Corso di Laurea in Fisica - UNITS

ISTITUZIONI DI FISICA

PER IL SISTEMA TERRA

Continuity and Transport equations

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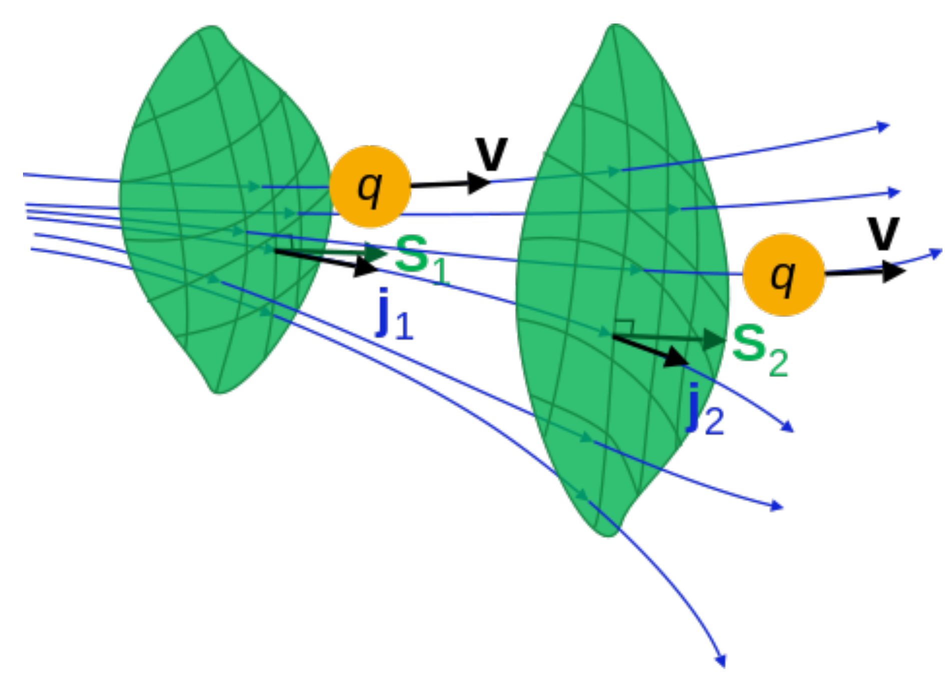
Continuity Equation - FD

- General differential form: ρ is the density of a quantity q, ***j*** is the flux of q, σ is the generation of q per unit volume per unit time

$$\frac{\partial \rho}{\partial t} + div(\boldsymbol{j}) = \sigma$$

- E.g. in fluid dynamics, the continuity equation states that, in any steady state process, the rate at which mass enters a system is equal to the rate at which mass leaves the system:

