increments over Δt are small, and ignoring Higher Order Terms

$$\Delta f = \frac{\partial f}{\partial t}\Delta t + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z + \boxed{\text{Higher Order Terms}}$$

Dividing by Δt and taking the small limit:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

Introducing the convention of d( )/dt ≡ D( )/Dt

$$\frac{Dx}{Dt} = u, \frac{Dy}{Dt} = v, \frac{Dz}{Dt} = w$$

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z}$$

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{V}\cdot\nabla(f)$$

Advection

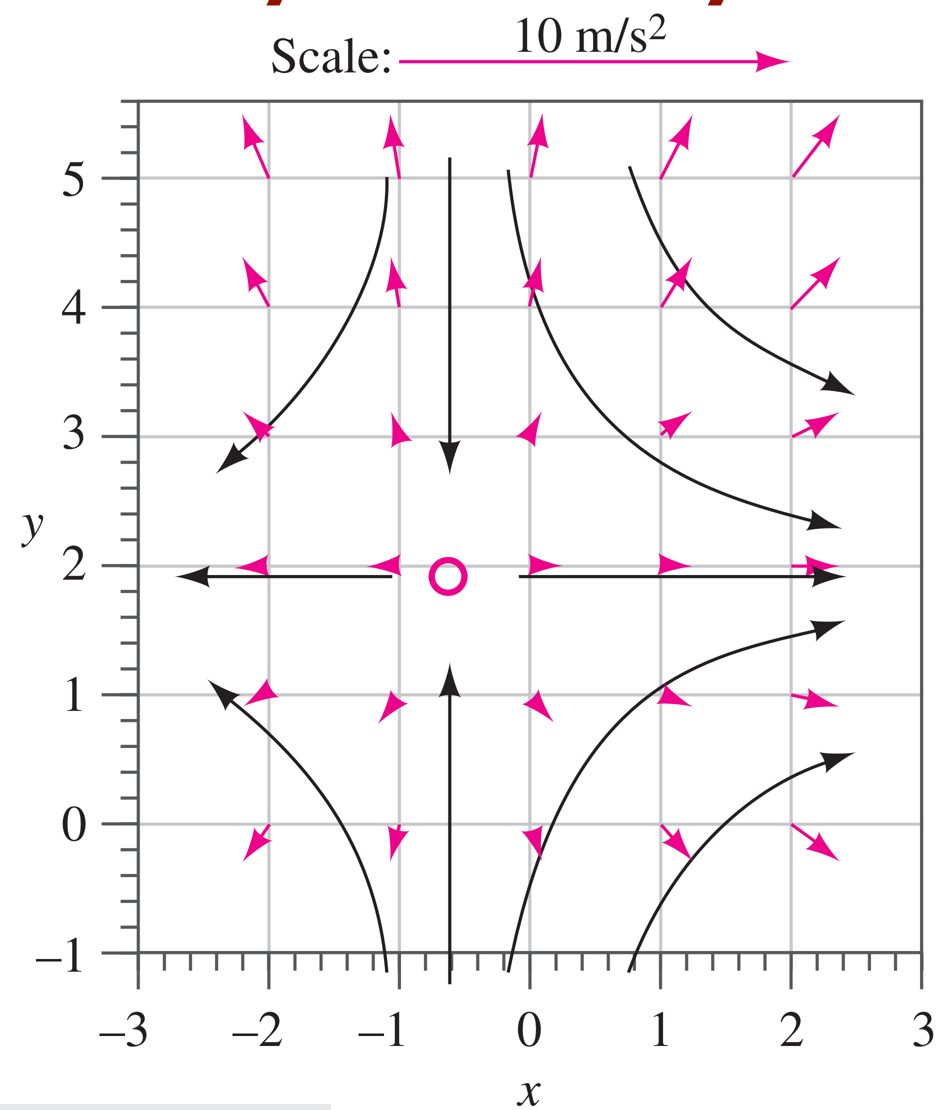
- In mathematics and continuum mechanics, including fluid dynamics, the substantive derivative (sometimes the Lagrangian derivative, material derivative), written D/Dt, is the rate of change of some property of a small parcel of fluid.
  - Note that if the fluid is moving, the substantive derivative is the rate of change of fluid within the small parcel, hence the other names like fluid following derivative.

- Advection is transport of a some conserved scalar quantity in a vector field.
  - Advective acceleration is nonlinear: source of many phenomenon and primary challenge in solving fluid flow problems.
  - Provides "transformation" between Lagrangian and Eulerian frames.

$$u\frac{\partial f}{\partial x}+v\frac{\partial f}{\partial y}+w\frac{\partial f}{\partial z}=\mathbf{V}\cdot\nabla(f)$$

Material Acceleration of a Steady Velocity Field

- Consider the same velocity field of first example. (*a*) Calculate the material acceleration at the point (x = 2 m, y = 3 m). (*b*) Sketch the material acceleration vectors at the same array of x- and y values as in Example A.



$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$= 0 + (0.5 + 0.8x)(0.8) + (15 - 0.8y)(0) + 0 = (0.4 + 0.64x)\ \mathrm{m/s^2}$$

and

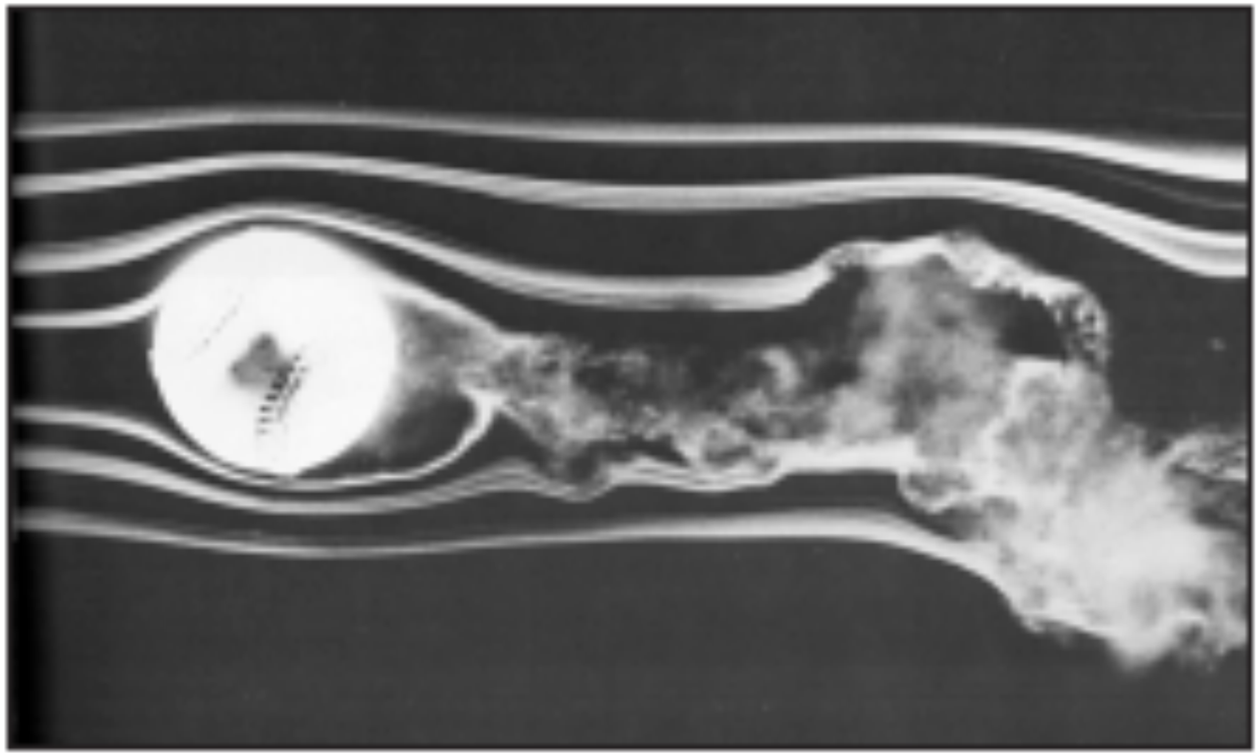
$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$= 0 + (0.5 + 0.8x)(0) + (1.5 - 0.8y)(-0.8) + 0 = (-1.2 + 0.64y)\ \mathrm{m/s^2}$$

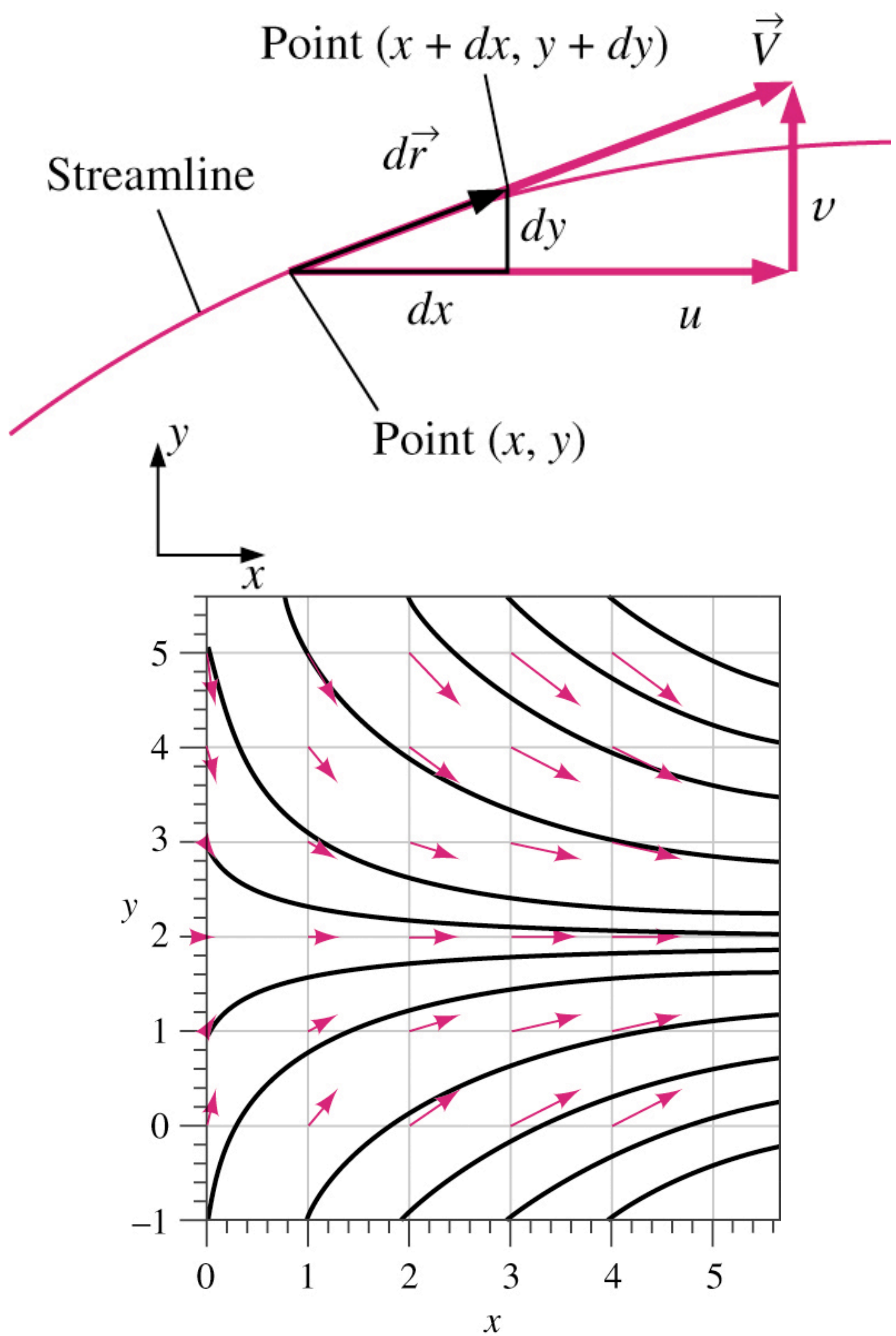
Flow Visualization

- Flow visualization is the visual examination of flow-field features.
- Important for both physical experiments and numerical (CFD) solutions.
- Numerous methods:
  - Streamlines and streamtubes
  - Pathlines
  - Streaklines
  - Timelines
  - Refractive techniques
  - Surface flow techniques

While quantitative study of fluid dynamics requires advanced mathematics, much can be learned from **flow visualization**



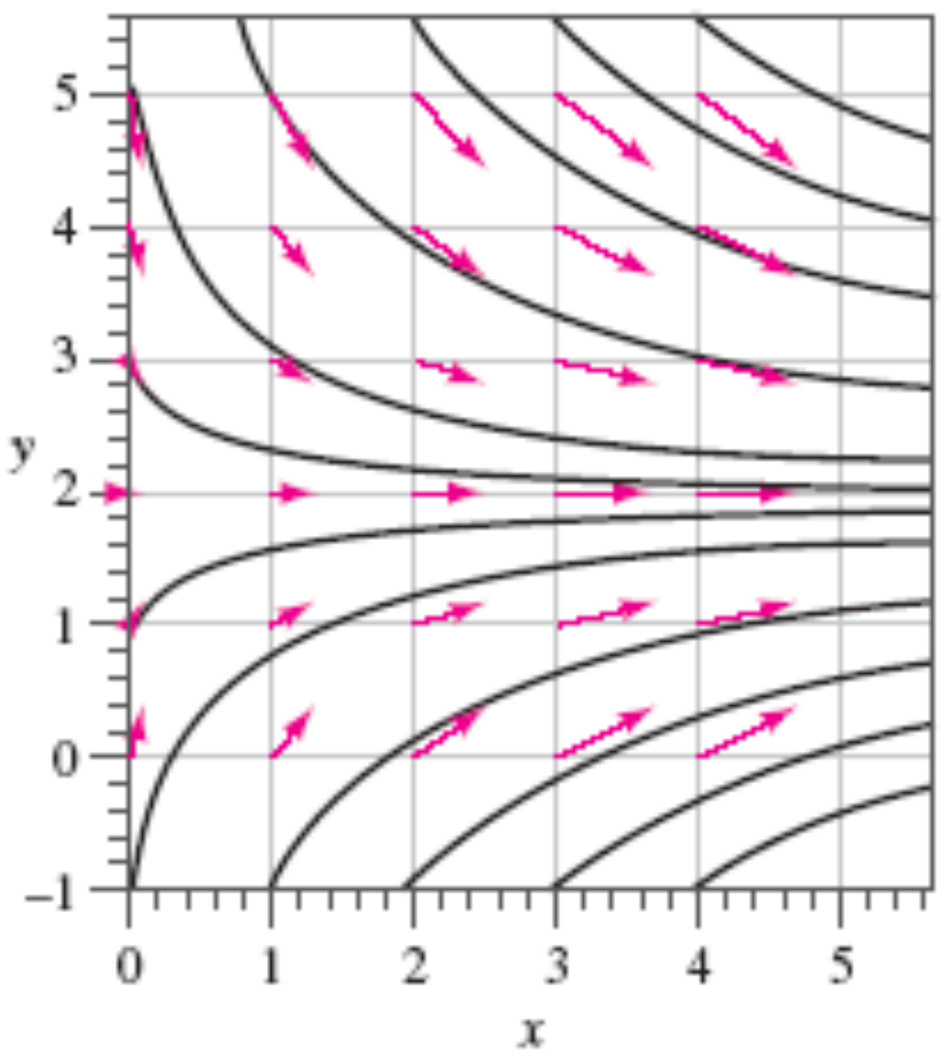
Streamlines



- A Streamline is a curve that is everywhere tangent to the instantaneous local velocity vector.
- Consider an arc length
$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$
- $d\vec{r}$ must be parallel to the local velocity vector
$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$$
- Geometric arguments results in the equation for a streamline
$$\frac{dr}{V} = \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

