Corso di Laurea in Fisica - UNITS

# Istituzioni di Fisica per il Sistema Terra

# Gravity (and Capillary) waves

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# Incompressible fluids

- In many cases of the flow of fluids their density may be supposed invariable, i.e. constant throughout the volume and its motion and we speak of incompressible flow: ρ = constant

  - Conservation of matter $$\nabla \cdot \boldsymbol{V} = 0$$

  - Euler equation $$\frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{\zeta} \times \boldsymbol{V} = -\nabla\left(\frac{1}{2}V^2 + \frac{P}{\rho} + gz\right)$$

- The conditions under which the fluid can be considered incompressible are:

$$\frac{\partial \rho}{\partial t} << \rho div(\boldsymbol{V}) \Rightarrow \frac{\Delta\rho}{\tau} << \frac{\rho V}{\lambda}$$

- i.e. $$\tau >> \frac{\lambda}{c}$$

$$\Delta\rho = \frac{\Delta P}{c^2} \approx \frac{1}{c^2}\left(\rho\frac{\partial V}{\partial t}\lambda\right) \approx \frac{1}{c^2}\left(\rho\frac{V}{\tau}\lambda\right)$$

- i.e. $$V << c$$

i.e. the time taken by a sound signal to traverse distances must be small compared with that during which flow changes appreciably

# Incompressible & Irrotational flow

- From Euler equations we have that only viscosity can generate vorticity if none exists initially. And if the flow is irrotational rot(***V***)=0, and thus ***V***=grad($\phi$), the flow is called potential.

- Euler equation rot (***V***)=0

and if it is also incompressible:

- Conservation of matter div (***V***)=0

the potential has to satisfy Laplace equation:

$$\nabla^2(\phi)=0$$

and we can separate the variables...

# Separation of variables + BC at bottom

- Let us consider a velocity potential propagating along the x-axis and uniform in the y- direction: all quantities are independent of y.

- We shall seek a solution which is a simple periodic function of time and of the coordinate x, i.e. we put:

$$\phi = F(z)\cos(kx - \omega t)$$

then $$\frac{d^2F}{dz^2} - k^2F = 0 \quad \Rightarrow \quad F(z) = \left[ Ae^{kz} + Be^{-kz} \right]$$

and if the liquid container has depth h,
there the vertical flow has to be 0:

$$v_z = \left.\frac{dF}{dz}\right|_{z=-h} = 0 \quad \Rightarrow \quad B = e^{-2kh}A$$

# BC at bottom

and this leads to:

$$F(z) = 2Ae^{-kh}\cosh\left[k(z+h)\right]$$

- Thus, at the bottom (z=-h) the cosh(0)=1, while at top it is cosh(kh), thus F grows as z goes from bottom to top values.

- If the container is infinitely deep (h goes to infinity) we have that B has to be 0 and the potential as well is going to 0:

$$F(z) = Ae^{kz}$$

# Gravity waves

- The free surface of a liquid in equilibrium in a gravitational field is a plane.
- If, under the action of some external perturbation, the surface is moved from its equilibrium position at some point, motion will occur in the liquid.
- This motion will be propagated over the whole surface in the form of waves, which are called gravity waves, since they are due to the action of the gravitational field.
- We shall here consider gravity waves in which the velocity of the moving fluid particles is so small that we may neglect the term $(\boldsymbol{V} \cdot \mathrm{grad})\boldsymbol{V}$ in comparison with $\partial/\partial t$ in Euler's equation.

# Gravity waves

The physical significance of this is easily seen:

- During a time interval of the order of the period, $\tau$, of the oscillations of the fluid particles in the wave, these particles travel a distance of the order of the amplitude, *a*, of the wave. Their velocity *V* is therefore of the order of a/$\tau$. It varies noticeably over time intervals of the order of $\tau$ and distances of the order of $\lambda$ in the direction of propagation (where $\lambda$ is the wavelength). Hence the time derivative of the velocity is of the order of $V/\tau$, and the space derivatives are of the order of $V/\lambda$.