$$\frac{\partial \rho}{\partial t} + div(\rho \boldsymbol{V}) = 0$$



Summary of Heat transfer Process

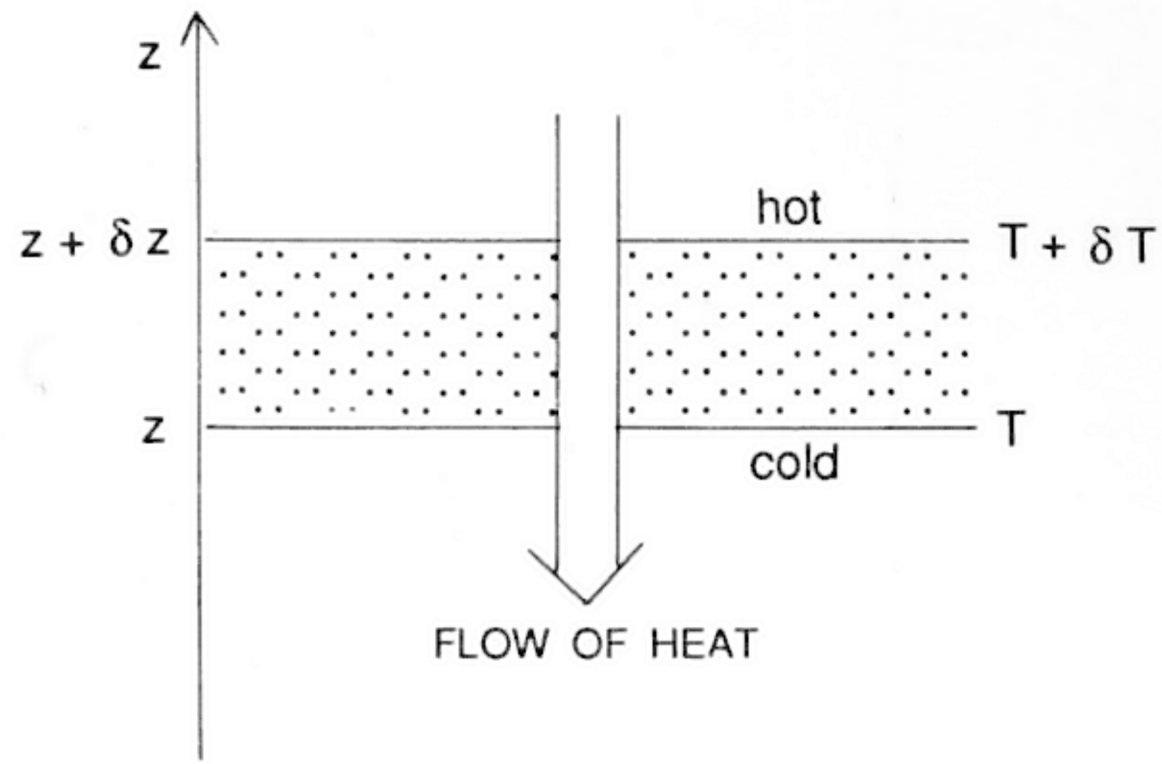
Difference in term of	Way of heat transfer		
	Conduction	**Convection**	**Radiation**
How is heat transferred	Heat flow from vibration (solid)/ collision (liquid & gas) between molecules due to temperature difference	Heat is carried by molecules that move, following the convection current due to temperature difference (density)	Heat is transferred in the form of electromagnetic waves
Medium	Solids & fluids(static)	Liquids or gases	Does not need a medium (vacuum)
Law involved in the heat transfer process	Fourier's law of heat conduction	Newton's law of cooling	Stefan-Boltzmann & Kirchhoff's laws

Heat flow

Imagine an infinitely wide and long solid plate with thickness δz .

Temperature above is $T + \delta T$

Temperature below is T



Heat flowing down is proportional to:

$$\frac{(T+\delta T)-T}{\delta z}$$

The rate of flow of heat per unit area up through the plate, Q, is:

$$Q=-k\left(\frac{T+\delta T-T}{\delta z}\right)$$

$$Q(z)=-k\frac{\delta T}{\delta z}$$

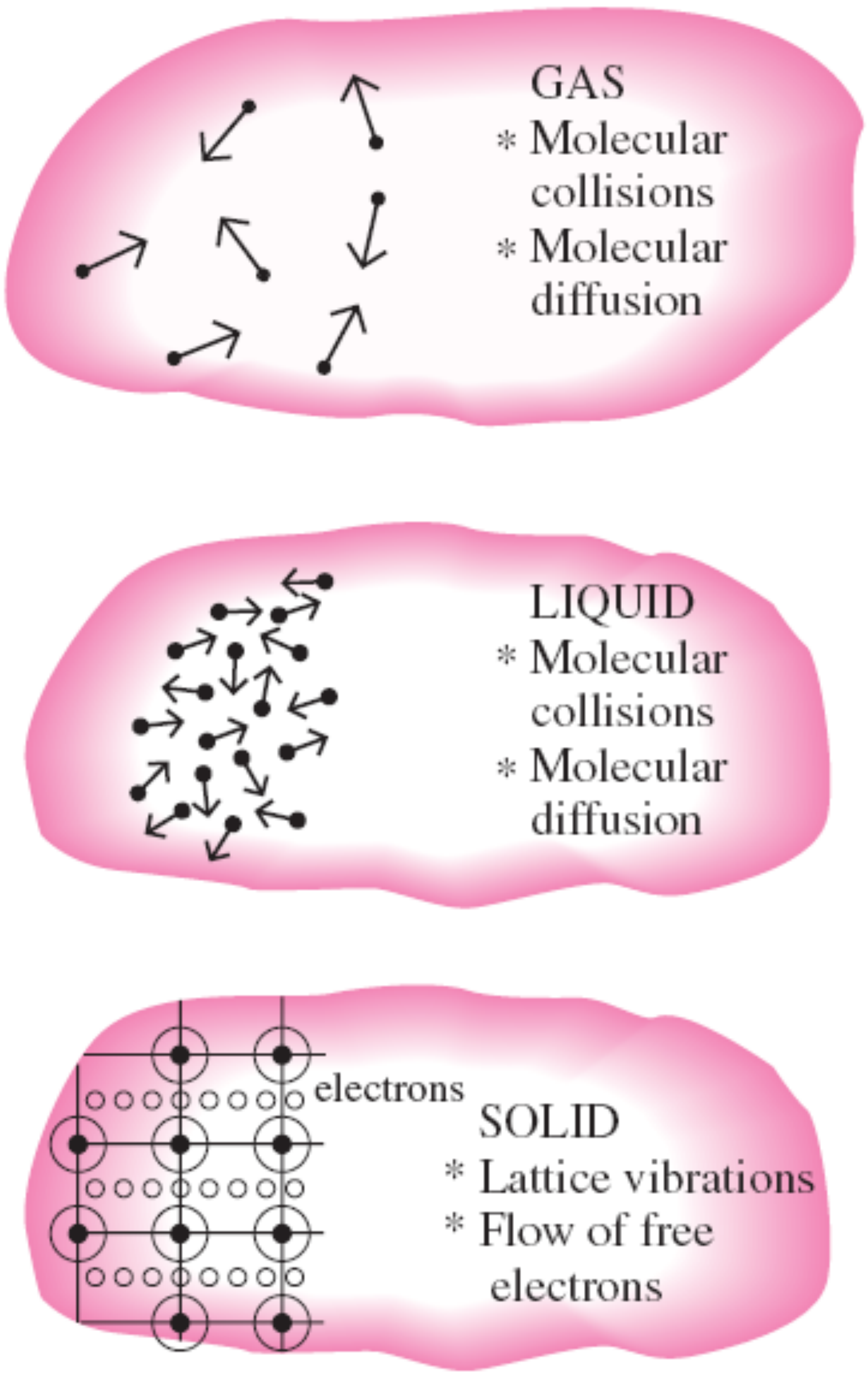
In the limit as δz goes to zero:

$$Q(z)=-k\frac{\partial T}{\partial z}$$

Heat flow

- Heat flow (or flux) Q is rate of flow of heat per unit area.
    - The units are watts per meter squared, $W\ m^{-2}$
    - Watt is a unit of power (amount of work done per unit time)
    - A watt is a joule per second
    - Typical continental surface heat flow is 40-80 $mW\ m^{-2}$