Streamlines in xy - analytical Solution

For the same velocity field of the example A, plot several streamlines in the right half of the flow ($x > 0$) and compare to the velocity vectors.



$$\frac{dy}{dx} = \frac{1.5 - 0.8y}{0.5 + 0.8x}$$

$$\frac{dy}{1.5 - 0.8y} = \frac{dx}{0.5 + 0.8x}$$

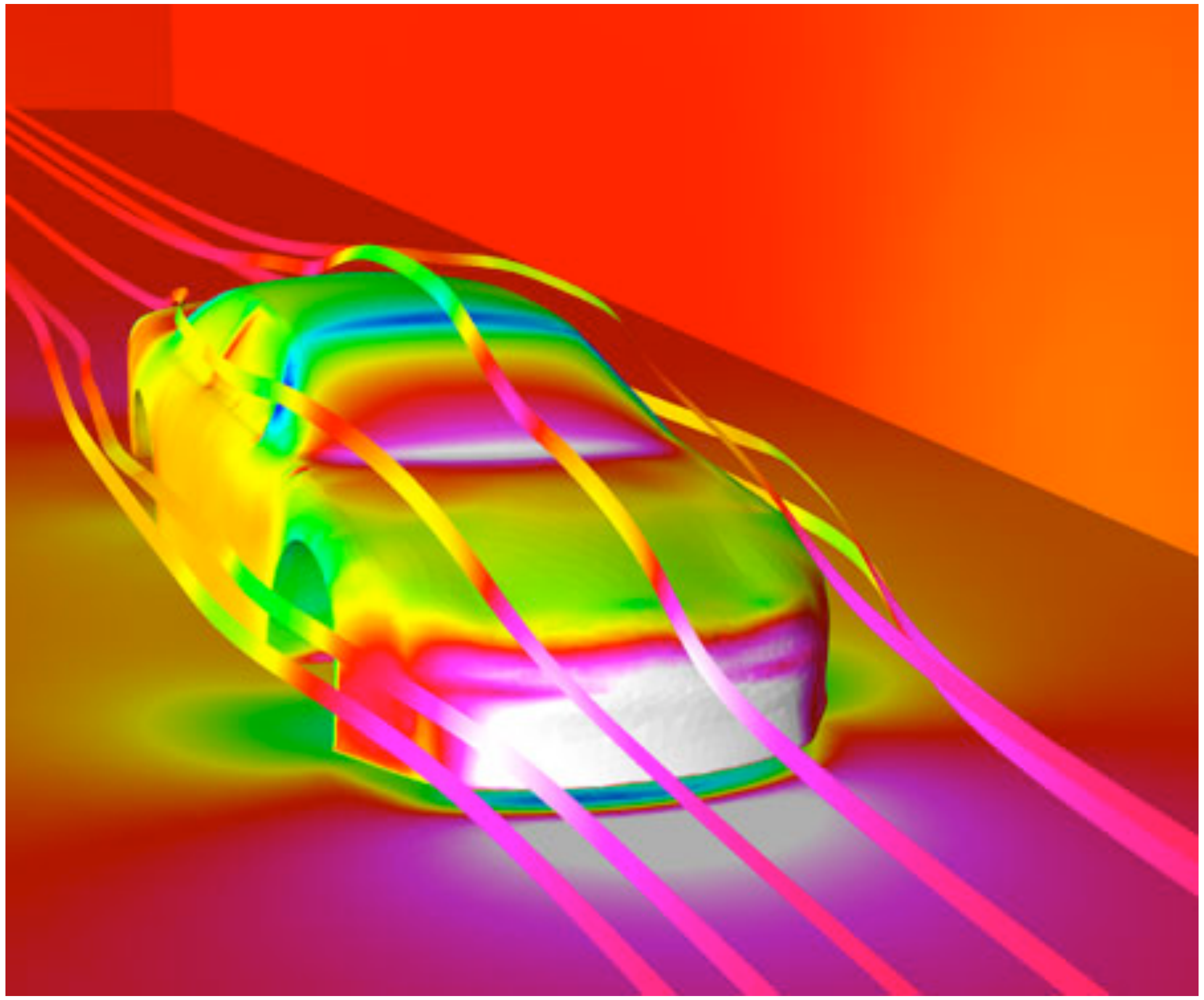
$$\rightarrow \int \frac{dy}{1.5 - 0.8y} = \int \frac{dx}{0.5 + 0.8x}$$

$$y = \frac{C}{0.8(0.5 + 0.8x)} + 1.875$$

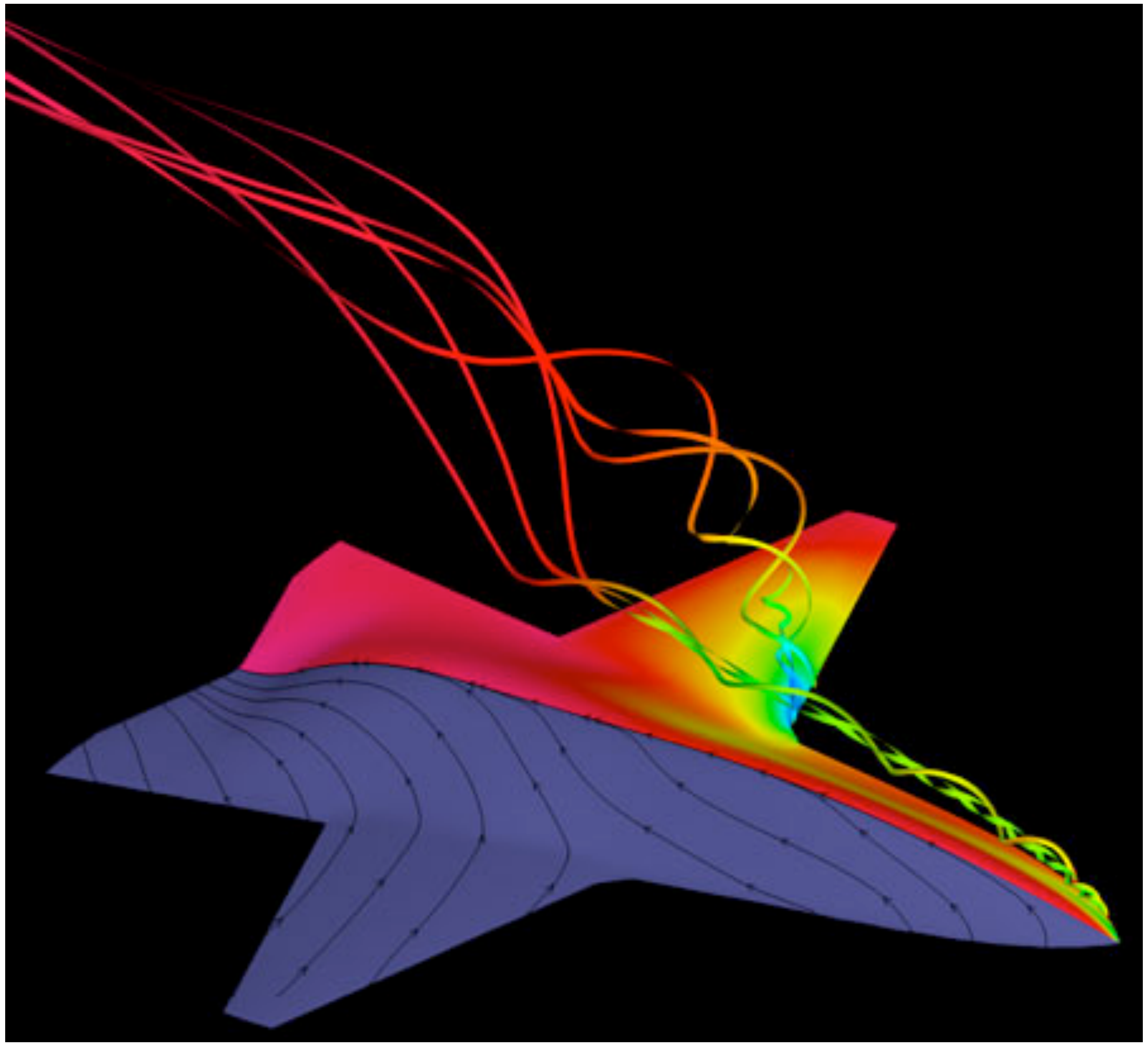
where C is a constant of integration that can be set to various values in order to plot the streamlines.

Streamlines

NASCAR surface pressure contours and streamlines

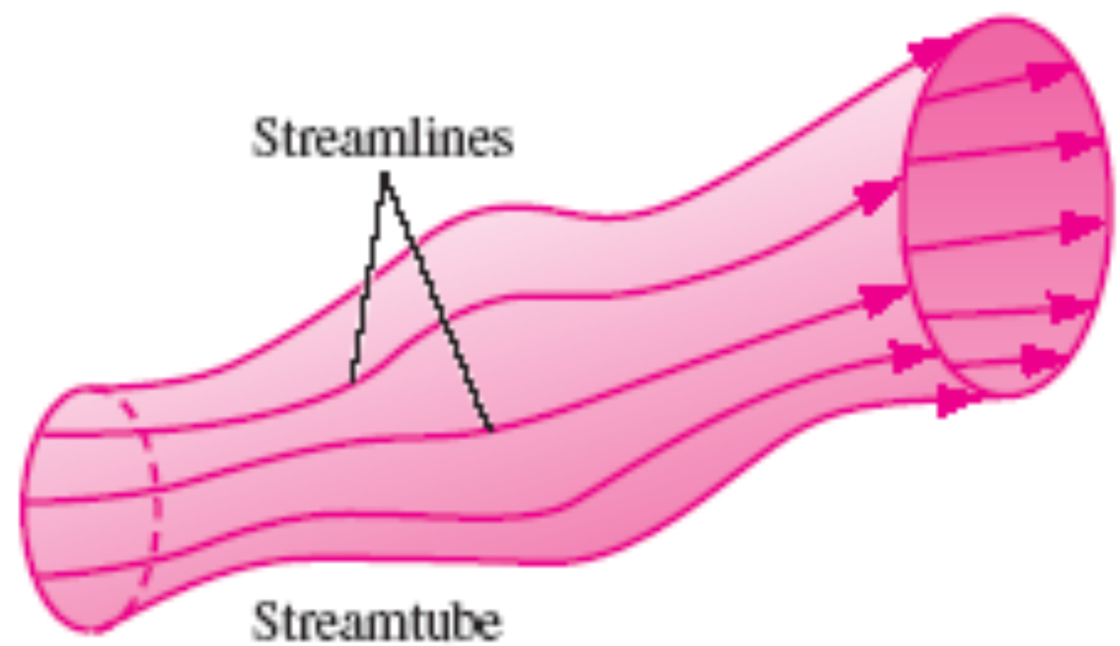


Airplane surface pressure contours, volume streamlines, and surface streamlines

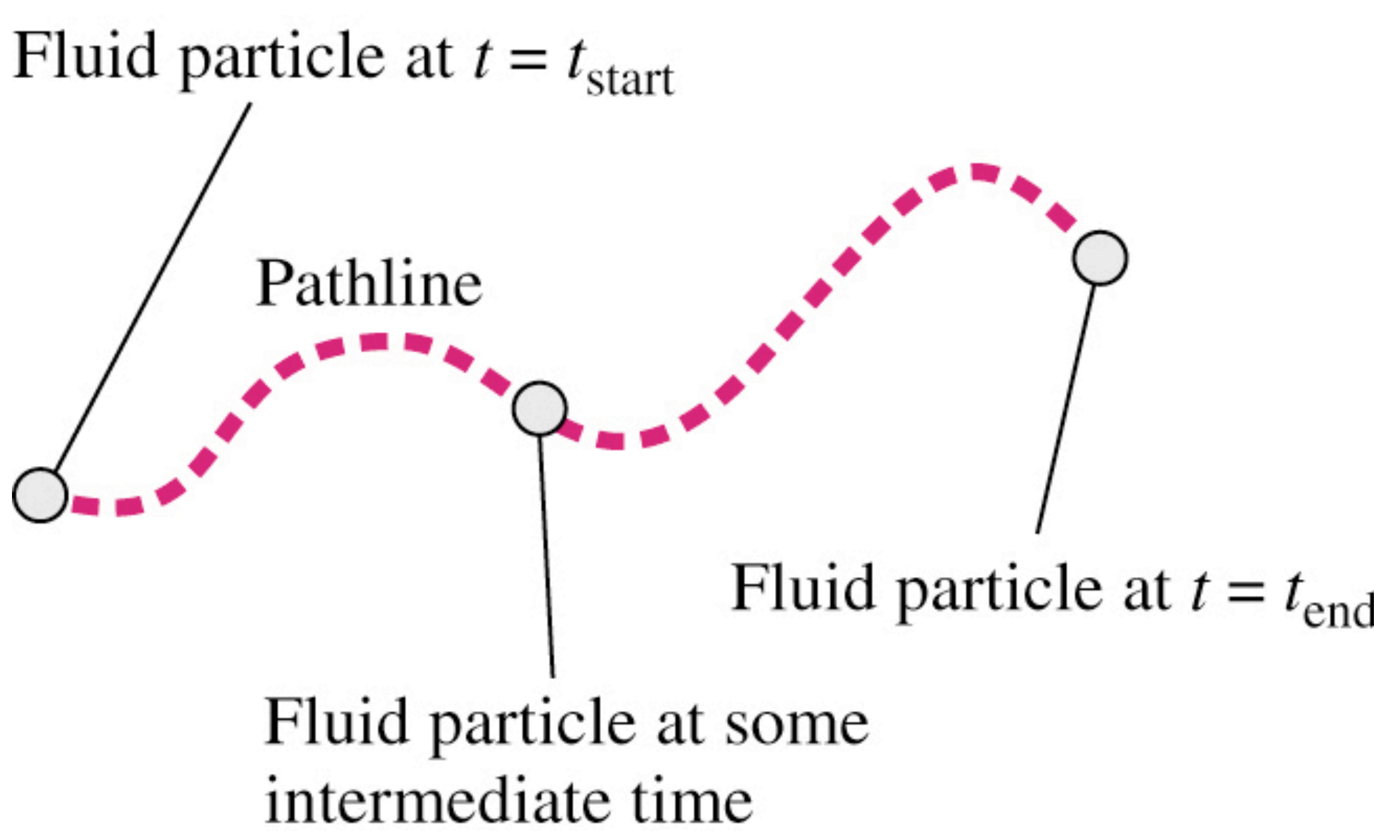


Streamtube

- A streamtube consists of a bundle of streamlines (both are instantaneous quantities).
  - Fluid within a streamtube must remain there and cannot cross the boundary of the streamtube.
  - In an unsteady flow, the streamline pattern may change significantly with time
  - ⇨ the mass flow rate passing through any cross-sectional slice of a given streamtube must remain the same