Thus the condition

$$(\mathbf{V} \cdot grad)\mathbf{V} << \frac{\partial \mathbf{V}}{\partial t}$$

is equivalent to

$$\frac{1}{\lambda}\left(\frac{a}{\tau}\right)^2 << \frac{a}{\tau}\frac{1}{\tau} \quad \Rightarrow \quad a << \lambda$$

i.e. the amplitude of the oscillations in the wave must be small compared with the wavelength

# Small amplitude gravity waves

- For waves whose amplitude of motion is smaller than the wavelength, all significant terms in the fluid equation are gradients, and the Euler equation can be expressed as:

$$grad\left(\frac{\partial \phi}{\partial t} + \frac{P}{\rho} + \Phi\right) = 0$$

- thus, in space:

$$\frac{\partial \phi}{\partial t} + \frac{P}{\rho} + \Phi = \text{constant}$$

- and assuming a gravitational potential $gz$, we obtain:

$$P = -\rho g z - \rho \frac{\partial \phi}{\partial t}$$

# Gravity waves: BC at the top

- Let us denote by $f$ the z coordinate of a point on the surface; $f$ is a function of x, y and t.

- In equilibrium $f$=0, so that $f$ gives the vertical displacement of the surface in its oscillations.

- Let a constant pressure $p_0$ act on the surface. Then we have at the surface:

$$p_0 = -\rho g f - \rho \frac{\partial \phi}{\partial t}$$

- The constant $p_0$ can be eliminated by redefining the potential, adding to it a quantity independent of the coordinates. We then obtain the condition at the surface as:

$$g f + \left. \frac{\partial \phi}{\partial t} \right|_{z=f} = 0$$

# Gravity waves: BC at the top

- Since the amplitude of the wave oscillations is small, the displacement *f* is small. Hence we can suppose, to the same degree of approximation, that the vertical component of the velocity of points on the surface is simply the time derivative of *f*:

$$V_z = \left.\frac{\partial \phi}{\partial z}\right|_{z=f} \cong \frac{\partial f}{\partial t} = -\left(\frac{1}{g}\frac{\partial^2 \phi}{\partial t^2}\right)$$

- Since the oscillations are small, we can take the value of the derivatives at $z=0$ instead of $z=f$. Thus we have finally the following system of equations to determine the motion in a gravitational field:

$$\Delta \phi = 0$$

- incompressibility

$$\left.\left(\frac{\partial \phi}{\partial z} + \frac{1}{g}\frac{\partial^2 \phi}{\partial t^2}\right)\right|_{z=0} = 0$$

- B.C.

# Gravity waves: dispersion

From the expression $$F(z) = 2Ae^{-kh}\cosh\left[k(z+h)\right]$$

the boundary at the top gives the dispersion relation for incompressible, irrotational, small amplitude "gravity" waves:

$$\omega^2 = kg\left[\tanh(kh)\right]$$

**Deep water**
(kh goes to infinity)

$$\omega^2 = kg$$

$$c = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}}$$

$$u = \frac{\partial\omega}{\partial k} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}\sqrt{\frac{g\lambda}{2\pi}} = \frac{1}{2}c$$

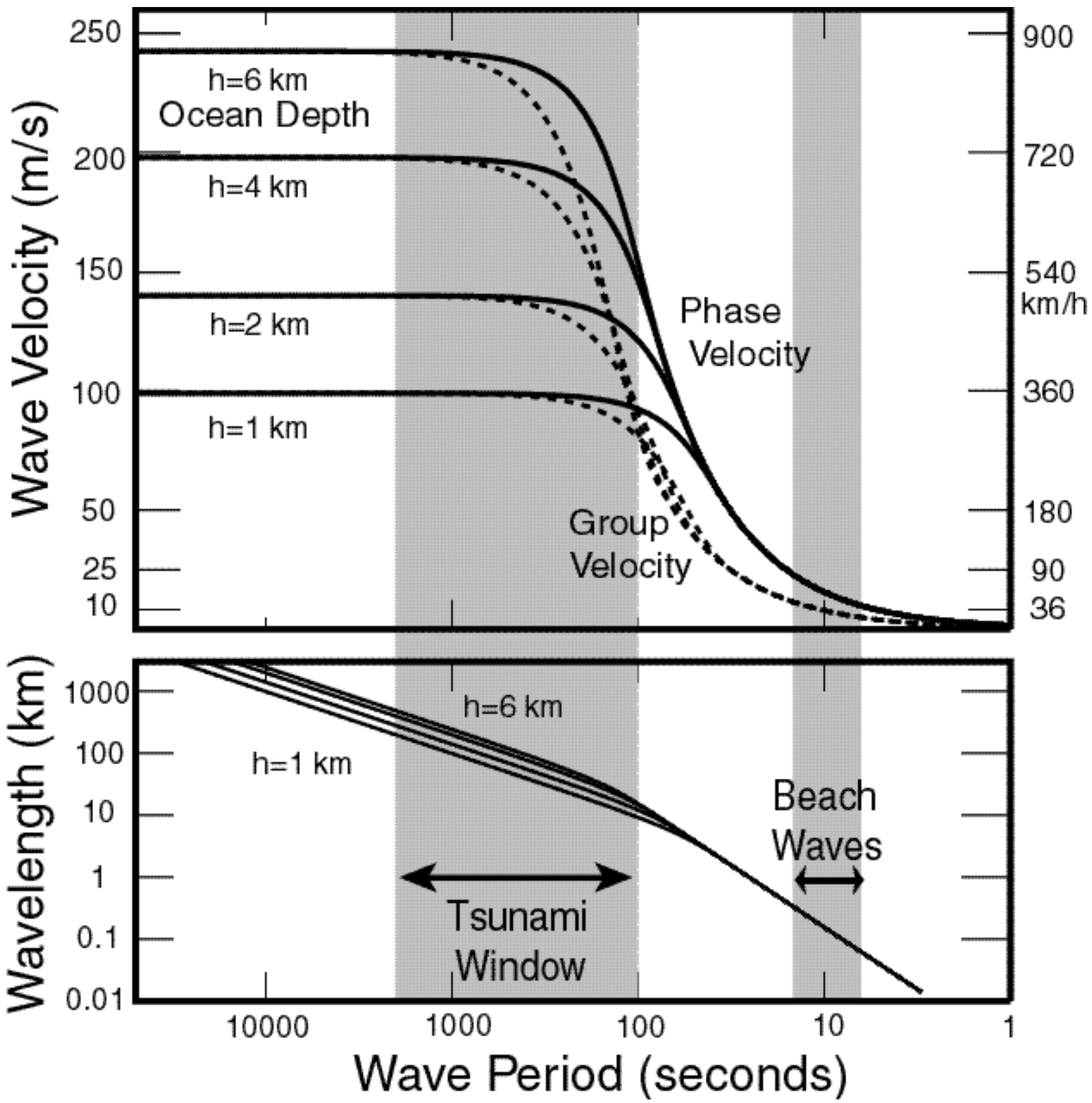
**Shallow water**
(kh goes to zero)