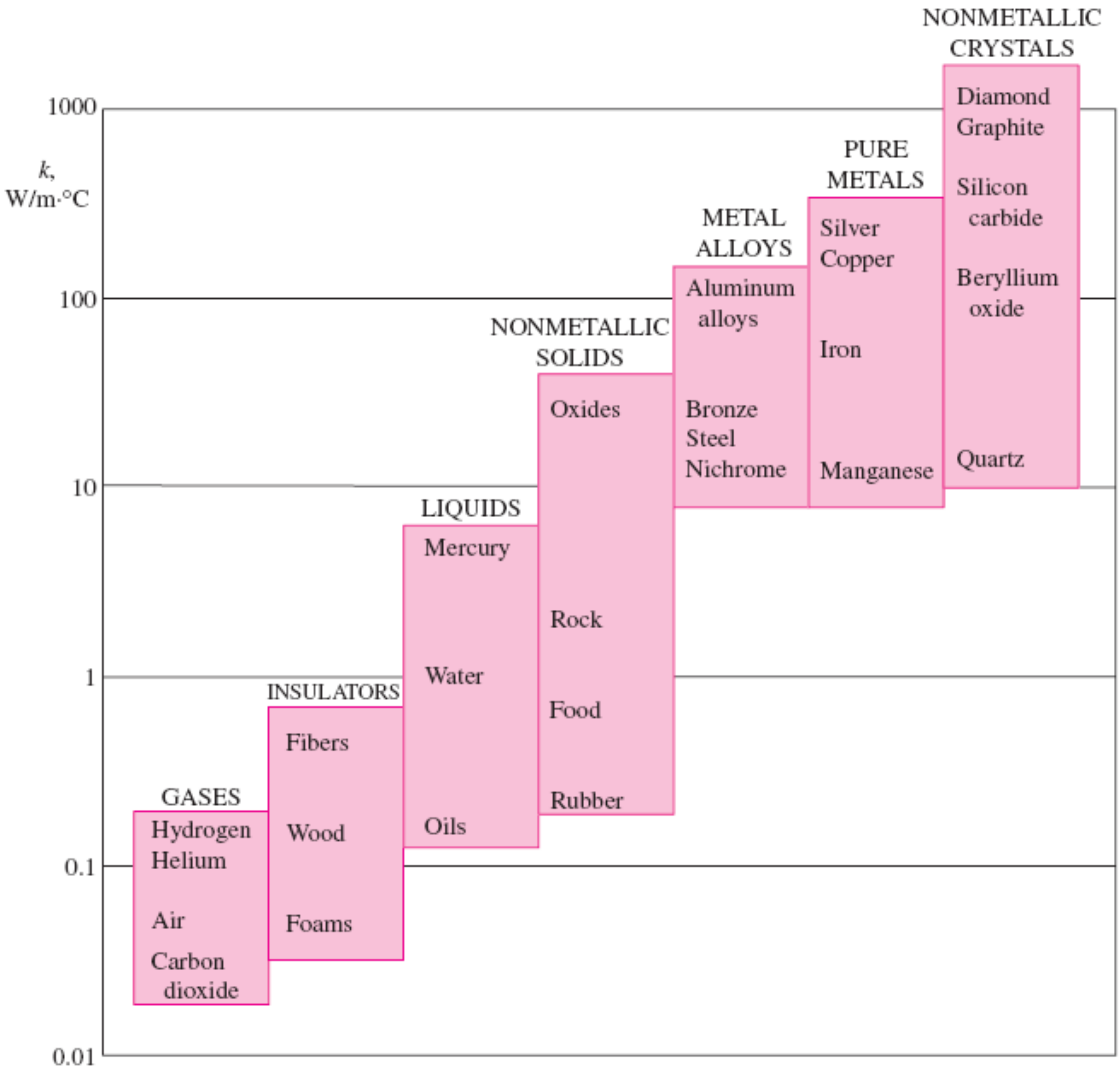
- Thermal conductivity k
    - The units are watts per meter per degree centigrade, $W\ m^{-1}\ {}^{\circ}C^{-1}$
    - Typical conductivity values in $W\ m^{-1}\ {}^{\circ}C^{-1}$:
        - Silver 420
        - Magnesium 160
        - Glass 1.2
        - Rock 1.7-3.3
        - Wood 0.1



The mechanisms of heat conduction in different phases of a substance.

The thermal conductivity of an alloy is usually much lower than the thermal conductivity of either metal of which it is composed

Pure metal or alloy	*k*, W/m · °C, at 300 K
Copper	401
Nickel	91
Constantan (55% Cu, 45% Ni)	23
Copper	401
Aluminum	237
Commercial bronze (90% Cu, 10% Al)	52