Pathlines



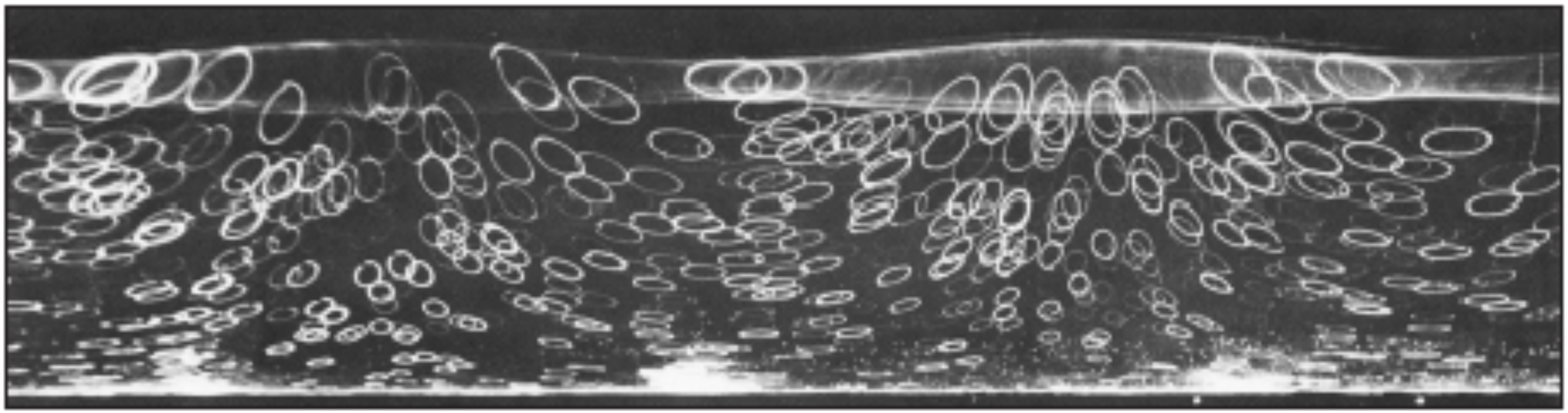
- A Pathline is the actual path traveled by an individual fluid particle over some time period.
- Same as the fluid particle's material position vector

$$\left(x_{particle}(t), y_{particle}(t), z_{particle}(t)\right)$$

- Particle location at time t:

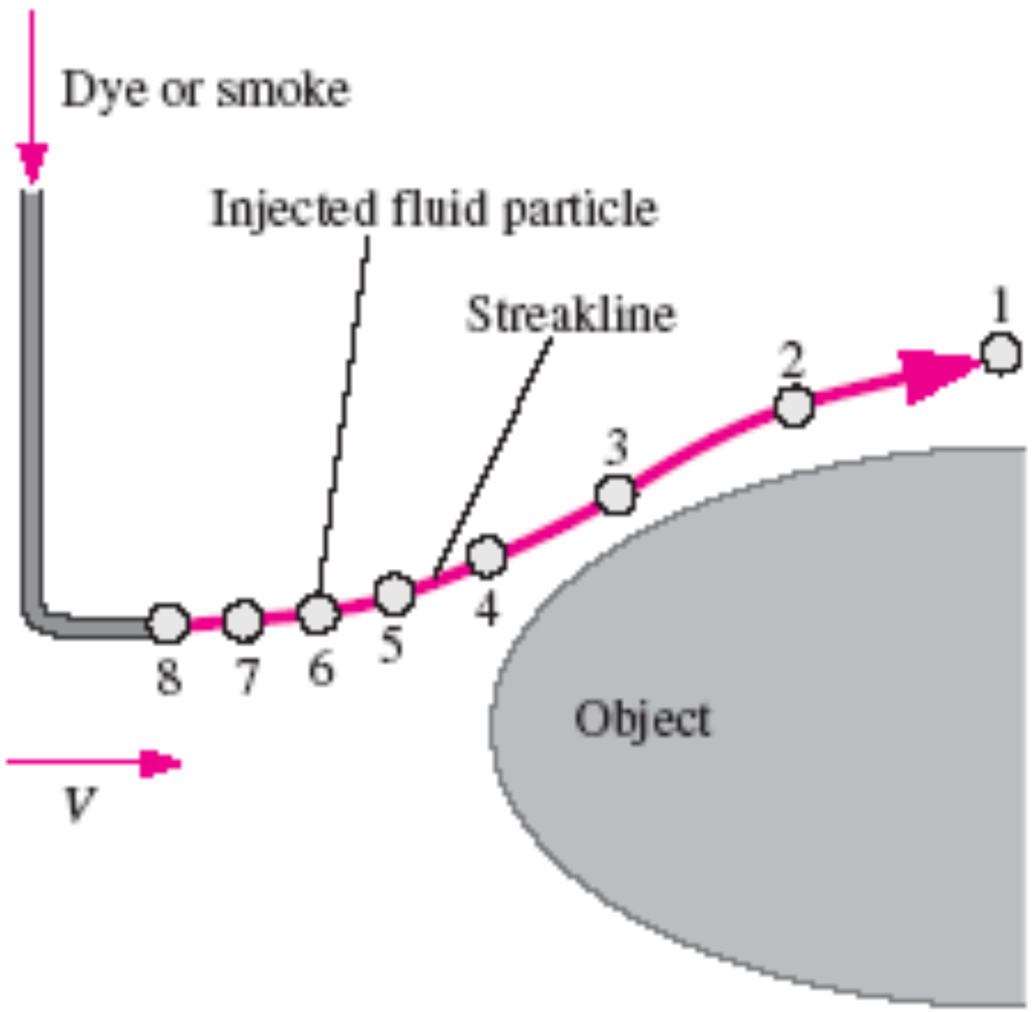
$$\vec{x} = \vec{x}_{start} + \int_{t_{start}}^{t} \vec{V} dt$$

Pathlines



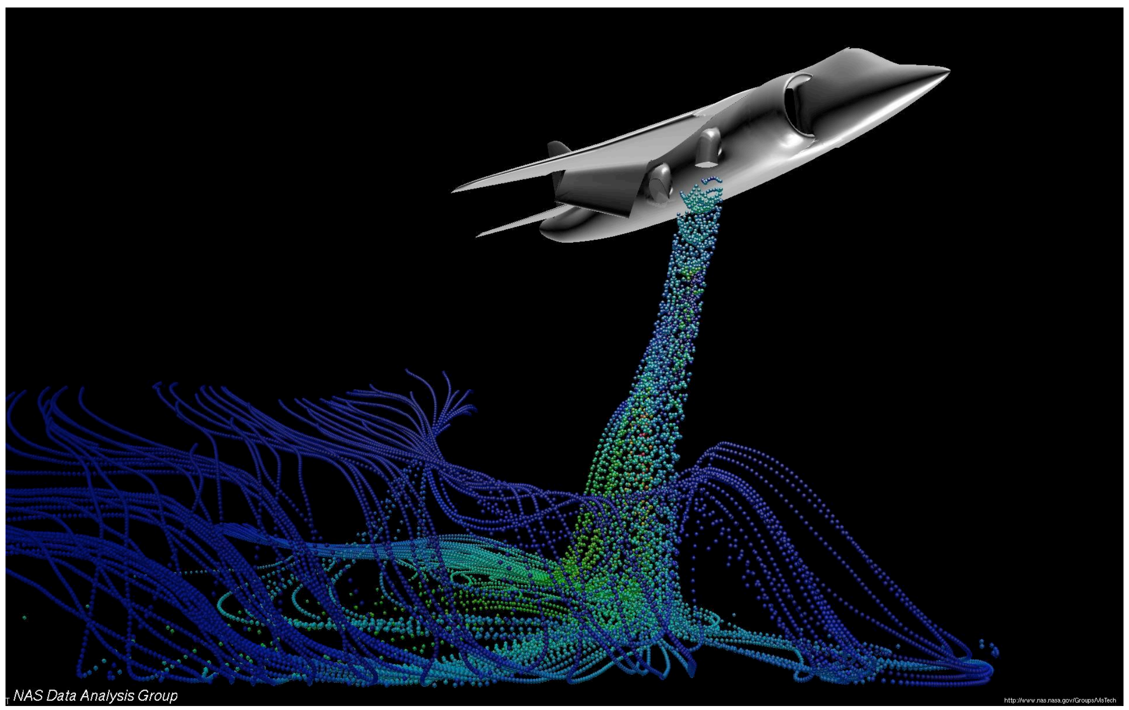
A modern experimental technique called **particle image velocimetry (PIV)** utilizes (tracer) particle pathlines to measure the velocity field over an entire plane in a flow (Adrian, 1991).

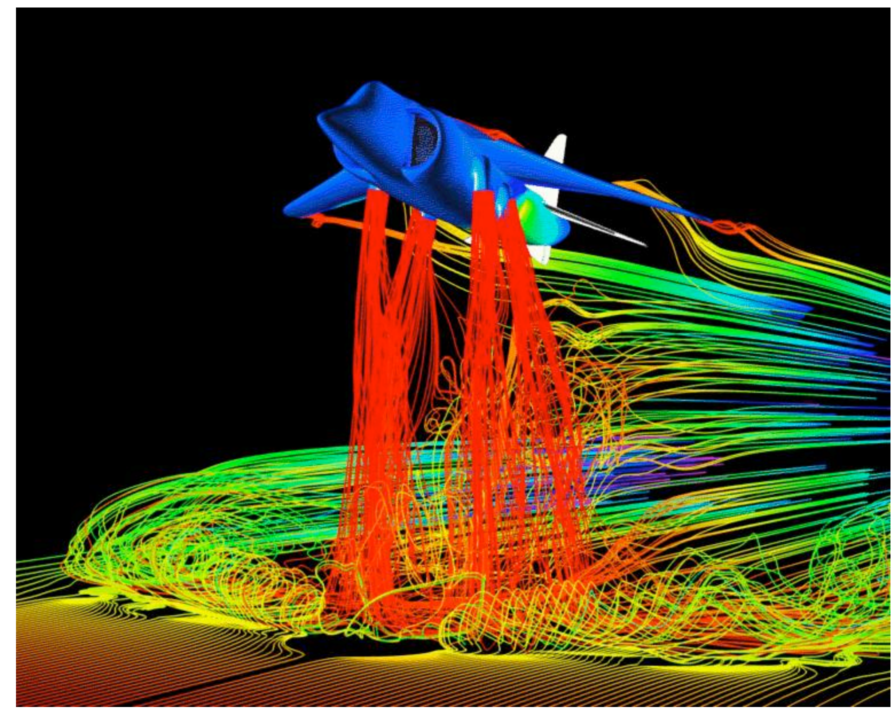
Streaklines



- A Streakline is the locus of fluid particles that have passed sequentially through a prescribed point in the flow.
- Easy to generate in experiments: dye in a water flow, or smoke in an airflow.