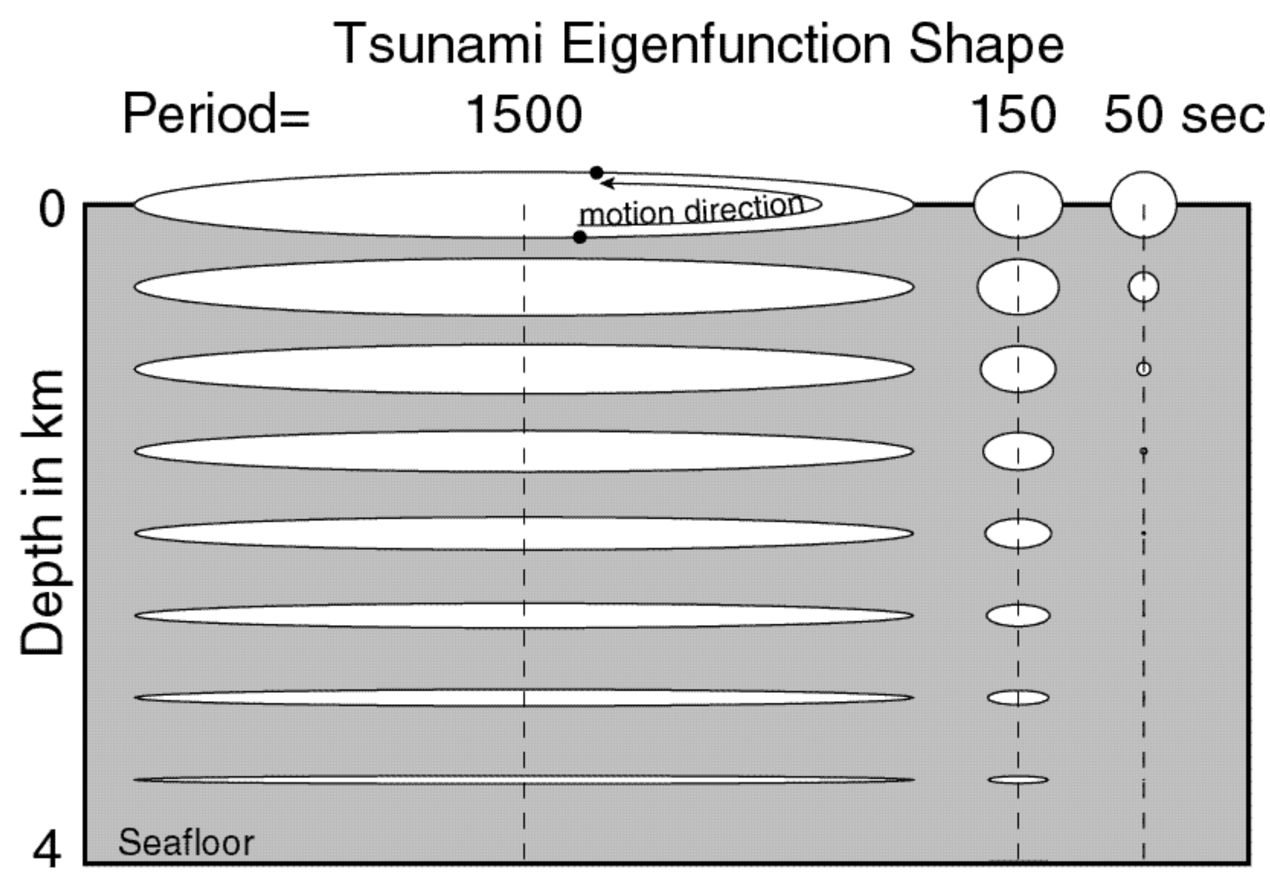
$$\omega^2 = k^2 gh$$

$$c = \sqrt{gh}$$

$$u = \frac{\partial\omega}{\partial k} = c = \sqrt{gh}$$

# Gravity waves eigenvalues & eigenfunctions



# Gravity waves in deep water

- The velocity distribution in the moving liquid is found by simply taking the space derivatives the velocity potential:

$$V_x = -Ake^{kz}\sin(kx-\omega t) \quad V_z = Ake^{kz}\cos(kx-\omega t)$$

- We see that the velocity diminishes exponentially as we go into the liquid. At any given point in space (i.e. for given x, z) the