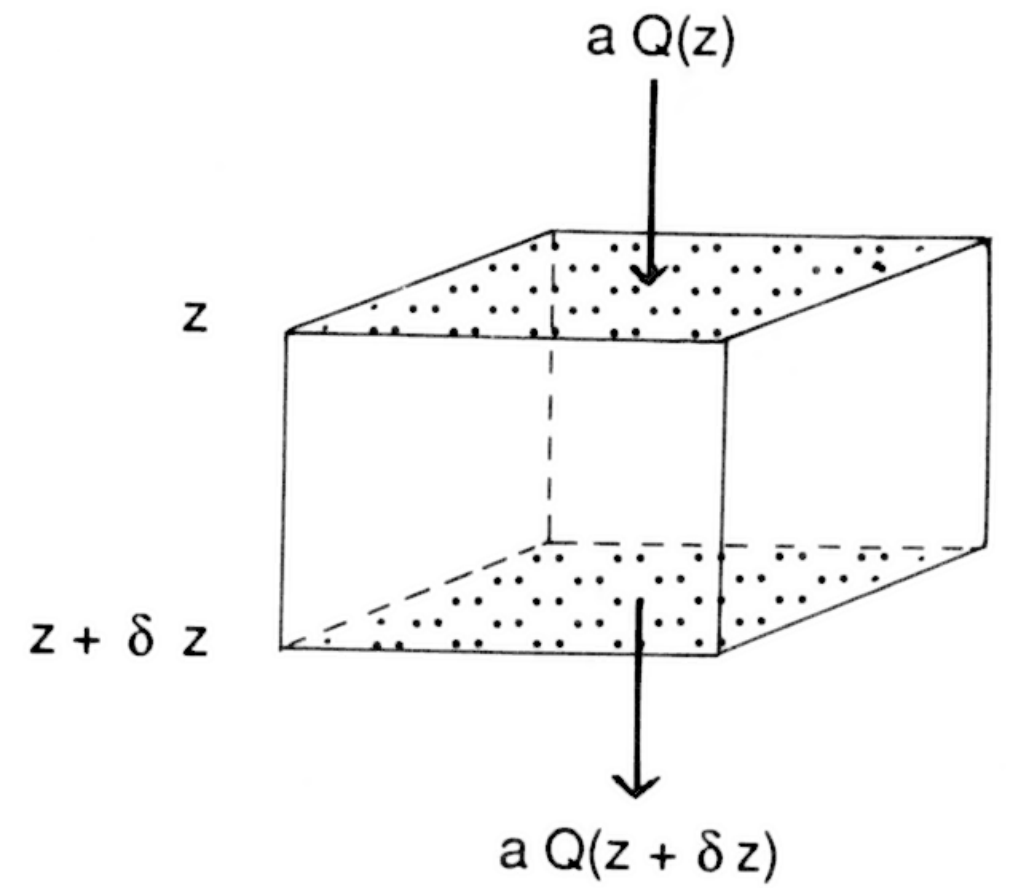
The range of thermal conductivity of various materials at room temperature.

Heat flow

Consider a small volume element of height δz and area a

Any change in the temperature of this volume in time δt depends on:

- Net flow of heat across the element's surface (can be in or out or both)
- Heat generated in the element
- Thermal capacity (specific heat) of the material



Heat flow

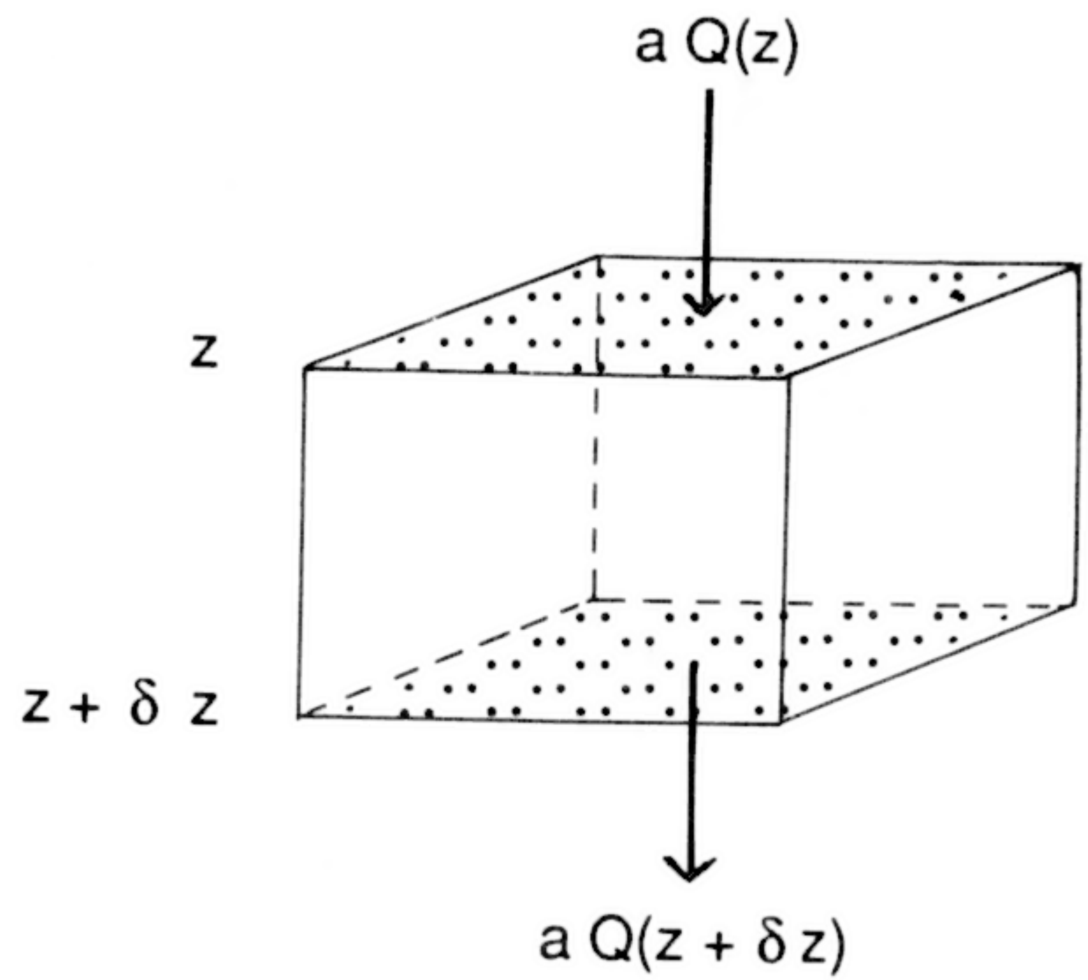
The heat per unit time **entering** the element across its face at z is aQ(z) .