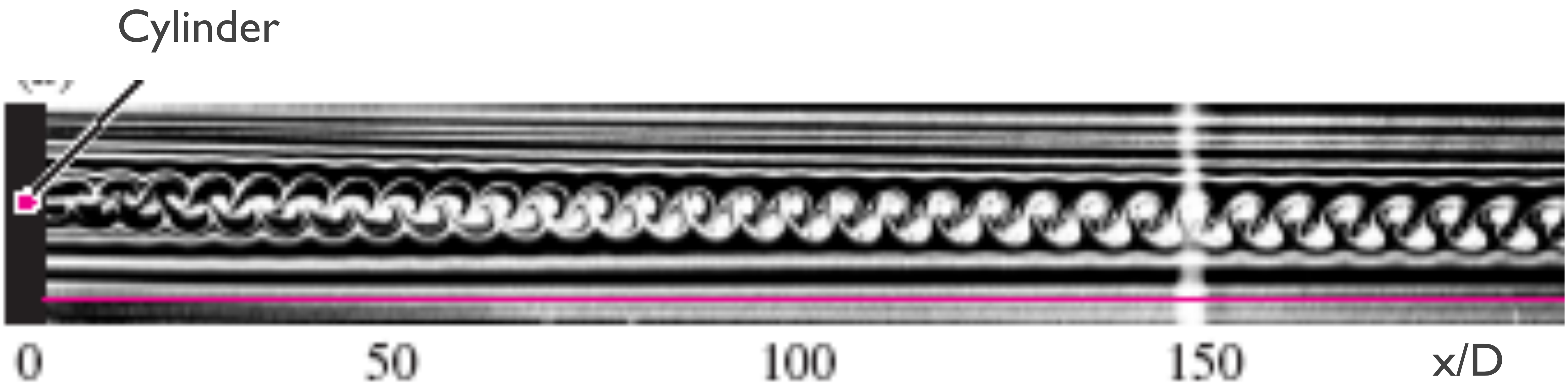
Streaklines





Streaklines

Karman Vortex street

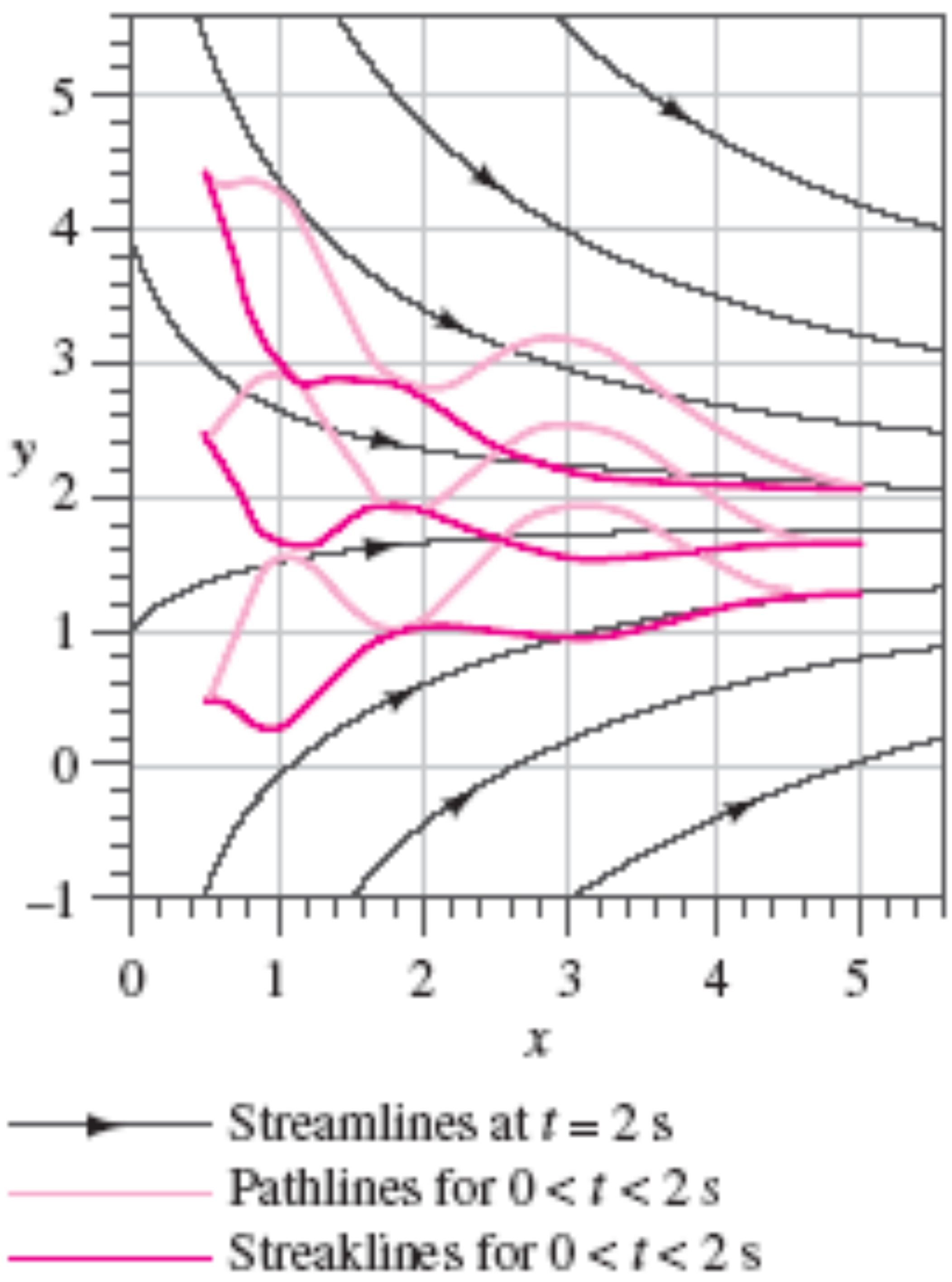


A smoke wire with mineral oil was heated to generate a rake of Streaklines

Comparisons

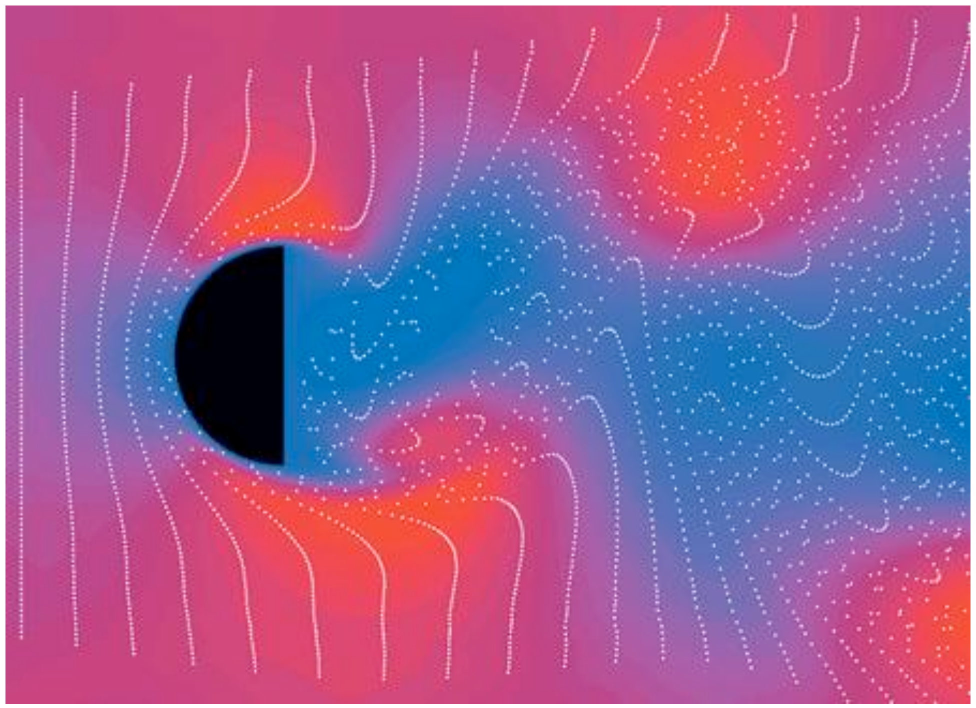
- For steady flow, streamlines, pathlines, and streaklines are identical.
- For unsteady flow, they can be very different.
  - Streamlines are an instantaneous picture of the flow field
  - Pathlines and Streaklines are flow patterns that have a time history associated with them.
  - Streakline: instantaneous snapshot of a time-integrated flow pattern.
  - Pathline: time-exposed flow path of an individual particle.

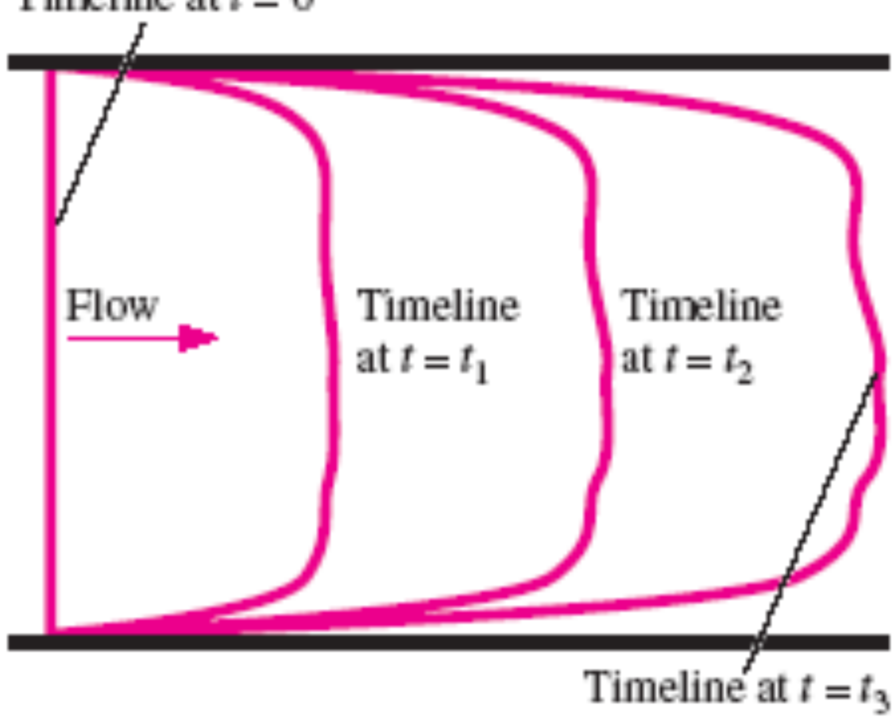
Comparisons



$$\vec{V} = (u, v) = (0.5 + 0.8x)\vec{i} + (1.5 + 2.5\sin(\omega t) - 0.8y)\vec{j}$$

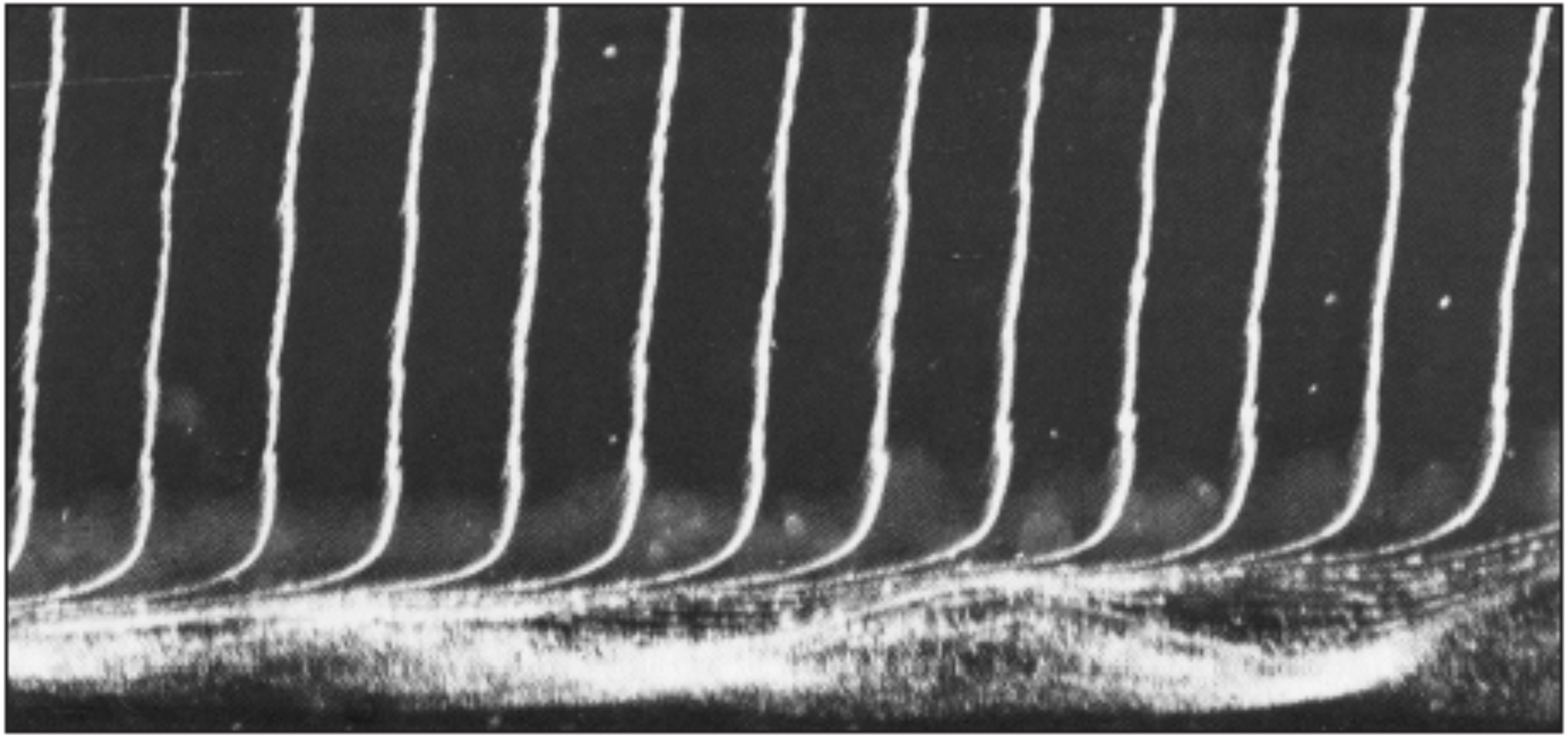
Timelines





- A Timeline is a set of adjacent fluid particles that were marked at the same (earlier) instant in time.
- Timelines can be generated using a hydrogen bubble wire.

Timelines

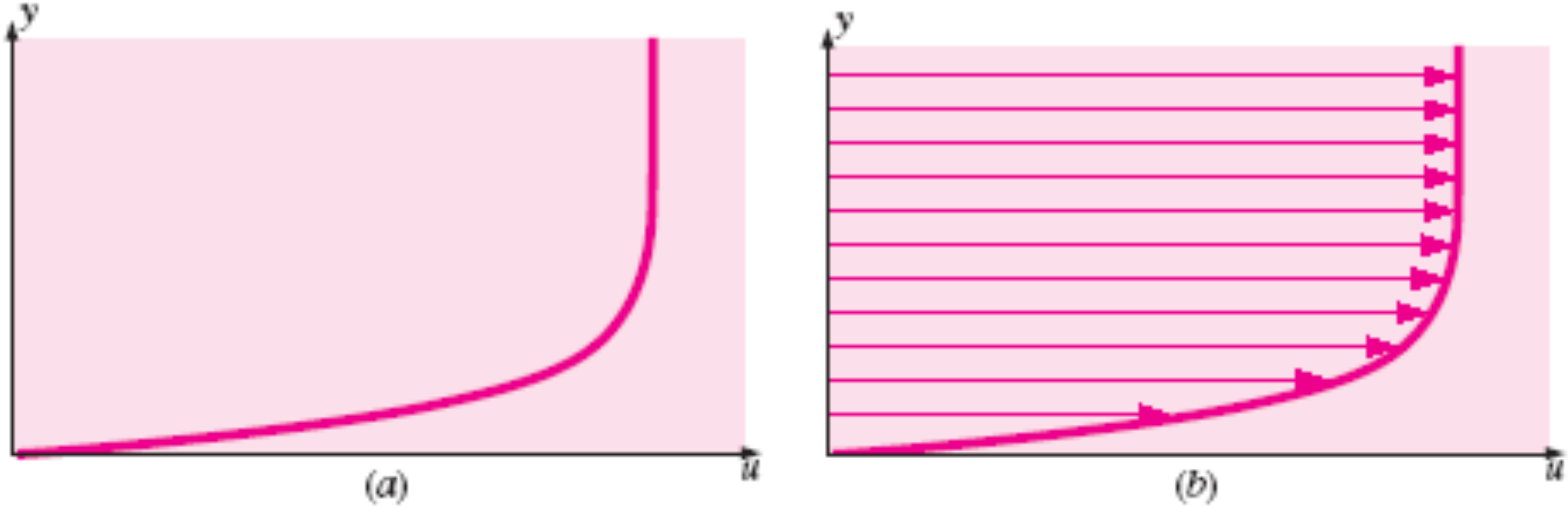


Timelines produced by a hydrogen bubble wire are used to visualize the boundary layer velocity profile shape.

Plots of Flow Data

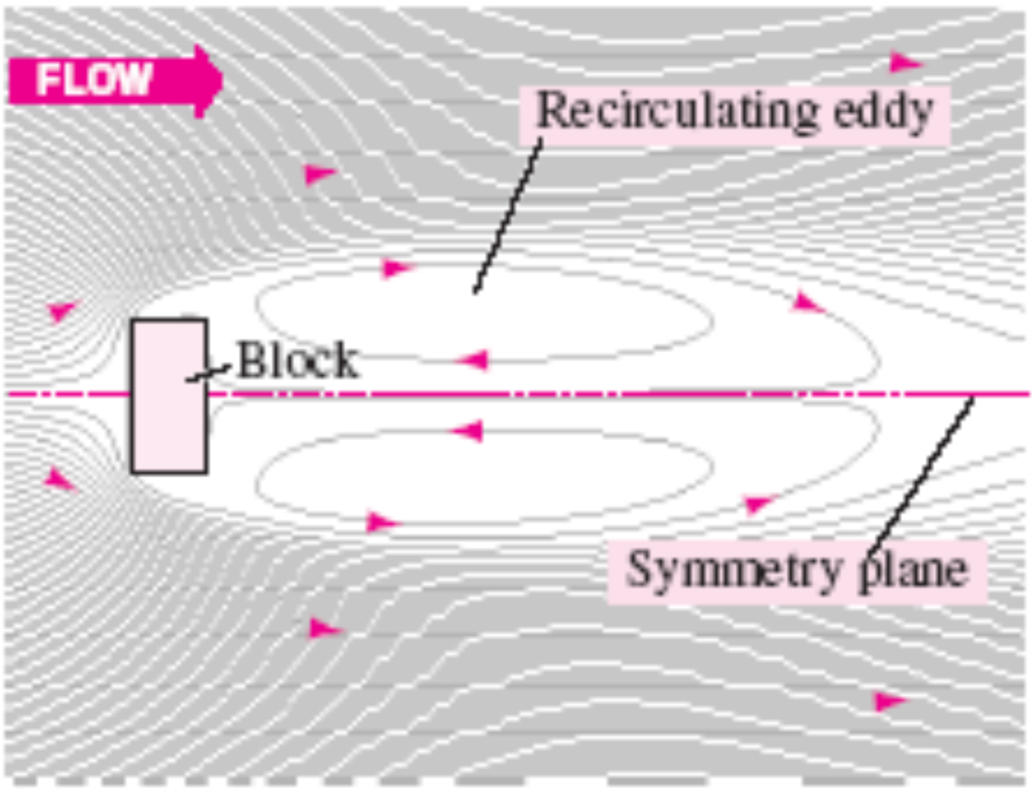
- Flow data are the presentation of the flow properties varying in time and/or space.
- A Profile plot indicates how the value of a scalar property varies along some desired direction in the flow field.
- A Vector plot is an array of arrows indicating the magnitude and direction of a vector property at an instant in time.
- A Contour plot shows curves of constant values of a scalar property for the magnitude of a vector property at an instant in time.