velocity vector rotates uniformly in the xz-plane, its magnitude remaining constant.

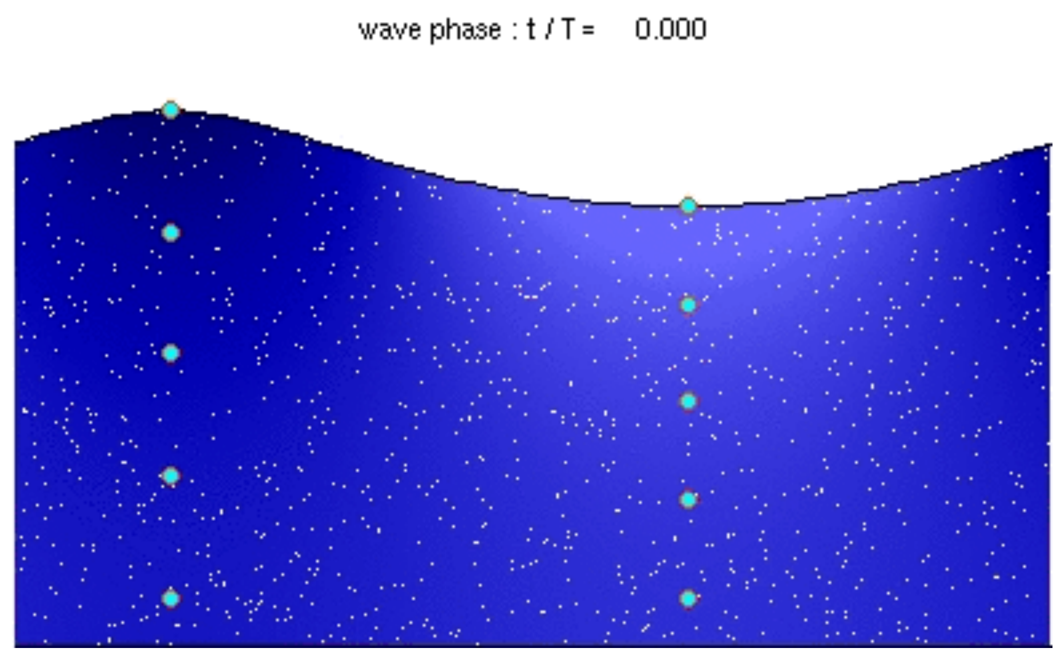
- Let us also determine the paths of fluid particles in the wave. We temporarily denote by x, z the coordinates of a moving fluid particle (and not of a point fixed in space), and by $x_0$, $z_0$ the values of x and z at the equilibrium position of the particle. Then $V_x$ = dx/dt, $V_z$ = dz/dt, and on the right-hand side  we may approximate by writing $x_0$, $z_0$ in place of x, z, since the oscillations are small.