The heat per unit time **leaving** the element across its face at z+δz is aQ(z+δz) .

Expand Q(z+δz) as Taylor series:

$$Q(z+\delta z) = Q(z) + \delta z \frac{\partial Q}{\partial z} + \frac{(\delta z)^2}{2!}\frac{\partial^2 Q}{\partial z^2} + \frac{(\delta z)^3}{3!}\frac{\partial^3 Q}{\partial z^3} + \ldots$$

The terms in $(\delta z)^2$ and above are small and can be neglected



The net change in heat in the element is (heat entering across z) minus (heat leaving across z+δz):

$$= aQ(z) - aQ(z+\delta z)$$

$$= -a\delta z \frac{\partial Q}{\partial z}$$

Suppose heat is generated in the volume element at a rate H per unit volume per unit time. The total amount of heat generated per unit time is then

$$H\ a\ \delta z$$

Radioactivity is the prime source of heat in rocks, but other possibilities include shear heating, latent heat, and endothermic/exothermic chemical reactions.

Combining this heating with the heating due to changes in heat flow in and out of the element gives us the total gain in heat per unit time (to first order in δz as:

$$Ha\delta z - a\delta z \frac{\partial Q}{\partial z}$$

This tells us how the amount of **heat** in the element changes, but not how much the **temperature** of the element changes.

The specific heat C_p of the material in the element determines the temperature increase due to a gain in heat.

Specific heat is defined as the amount of heat required to raise 1 kg of material by 1°C.

Specific heat is measured in units of J kg^{-1} $°C^{-1}$.

If material has density ρ and specific heat C_p, and undergoes a temperature increase of δT in time δt, the rate at which heat is gained is:

$$C_p a \delta z \rho \frac{\delta T}{\delta t}$$

We can equate this to the rate at which heat is gained by the element:

$$C_p a \delta z \rho \frac{\delta T}{\delta t} = H a \delta z - a \delta z \frac{\partial Q}{\partial z}$$

$$C_p a\delta z\rho\frac{\delta T}{\delta t} = Ha\delta z - a\delta z\frac{\partial Q}{\partial z}$$

Simplifies to:

$$C_p\rho\frac{\delta T}{\delta t} = H - \frac{\partial Q}{\partial z}$$

In the limit as δt goes to zero:

$$C_p\rho\frac{\delta T}{\delta t} = H - \frac{\partial Q}{\partial z}$$

Several slides back we defined Q as:

$$Q(z) = -k\frac{\partial T}{\partial z}$$

$$C_p\rho\frac{\partial T}{\partial t} = H + k\frac{\partial^2 T}{\partial z^2} \longrightarrow \frac{\partial T}{\partial t} = \frac{k}{C_p\rho}\frac{\partial^2 T}{\partial z^2} + \frac{H}{C_p\rho}$$

1D heat conduction equation

Heat equation

The term $k/(c_p\rho)$ is known as the thermal diffusivity α. The thermal diffusivity expresses the ability of a material to diffuse heat by conduction.