Profile plot

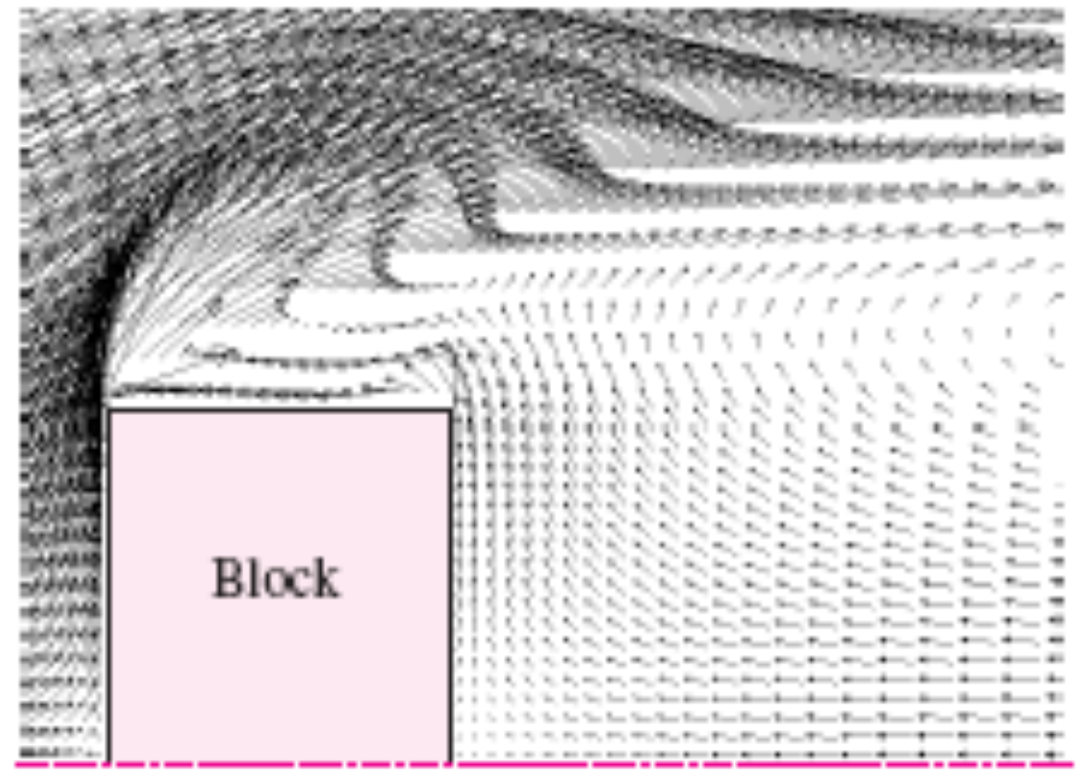


Profile plots of the horizontal component of velocity as a function of vertical distance; flow in the boundary layer growing along a horizontal flat plate.

Vector plot

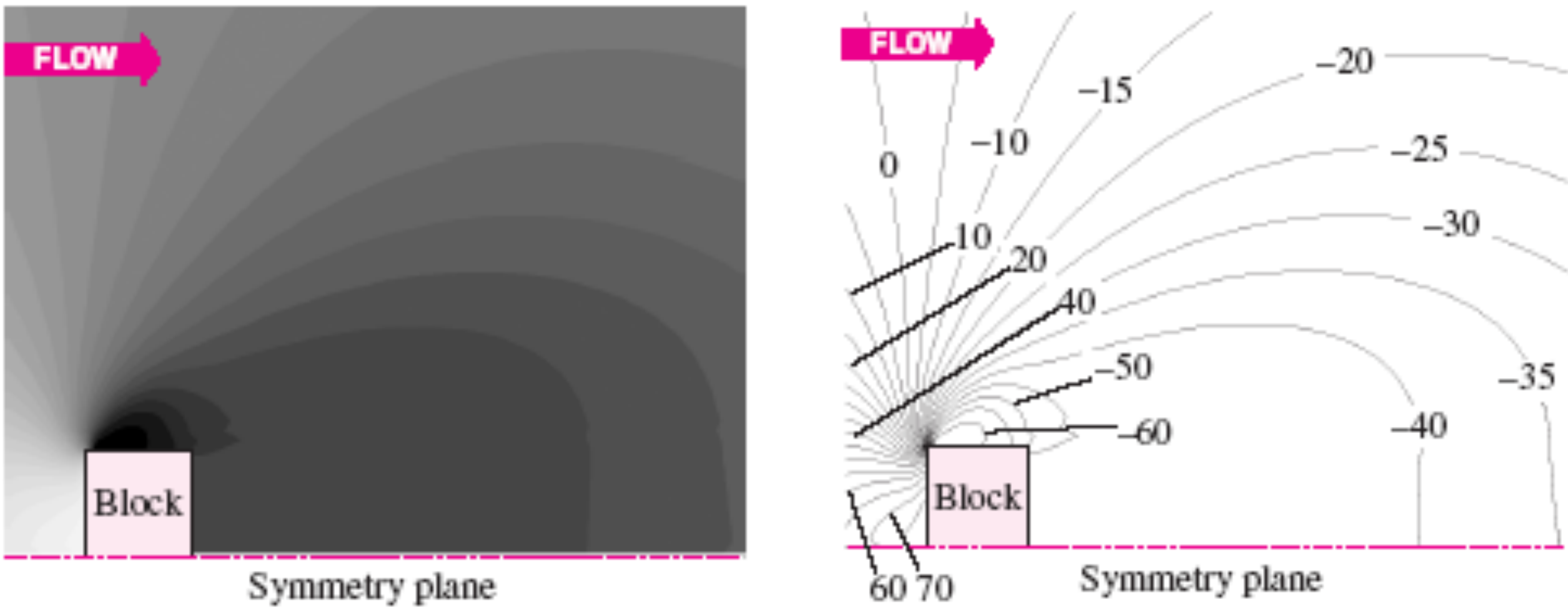


(a)



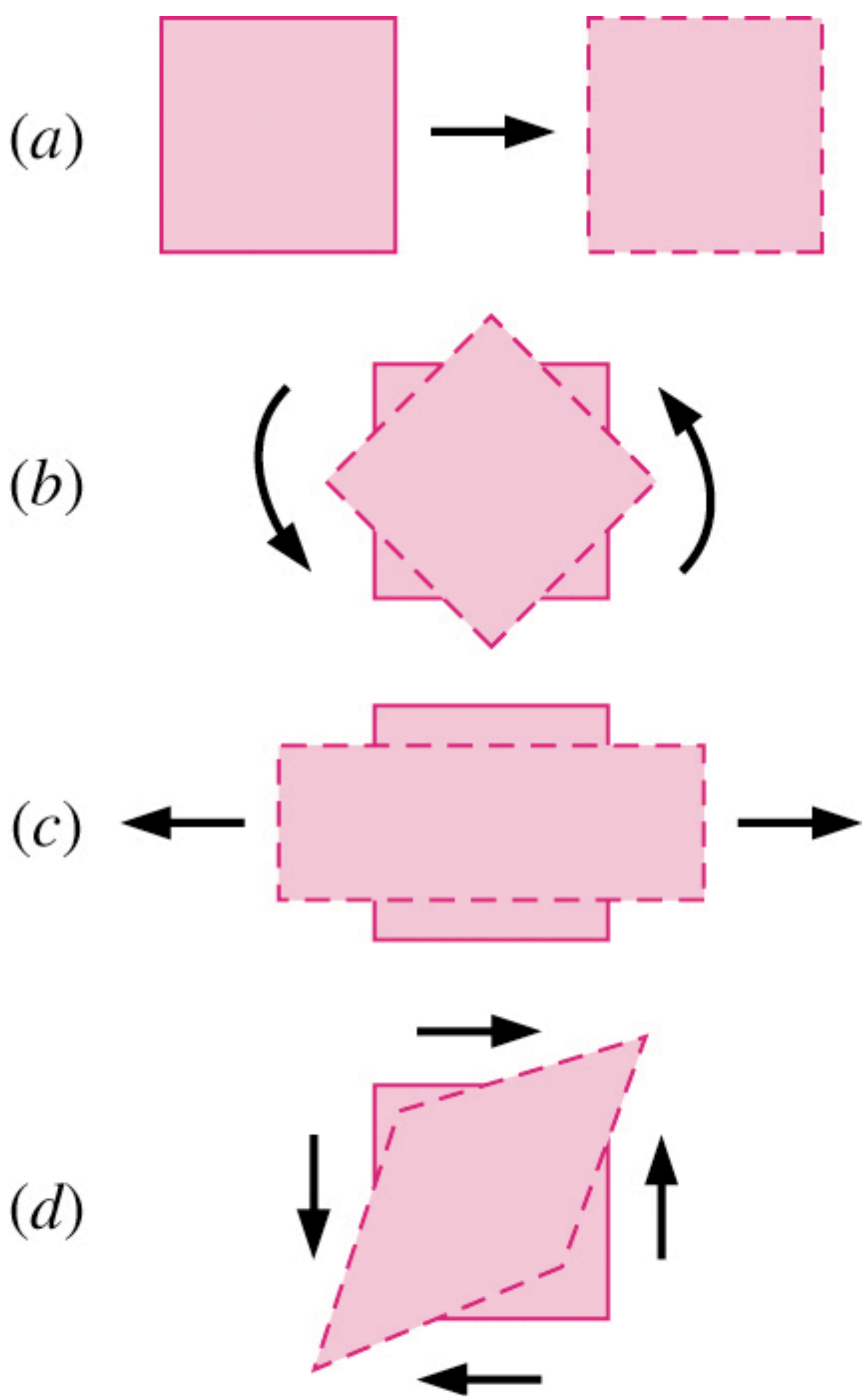
(c)
Symmetry plane

Contour plot



Contour plots of the pressure field due to flow impinging on a block.

Kinematic Description

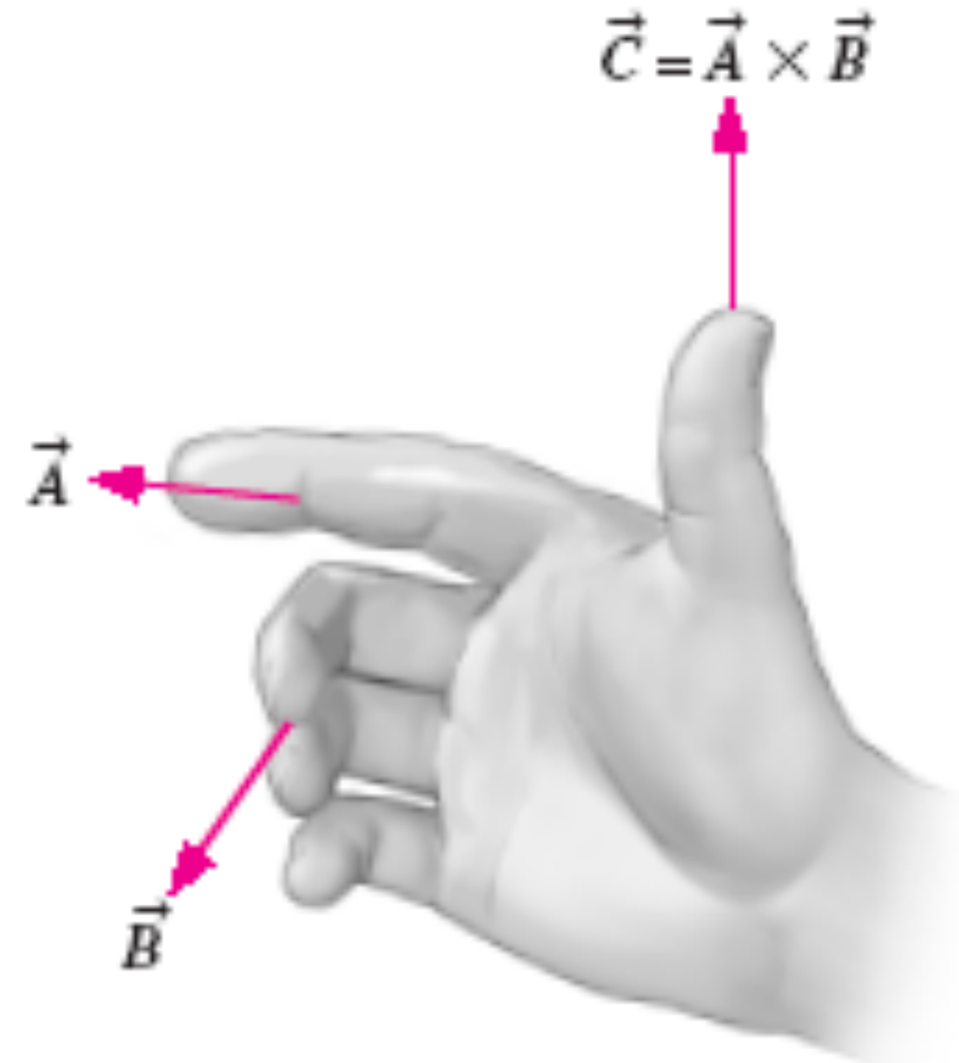


- In fluid dynamics, an element may undergo four fundamental types of motion:
  - Translation
  - Rotation
  - Linear strain
  - Shear strain
- Because fluids are in constant motion, motion and deformation is best described in terms of rates
  - velocity: rate of translation
  - angular velocity: rate of rotation
  - linear strain rate: rate of linear strain
  - shear strain rate: rate of shear strain

Rate of Translation and Rotation

- To be useful, these rates must be expressed in terms of velocity and derivatives of velocity
- The rate of translation vector is described as the velocity vector. In Cartesian coordinates:

$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$$

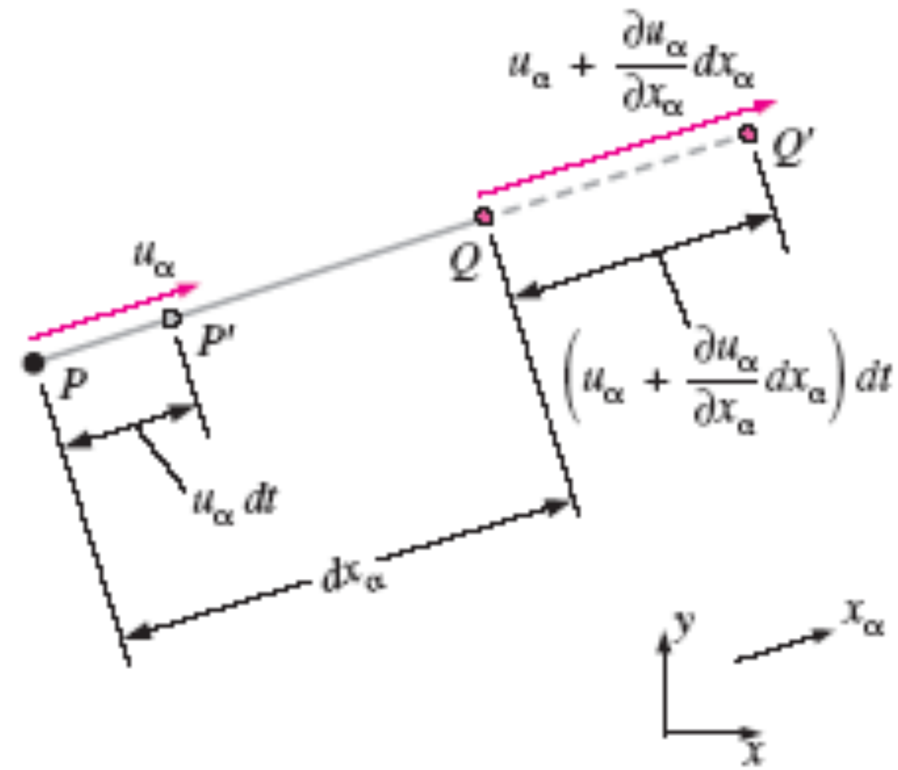


Rule of thumb for rotation

Linear Strain Rate

- Linear Strain Rate is defined as the rate of increase in length per unit length.
- In Cartesian coordinates:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \varepsilon_{yy} = \frac{\partial v}{\partial y}, \varepsilon_{zz} = \frac{\partial w}{\partial z}$$