# Gravity waves in deep water

- An integration with respect to time then gives:

$$x - x_0 = -A\frac{k}{\omega}e^{kz_0}\cos(kx_0 - \omega t) \quad z - z_0 = -A\frac{k}{\omega}e^{kz_0}\sin(kx_0 - \omega t)$$

- Thus the fluid particles describe circles about the points ($x_0$, $z_0$) with a radius which diminishes exponentially with increasing depth.



# Long Gravity waves

- Let us now discuss the limiting case of waves whose length is large compared with the depth. These are called long waves.

- Let us examine the propagation of long waves in a channel that is supposed to be along the x-axis, and of infinite length. The cross-section of the channel may have any shape, and may vary along its length. We denote the cross-sectional area of the liquid in the channel by $S = S(x,t)$. The depth and width of the channel are supposed small in comparison with the wavelength.

- We shall here consider longitudinal waves, in which the liquid moves along the channel. In such waves the velocity component $v_x$ along the channel is large compared with the components $v_y$, $v_z$. We denote $v_x$ by $v$ simply, and omit small terms.

# Long Gravity waves - alternative

- The x-component of Euler's equation can then be written in the form:

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

- and the z-component of Euler's equation can then be written in the form:

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g$$

- we omit terms quadratic in the velocity, since the amplitude of the wave is again supposed small. From the second equation we have, since the pressure at the free surface (z=*f*) must be $p_0$:

$$p = p_0 + \rho g(f - z)$$

- Substituting this expression in the first equation, we obtain

$$\frac{\partial v}{\partial t} = -g\frac{\partial f}{\partial x}$$

# Long Gravity waves

- The second equation needed to determine the two unknowns v and f, that can be derived similarly to the equation of continuity; it is essentially the equation of continuity for the case in question. Let us consider a volume of liquid bounded by two plane cross-sections of the channel at a distance dx apart. In unit time a volume $(Sv)_x$ of liquid flows through one plane, and a volume $(Sv)_{x+dx}$ through the other. Hence the volume of liquid between the two planes changes, and since the liquid is incompressible, this change must be due simply to the change in the level of the liquid. The change per unit time is:

$$\frac{\partial S}{\partial t} dx = -\frac{\partial (Sv)}{\partial x} dx$$

- Let $S_o$ be the equilibrium cross-sectional area of the liquid in the channel.
Then $S = S_o + S'$, where S' is the change in the cross-sectional area caused by the wave. Since the change in the liquid level is small, we can write S' in the form bf, where b is the width of the channel at the surface of the liquid.

$$b\frac{\partial f}{\partial t} + \frac{\partial (S_0 v)}{\partial x} = 0$$

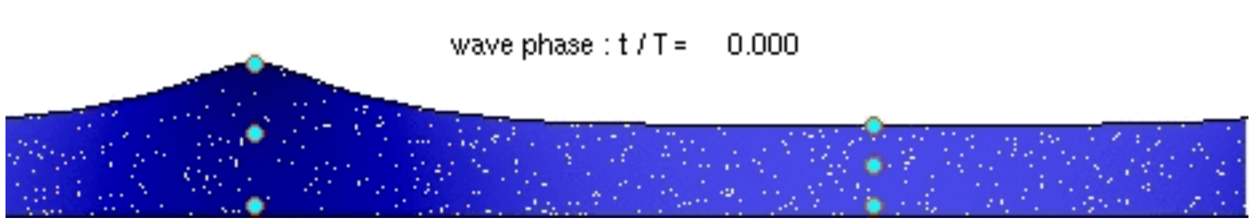
# Long Gravity waves

- From Euler and Continuity equations for a channel with a constant cross section $S_0$, and height $b$, one can obtain:

$$\frac{\partial^2 f}{\partial t^2} - \frac{gS_0}{b}\frac{\partial^2 f}{\partial x^2} = 0$$

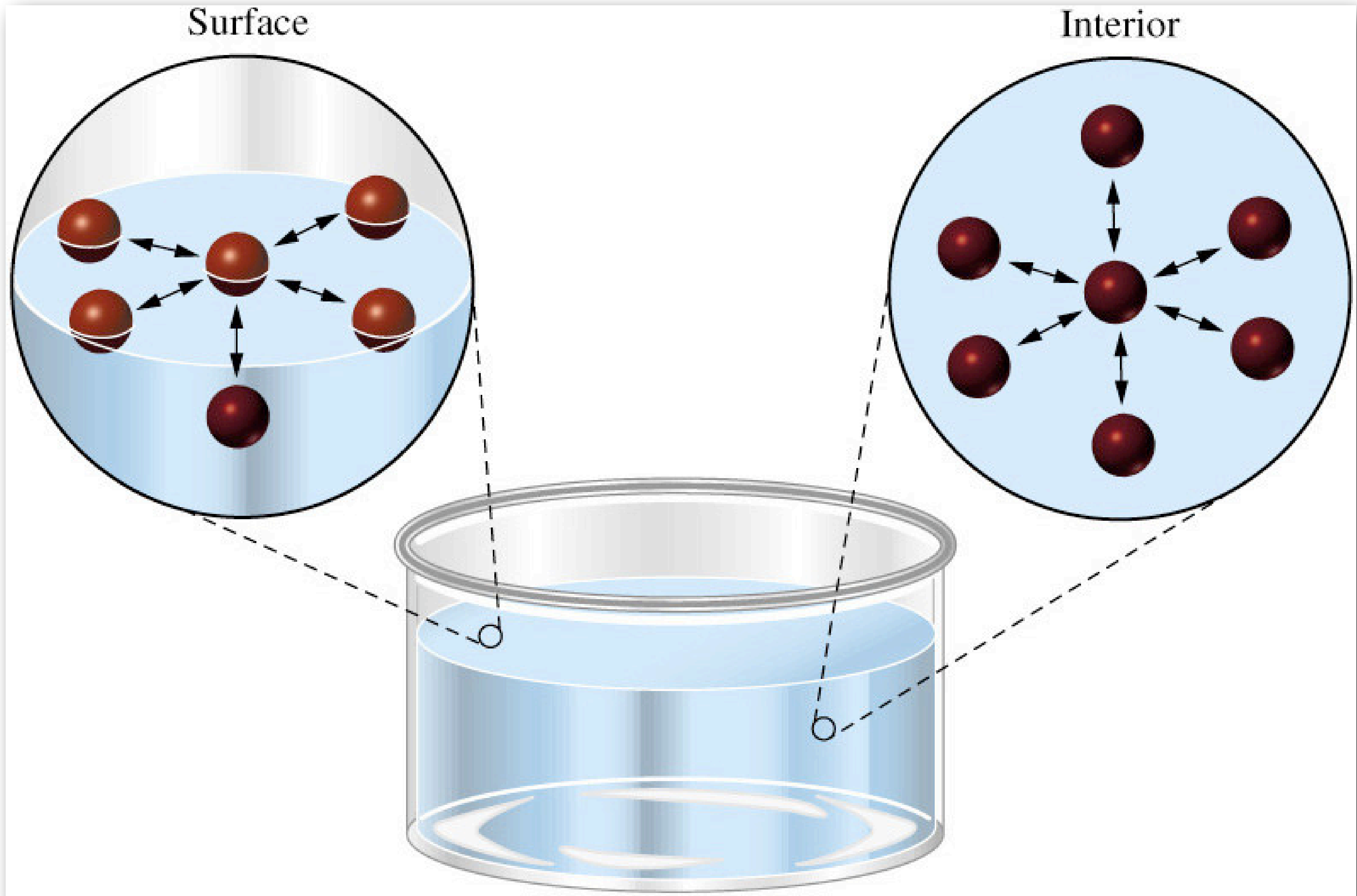
- This is called a wave equation and corresponds to the propagation of waves with a velocity c(u) which is independent of frequency and is :