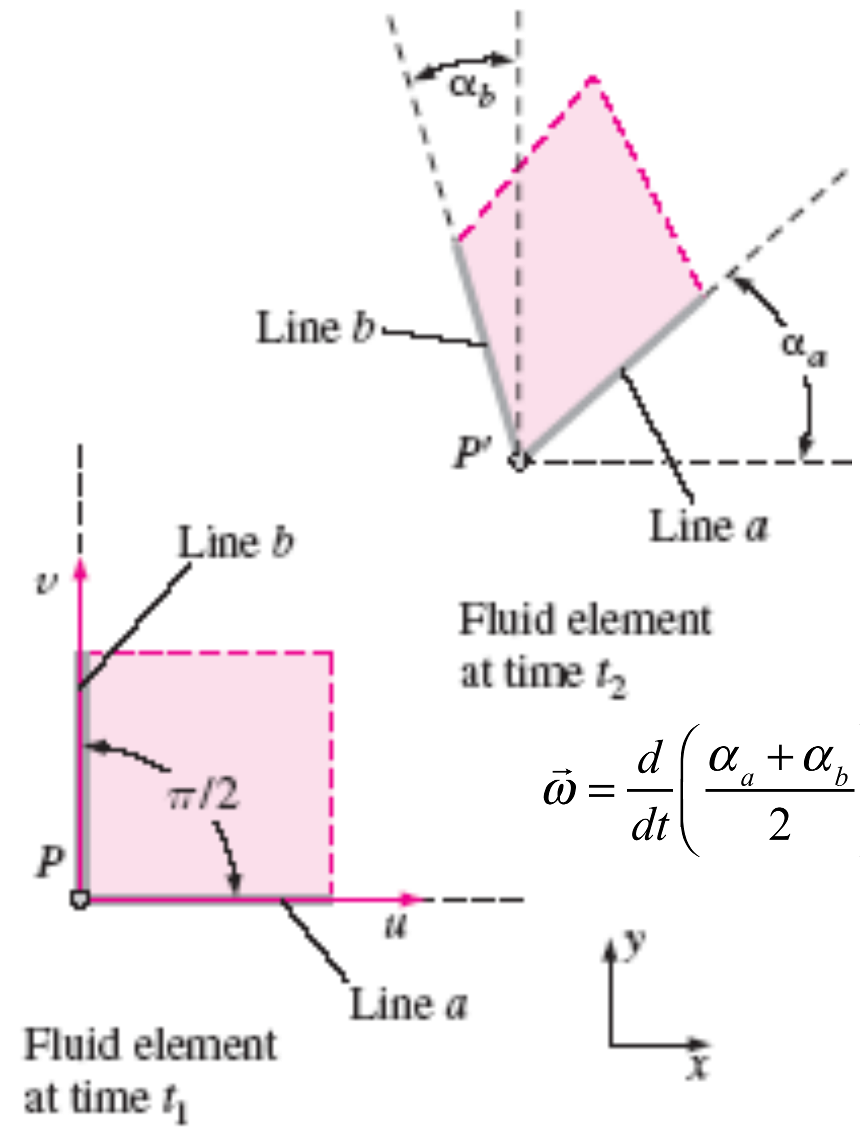
- Volumetric strain rate in Cartesian coordinates

$$\frac{1}{V}\frac{DV}{Dt} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

- Since the volume of a fluid element is constant for an incompressible flow, the volumetric strain rate must be zero.

Rate of Translation and Rotation

- Rate of rotation at a point is defined as the average rotation rate of two initially perpendicular lines that intersect at that point. The rate of rotation vector in Cartesian coordinates:

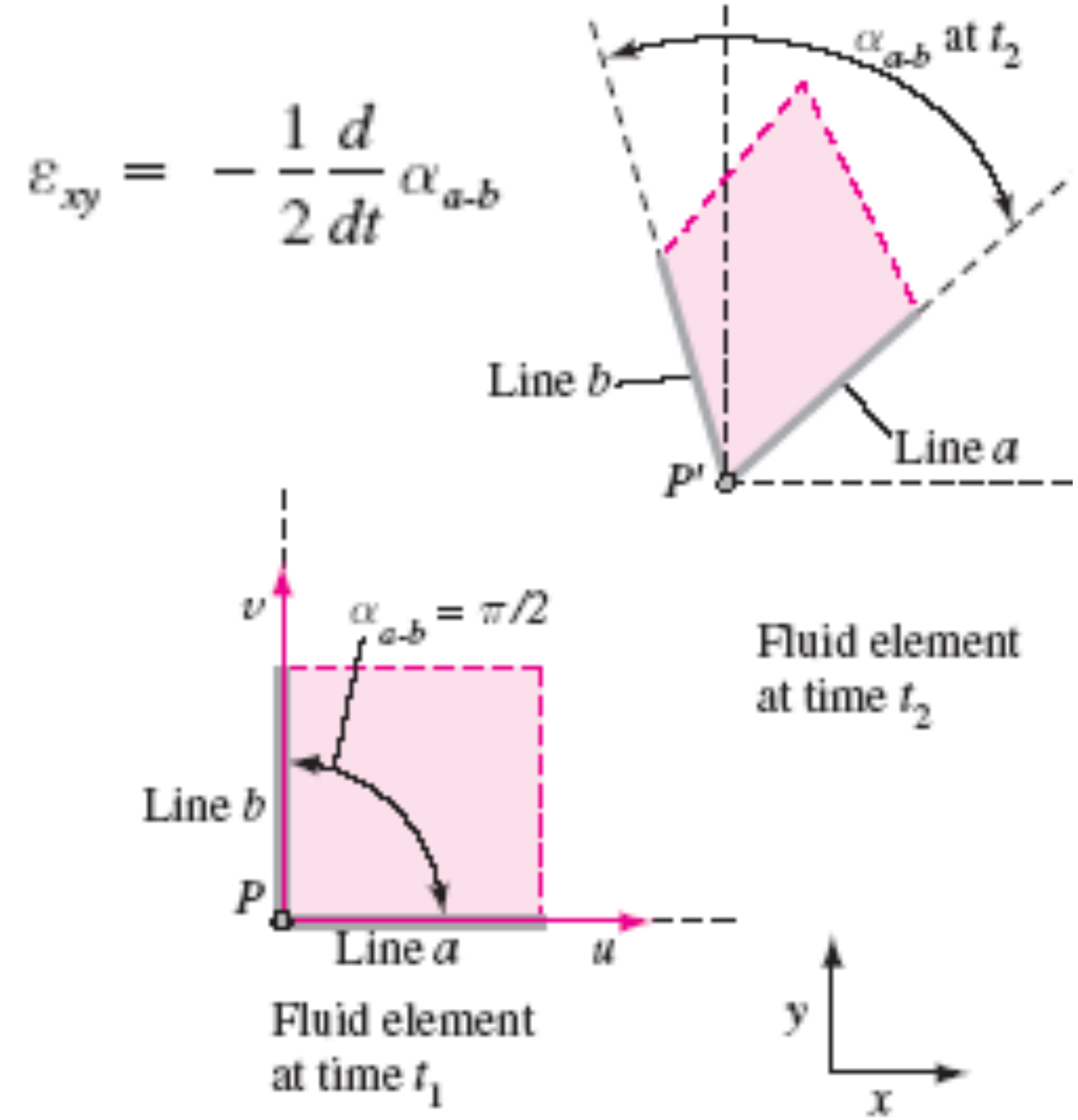


$$\vec{\omega} = \frac{d}{dt}\left(\frac{\alpha_a + \alpha_b}{2}\right) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\vec{\omega} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

Shear Strain Rate

- Shear Strain Rate at a point is defined as half of the rate of decrease of the angle between two initially perpendicular lines that intersect at a point.



- Shear strain rate can be expressed in Cartesian coordinates as:

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

Shear Strain Rate

We can combine linear strain rate and shear strain rate into one symmetric second-order tensor called the **strain-rate tensor**

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right) & \dfrac{\partial v}{\partial y} & \dfrac{1}{2}\left(\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z}\right) & \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\right) & \dfrac{\partial w}{\partial z} \end{pmatrix}$$

Shear Strain Rate

- Purpose of our discussion of fluid element kinematics:
  - Better appreciation of the inherent complexity of fluid dynamics
  - Mathematical sophistication required to fully describe fluid motion
- Strain-rate tensor is important for numerous reasons. For example,
  - Develop relationships between fluid stress and strain rate.

Vorticity and Rotationality

- The vorticity vector is defined as the curl of the velocity vector $\vec{\zeta} = \vec{\nabla} \times \vec{V}$, *a measure of rotation of a fluid particle.*
- Vorticity is equal to twice the angular velocity of a fluid particle
  Cartesian coordinates $\vec{\zeta} = 2\vec{\omega}$

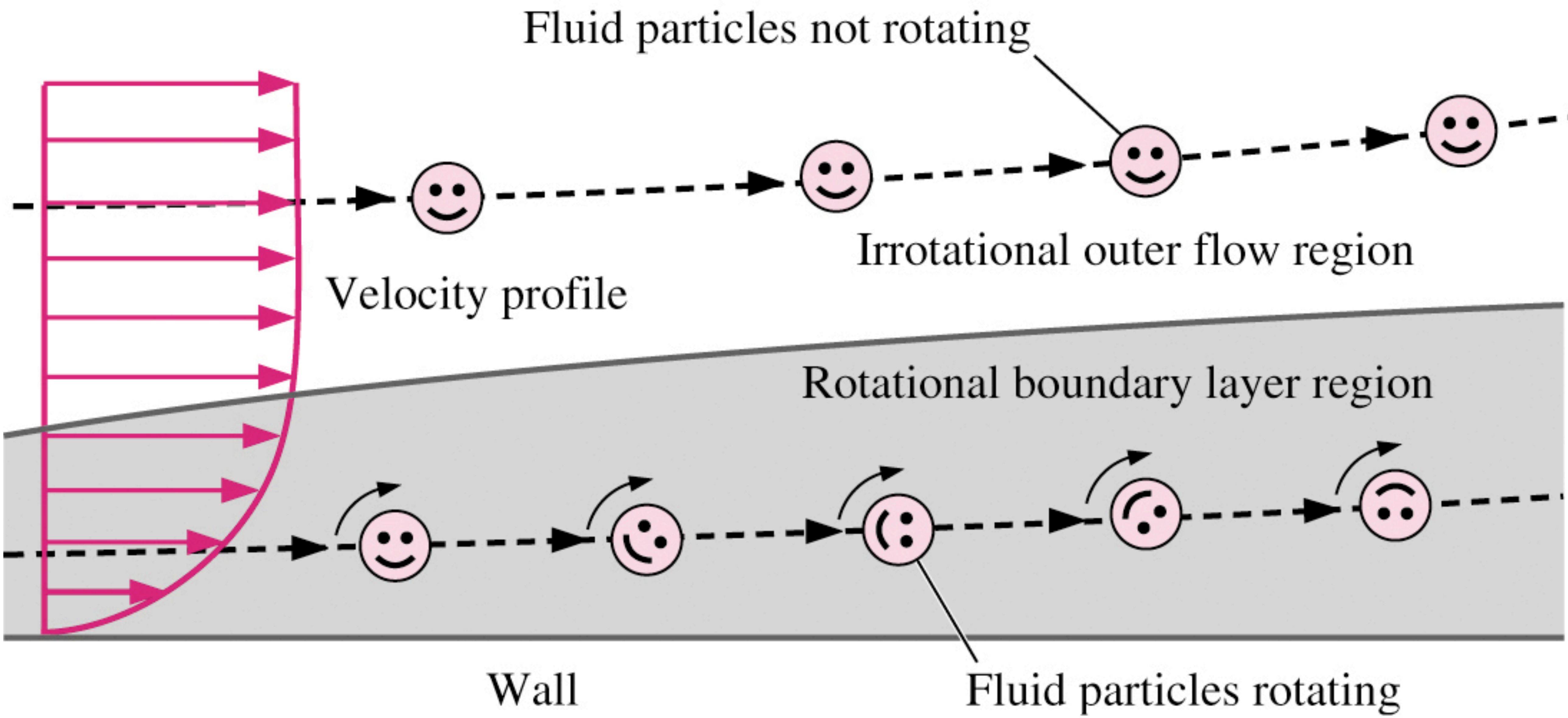
$$\vec{\zeta} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)\vec{k}$$

  Cylindrical coordinate

$$\vec{\zeta} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \right)\vec{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right)\vec{e}_\theta + \left(\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta} \right)\vec{e}_z$$

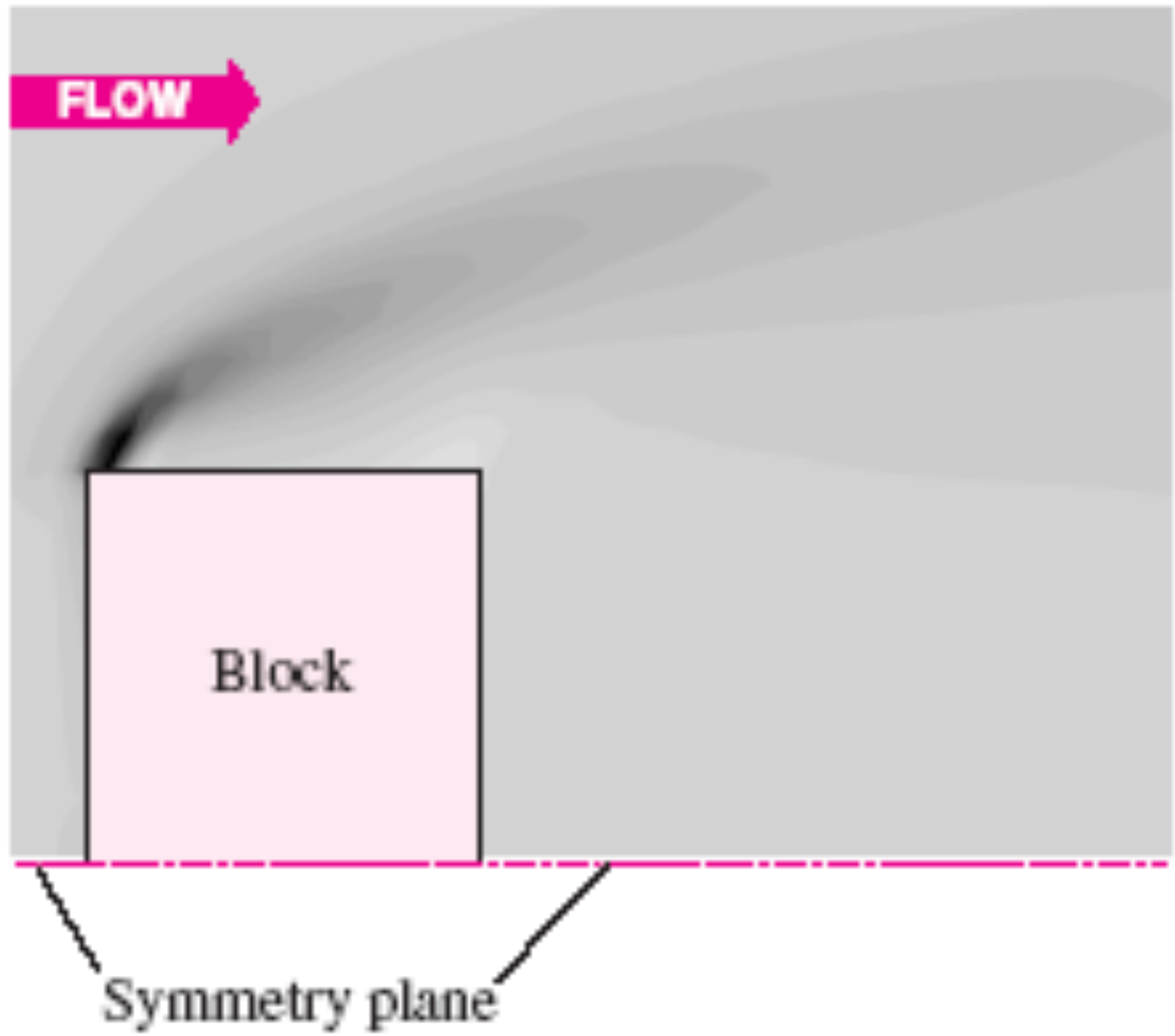
- In regions where $\zeta = 0$, the flow is called irrotational
- Elsewhere, the flow is called rotational