$$c = u = \sqrt{\frac{gS_0}{b}} \approx \sqrt{gh}$$



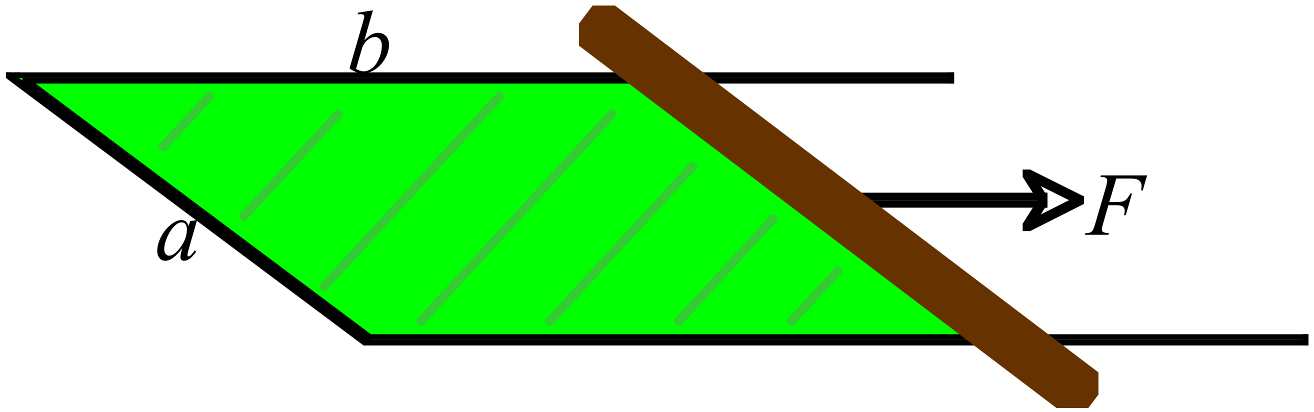
# Surface tension

# Surface tension

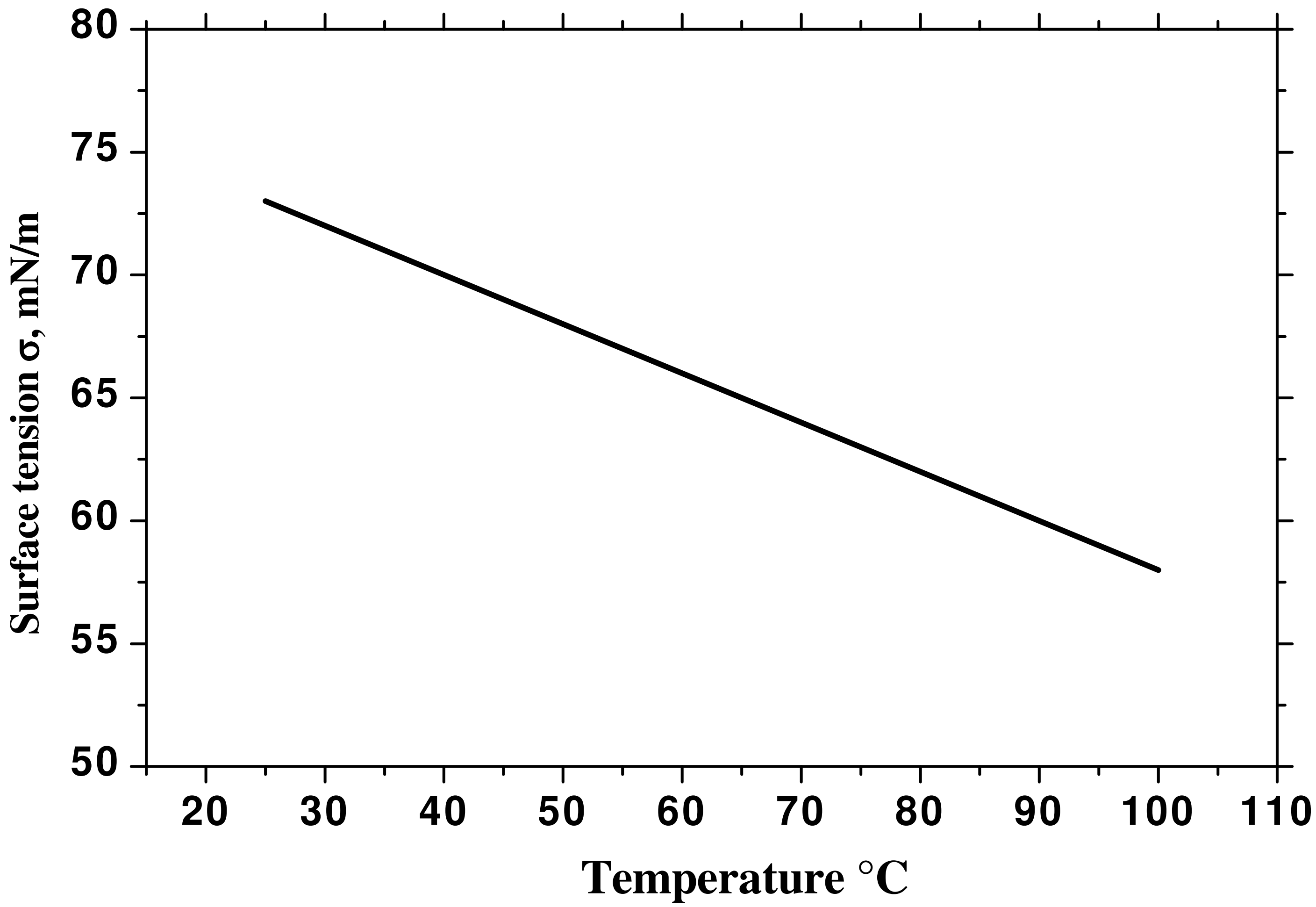



- Molecules have a tendency to be drawn into the interior of a liquid to the greatest extent possible, leaving a minimum of surface area.
- Because a sphere has a smaller ratio of surface area to volume than any other three-dimensional figure, free-falling liquids tend for form spherical drops.