Vorticity and Rotationality



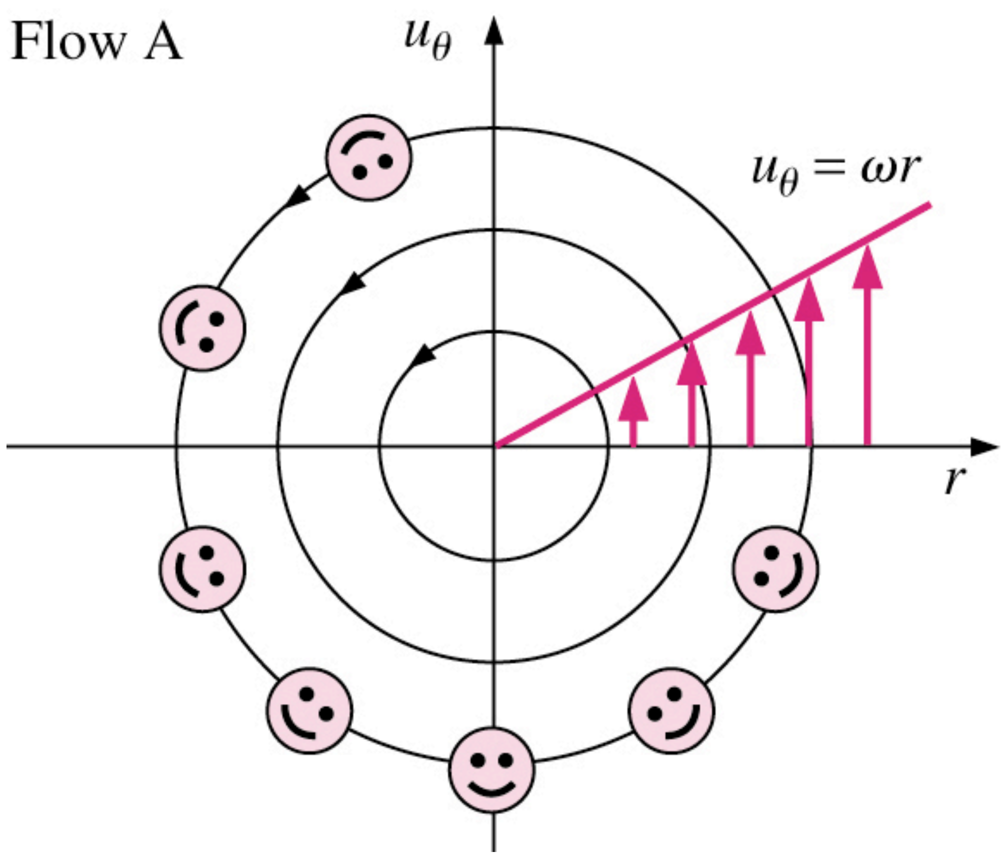
Contour plot of the vorticity field ζ_z

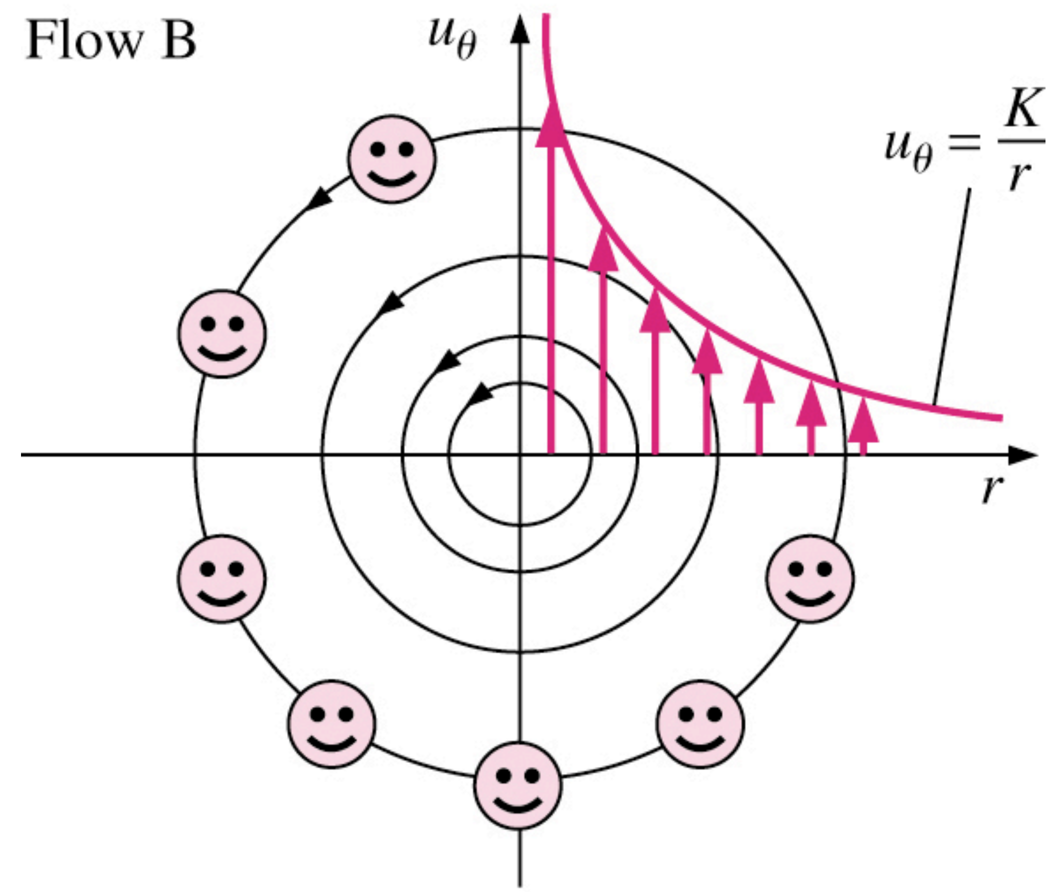
Dark regions represent large negative vorticity, and light regions represent large positive vorticity.



Comparison of Two Circular Flows

Special case: consider two flows with circular streamlines





(b)

$$u_r = 0, u_\theta = \omega r$$

$$\vec{\zeta} = \frac{1}{r}\left(\frac{\partial\left(r u_\theta\right)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\vec{e}_z = \frac{1}{r}\left(\frac{\partial\left(\omega r^2\right)}{\partial r} - 0\right)\vec{e}_z = 2\omega\vec{e}_z$$

$$u_r = 0, u_\theta = \frac{K}{r}$$

$$\vec{\zeta} = \frac{1}{r}\left(\frac{\partial\left(r u_\theta\right)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\vec{e}_z = \frac{1}{r}\left(\frac{\partial\left(K\right)}{\partial r} - 0\right)\vec{e}_z = 0\vec{e}_z$$

Comparison

(a)

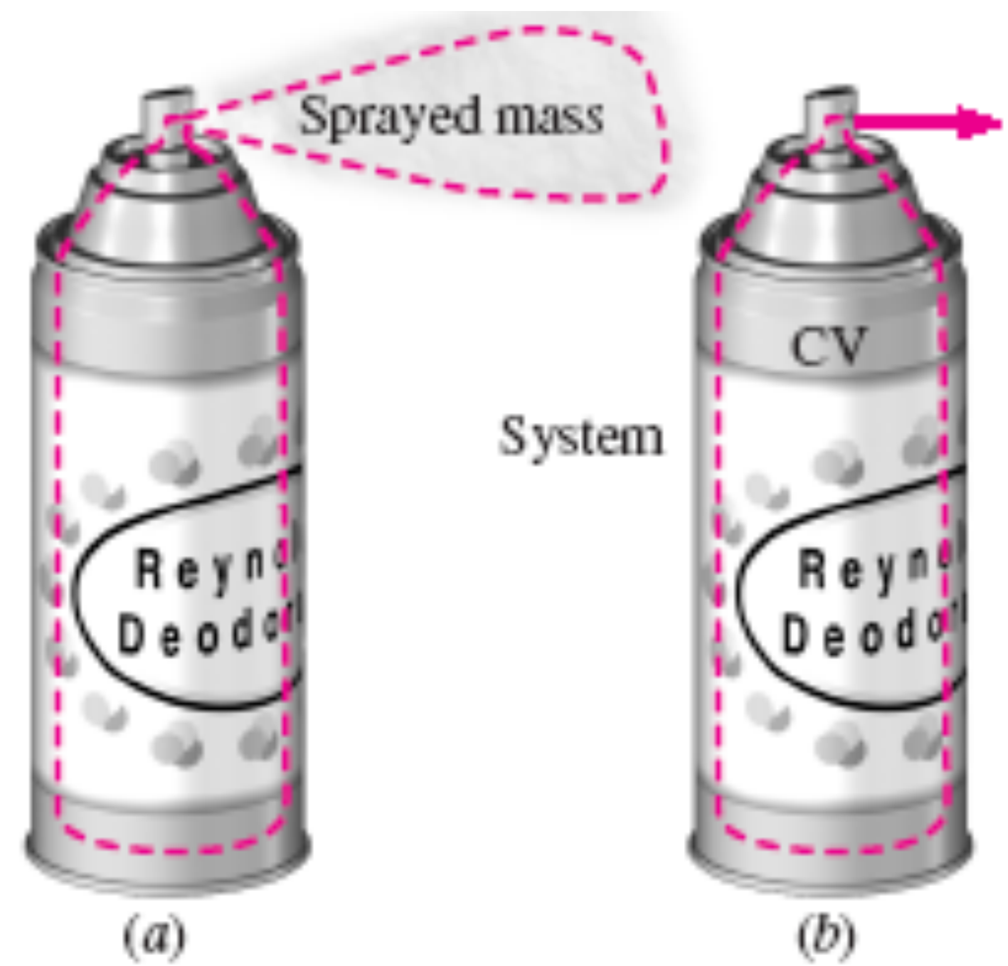
(b)

A merry-go-round or roundabout

A Ferris wheel

Reynolds—Transport Theorem (RTT)

- A system is a quantity of matter of fixed identity. *No mass can cross a system boundary.*
- A control volume is a region in space chosen for study. Mass can cross a control surface.

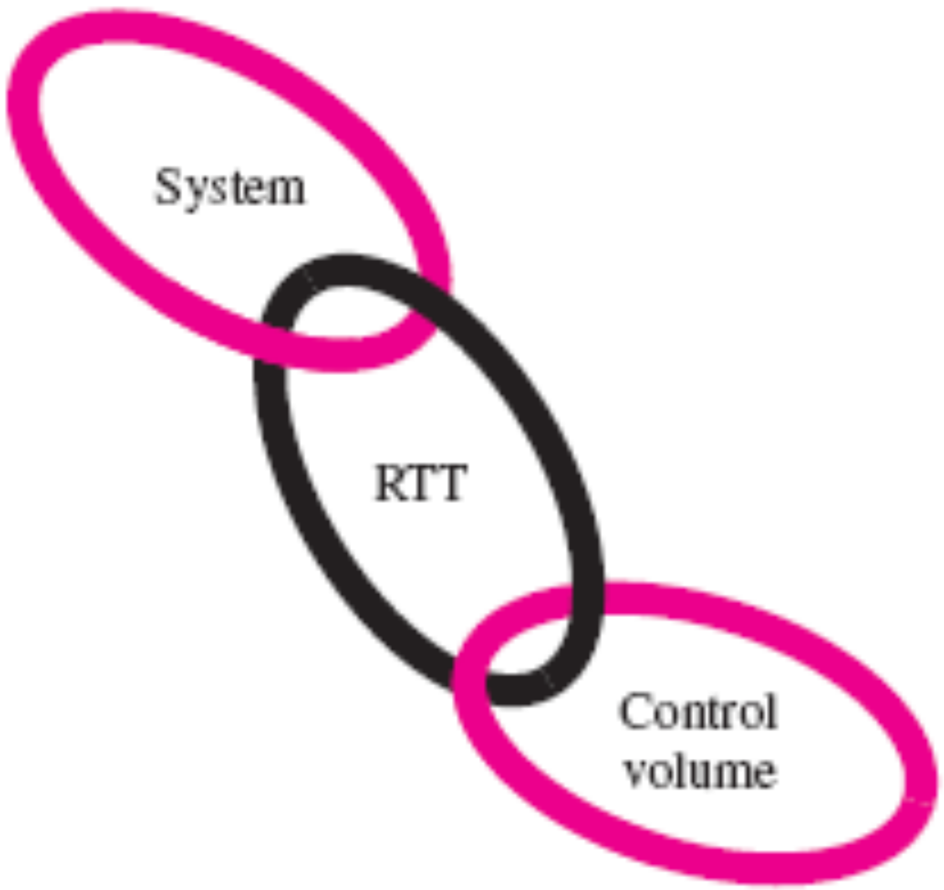


System deformable

CV fixed, nondeformable

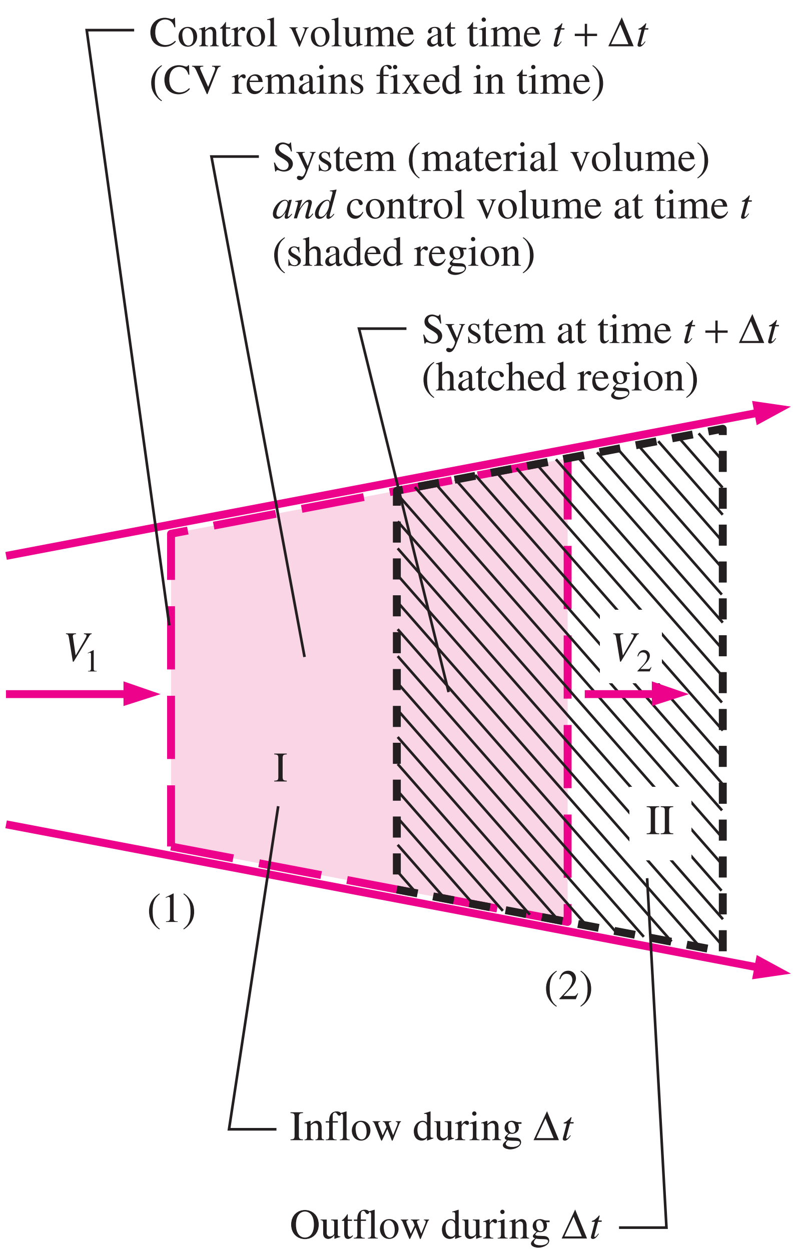
Reynolds—Transport Theorem (RTT)

- The fundamental conservation laws (conservation of mass, energy, and momentum) apply directly to systems.
- However, in most fluid mechanics problems, control volume analysis is preferred over system analysis (for the same reason that the Eulerian description is usually preferred over the Lagrangian description).
- Therefore, we need to transform the conservation laws from a system to a control volume. This is accomplished with the Reynolds transport theorem (RTT).



Reynolds—Transport Theorem (RTT)

Let B represent any extensive property (such as mass, energy, or momentum), and let b=B/m represent the corresponding intensive property. Noting that extensive properties are additive, the extensive property B of the system at times t and t +Δt can be expressed as:



At time t: Sys = CV
At time $t + \Delta t$: Sys = CV − I + II

$$B_{\text{sys},\, t} = B_{\text{CV},\, t} \qquad \text{(the system and CV concide at time } t\text{)}$$

$$B_{\text{sys},\, t+\Delta t} = B_{\text{CV},\, t+\Delta t} - B_{\text{I},\, t+\Delta t} + B_{\text{II},\, t+\Delta t}$$

$$\frac{B_{\text{sys},\, t+\Delta t} - B_{\text{sys},\, t}}{\Delta t} = \frac{B_{\text{CV},\, t+\Delta t} - B_{\text{CV},\, t}}{\Delta t} - \frac{B_{\text{I},\, t+\Delta t}}{\Delta t} + \frac{B_{\text{II},\, t+\Delta t}}{\Delta t}$$

$$B_{\text{I},\, t+\Delta t} = b_1 m_{\text{I},\, t+\Delta t} = b_1 \rho_1 V_{\text{I},\, t+\Delta t} = b_1 \rho_1 V_1 \,\Delta t\, A_1$$

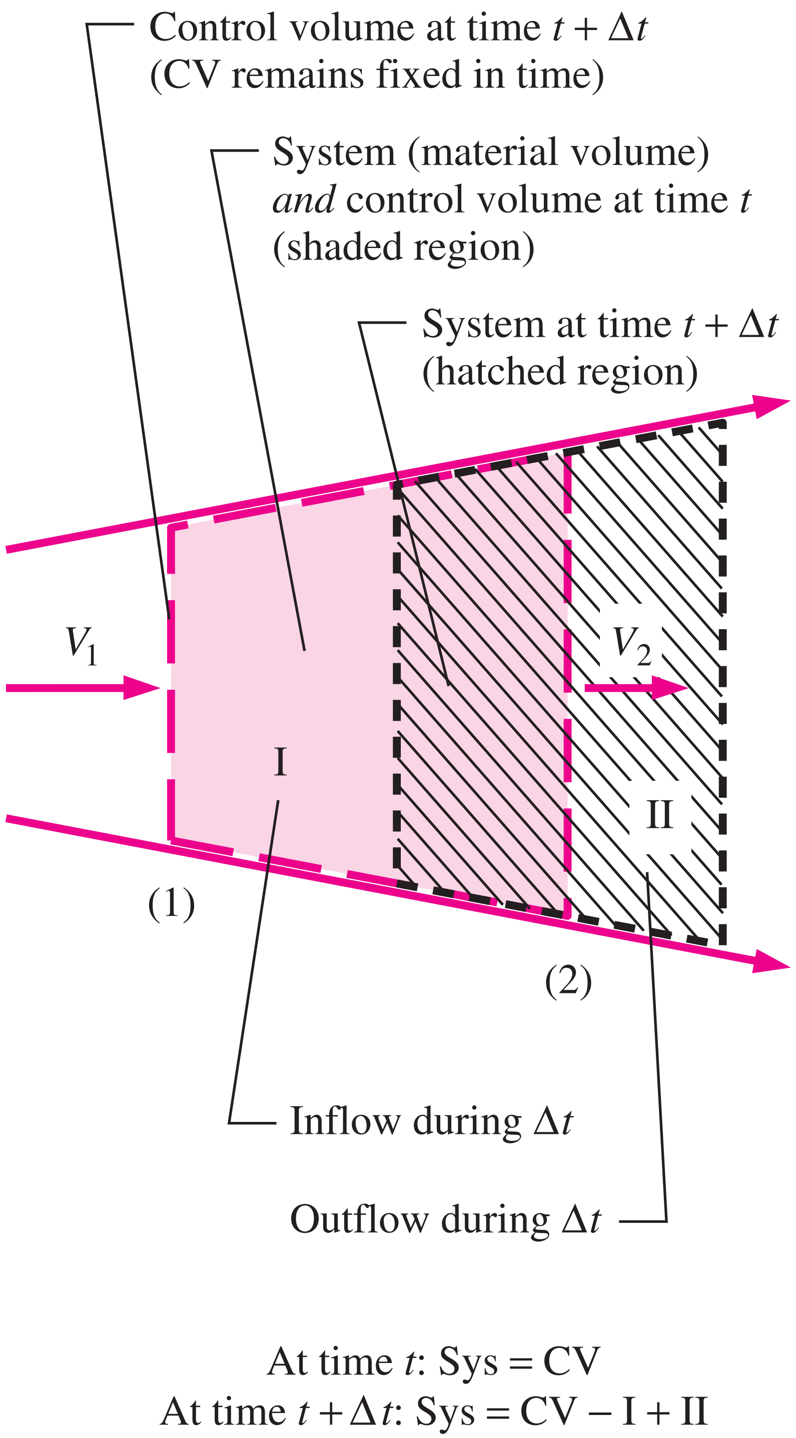
$$B_{\text{II},\, t+\Delta t} = b_2 m_{\text{II},\, t+\Delta t} = b_2 \rho_2 V_{\text{II},\, t+\Delta t} = b_2 \rho_2 V_2 \,\Delta t\, A_2$$

$$\dot{B}_{\text{in}} = \dot{B}_{\text{I}} = \lim_{\Delta t \to 0} \frac{B_{\text{I},\, t+\Delta t}}{\Delta t} = \lim_{\Delta t \to 0} \frac{b_1 \rho_1 V_1 \,\Delta t\, A_1}{\Delta t} = b_1 \rho_1 V_1 A_1$$

$$\dot{B}_{\text{out}} = \dot{B}_{\text{II}} = \lim_{\Delta t \to 0} \frac{B_{\text{II},\, t+\Delta t}}{\Delta t} = \lim_{\Delta t \to 0} \frac{b_2 \rho_2 V_2 \,\Delta t\, A_2}{\Delta t} = b_2 \rho_2 V_2 A_2$$

Reynolds—Transport Theorem (RTT)

$$\frac{dB_{\mathrm{sys}}}{dt} = \frac{dB_{\mathrm{CV}}}{dt} - \dot{B}_{\mathrm{in}} + \dot{B}_{\mathrm{out}}$$



$$\frac{dB_{\mathrm{sys}}}{dt} = \frac{dB_{\mathrm{CV}}}{dt} - b_1\rho_1 V_1 A_1 + b_2\rho_2 V_2 A_2$$

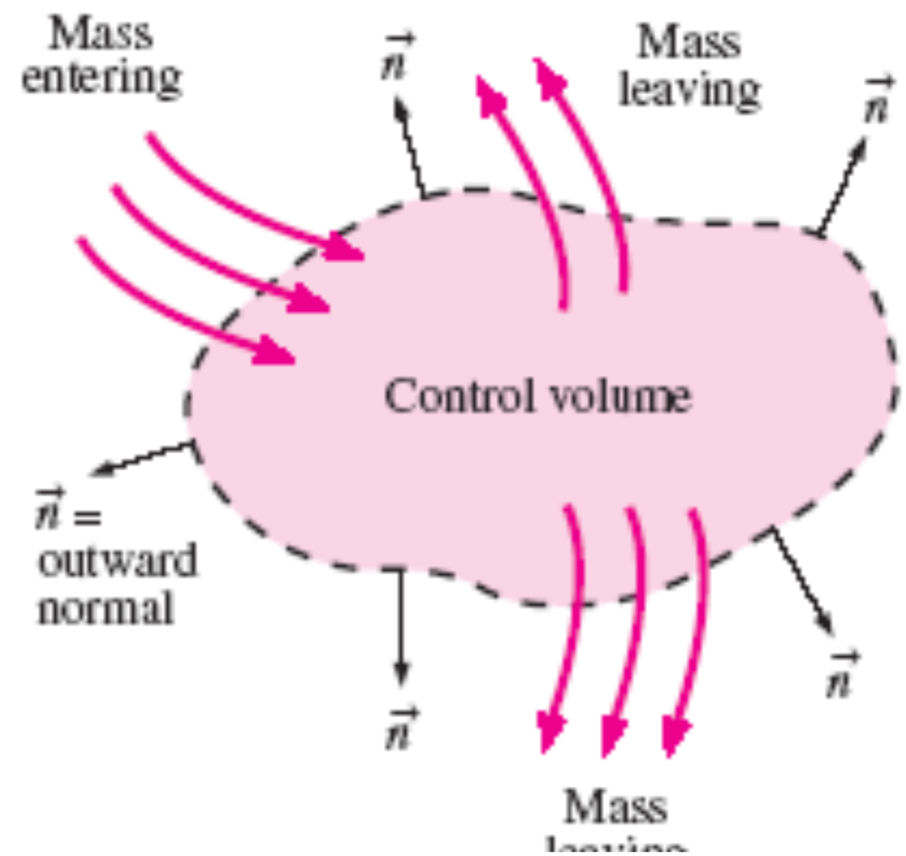
The equation states that the time rate of change of the property B of the system is equal to the time rate of change of B of the control volume plus the net flux of B out of the control volume by mass crossing the control surface.

This is the desired relation since it relates the change of a property of a system to the change of that property for a control volume.

Note that it applies at any instant in time, where it is assumed that the system and the control volume occupy the same space at that particular instant in time.

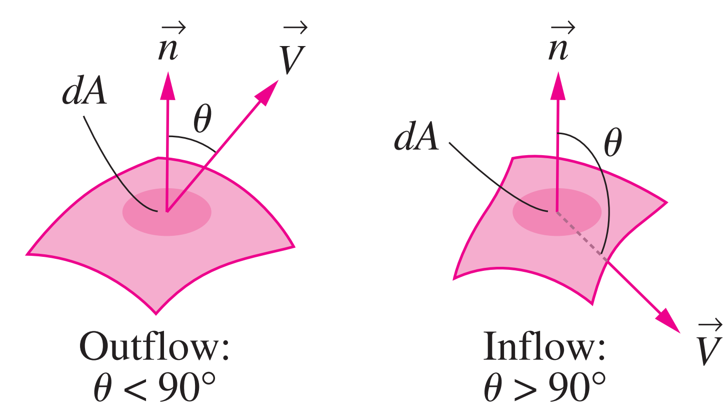
Reynolds—Transport Theorem (RTT)

- In general, however, we may have several inlet and outlet ports, and the velocity may not be normal to the control surface at the point of entry. Also, the velocity may not be uniform. To generalize the process, we consider a differential surface area dA on the control surface and denote its unit outer normal



$$\dot{B}_{\text{net}} = \dot{B}_{\text{out}} - \dot{B}_{\text{in}} = \int_{\text{CS}} \rho b \vec{V} \cdot \vec{n}\, dA \quad \text{(inflow if negative)}$$

the differential is positive for mass flowing out of the control volume, and negative for mass flowing into the control volume, and its integral over the entire control surface gives the rate of net outflow of the property B by mass.



Reynolds—Transport Theorem (RTT)

The properties within the control volume may vary with position, in general. In such a case, the total amount of property B within the control volume must be determined by integration:

$$B_{\mathrm{CV}} = \int_{\mathrm{CV}} \rho b \, dV$$

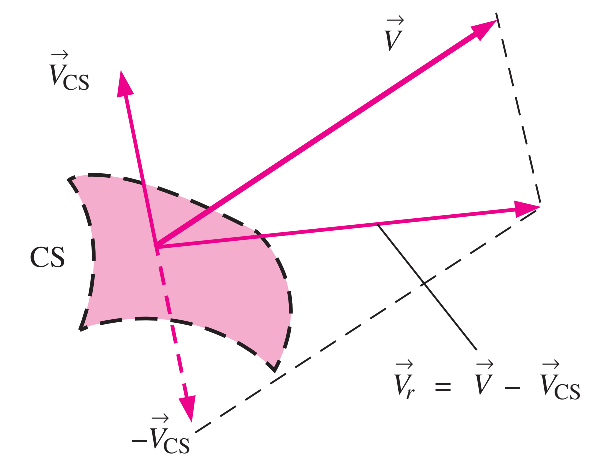
Therefore, the *system-to-control-volume transformation* for a fixed control volume:

$$\frac{dB_{\mathrm{sys}}}{dt} = \frac{d}{dt}\int_{\mathrm{CV}} \rho b \, dV + \int_{\mathrm{CS}} \rho b \vec{V} \cdot \vec{n} \, dA$$

Reynolds—Transport Theorem (RTT)

- Note that for a control volume that moves and/or deforms with time, the time derivative must be applied after integration many practical systems such as turbine and propeller blades involve nonfixed control volumes. Fortunately, is also valid for moving and/or deforming control volumes provided that the absolute fluid velocity in the last term is replaced by the relative velocity

$$\vec{V}_r = \vec{V} - \vec{V}_{CS}$$



- General RTT, nonfixed CV (integral analysis):

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b \, dV + \int_{CS} \rho b \vec{V}_r \cdot \vec{n} \, dA$$

Alternate RTT, nonfixed CV: $$\frac{dB_{sys}}{dt} = \int_{CV} \frac{\partial}{\partial t}(\rho b) \, dV + \int_{CS} \rho b \vec{V} \cdot \vec{n} \, dA$$

RTT Special Cases

For steady flow, the time derivative drops out,

$$\frac{dB_{sys}}{dt} = \int_{CV} \underbrace{\cancel{\frac{\partial}{\partial t}(\rho b)}}_{0} d\cancel{V} + \int_{CS} \rho b \vec{V}_r \bullet \vec{n}\, dA = \int_{CS} \rho b \vec{V}_r \bullet \vec{n}\, dA$$

For control volumes with well-defined inlets and outlets

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b\, d\cancel{V} + \sum_{out} \rho_{avg} b_{avg} V_{r,avg} A - \sum_{in} \rho_{avg} b_{avg} V_{r,avg} A$$

Reynolds—Transport Theorem (RTT)

- Interpretation of the RTT:
  - Time rate of change of the property B of the system is equal to (Term 1) + (Term 2)
  - Term 1: the time rate of change of B of the control volume
  - Term 2: the net flux of B out of the control volume by mass crossing the control surface

$$\frac{dB_{sys}}{dt} = \int_{CV} \frac{\partial}{\partial t}\left(\rho b\right) d\forall + \int_{CS} \rho b \vec{V} \bullet \vec{n}\, dA$$

- We will apply RTT to conservation of mass, energy, linear momentum, and angular momentum.

	Mass	Momentum	Energy	Angular momentum
B, Extensive properties	m	$m\vec{V}$	E	$\vec{H}$
b, Intensive properties	1	$\vec{V}$	e	$\left(\vec{r}\times\vec{V}\right)$

There is a direct analogy between the transformation from Lagrangian to Eulerian descriptions (for differential analysis using infinitesimally small fluid elements) and the transformation from systems to control volumes (for integral analysis using large, finite flow fields).