# Measurement of surface tension




- The work done to pull a thin film of fluid has to be equal to the increase in energy: $Fdx=2\sigma adx$



# Capillary waves

- When the surface of a liquid is curved, the surface tension is acting as a restoring force

# New BC at top

- The condition that requires to be modified is the free-surface dynamic boundary condition: in the presence of surface tension, the gauge pressure on the free surface will be nonzero and will be balanced by surface tension. After linearization, the new term, dependent of the radius of curvature at the surface, will be:

$$gf + \frac{\sigma}{\rho}\frac{\partial^2 \phi}{\partial x^2}\bigg|_{z=f} + \frac{\partial \phi}{\partial t}\bigg|_{z=f} = 0$$

leading to the new dispersion relation:

$$\omega^2 = \left( kg + k^3 \frac{\sigma}{\rho} \right) \tanh(kh)$$

that shows that surface tension is more significant for large k, i.e. wavelengths smaller than the capillary length $(\sigma/\rho g)^{1/2}$, that is 2-3 mm for water!

# Gravity capillary waves dispersion

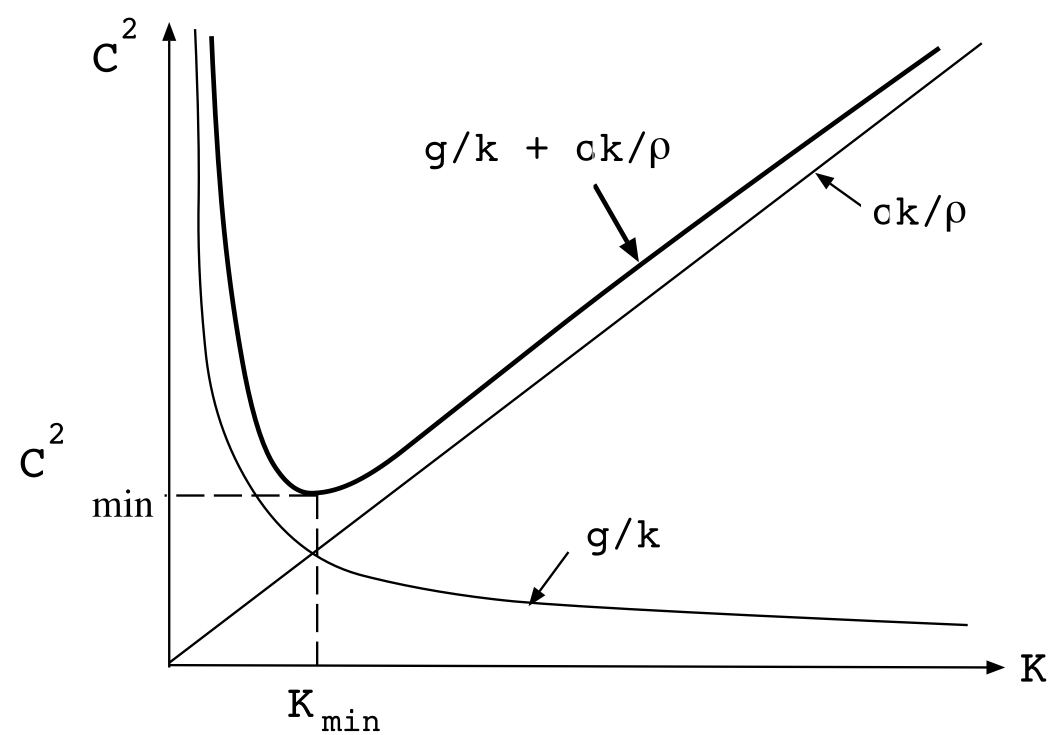
$$\omega^2 = \left( kg + k^3 \frac{\sigma}{\rho} \right) \tanh(kh)$$

neglecting gravity in deep water

$$\omega^2 = k^3 \frac{\sigma}{\rho}$$

$$c = \sqrt{\frac{\sigma}{\rho} k} \quad u = \frac{\partial \omega}{\partial k} = \frac{3}{2} c$$

that shows that there is anomalous dispersion




and the $k_{min} = (\rho g/\sigma)^{1/2}$, associated to a wavelength of 1.73 cm for the water, corresponds to a minimum for phase velocity (23.2 cm/s).

- Capillary waves on water have usually wavelengths less than 4mm and frequencies higher than 70Hz, thus easily excited by a tuning fork