The heat conduction equation can be generalized to 3 dimensions:

$$\frac{\partial T}{\partial t} = \frac{k}{c_p \rho} \frac{\partial^2 T}{\partial z^2} + \frac{H}{c_p \rho}$$

The symbol in the center is the gradient operator squared, aka the Laplacian operator. It is the dot product of the gradient with itself.

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \qquad \nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla T = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right) \qquad \nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

$$\frac{\partial T}{\partial t} = \frac{k}{C_p \rho} \nabla^2 T + \frac{H}{C_p \rho}$$

This simplifies in many special situations.

For a **steady-state** situation, there is no change in temperature with time. Therefore:

$$\nabla^2 T = -\frac{H}{k}$$

In the **absence** of heat generation, H=0:

$$\frac{\partial T}{\partial t} = \frac{k}{C_p \rho} \nabla^2 T$$

Scientists in many fields recognize this as the classic "**diffusion**" equation.

Continuity and Heat Equation

- Conservation of energy says that energy cannot be created or destroyed: there is a continuity equation for energy *U*, is heat per unit volume, and its flow:

$$U = \rho C_p T$$

$$\frac{\partial U}{\partial t} + div(\boldsymbol{Q}) = 0$$

- When heat flows inside a medium, the continuity equation can be combined with Fourier's law, where *k* is thermal conductivity (W/(m K))

$$\boldsymbol{Q} = -k \; grad(T)$$

Continuity and Heat Equation

- When heat flows inside a solid, the continuity equation can be combined with Fourier's law to arrive at the heat equation, defining α (m^2/s) the heat diffusivity:

$$\frac{\partial T}{\partial t} - \frac{k}{\rho C_p}\Delta(T) = \frac{\partial T}{\partial t} - \alpha\Delta(T) = 0$$

- The equation of heat flow may also have source terms: Although energy cannot be created or destroyed, heat can be created from other types of energy, for example via friction or joule heating:

$$\frac{\partial T}{\partial t} - \alpha\Delta(T) = \sigma$$

Transport Equation

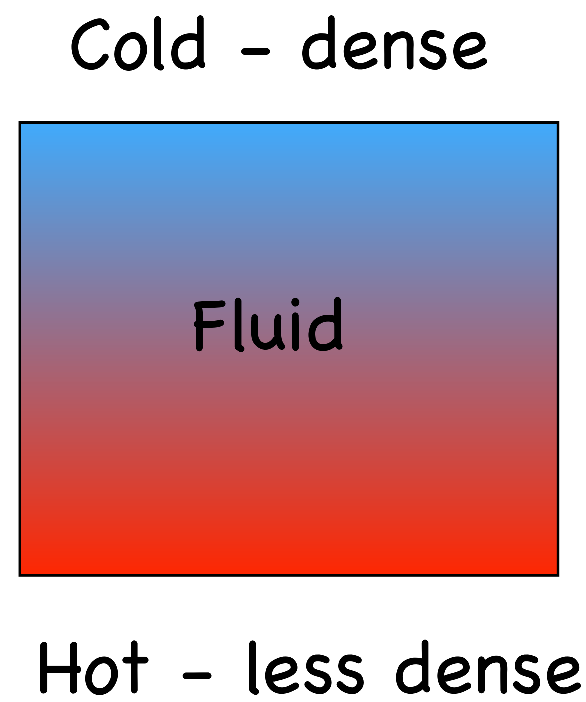
- The convection–diffusion equation is a combination of the diffusion and advection equations, and describes physical phenomena where particles, energy, or other physical quantities are transferred inside a physical system due to two processes: advection and diffusion.

$$\frac{\partial \rho}{\partial t} + div(\boldsymbol{j} - Dgrad(\rho)) = \sigma$$

- It can be derived in a straightforward way from the continuity equation, which states that the rate of change for a scalar quantity in a differential control volume is given by flow and diffusion into and out of that part of the system along with any generation or consumption inside the control volume

Convection

- Convection arises because fluids expand and decrease in density when heated
- The situation on the right is **gravitationally unstable** – hot fluid will tend to rise
- But viscous forces oppose fluid motion, so there is a competition between viscous and (thermal) buoyancy forces



- So convection will only initiate if the buoyancy forces are big enough

Peclet Number

- It would be nice to know whether we have to worry about the advection of heat in a particular problem
- One way of doing this is to compare the relative timescales of heat transport by conduction and advection:

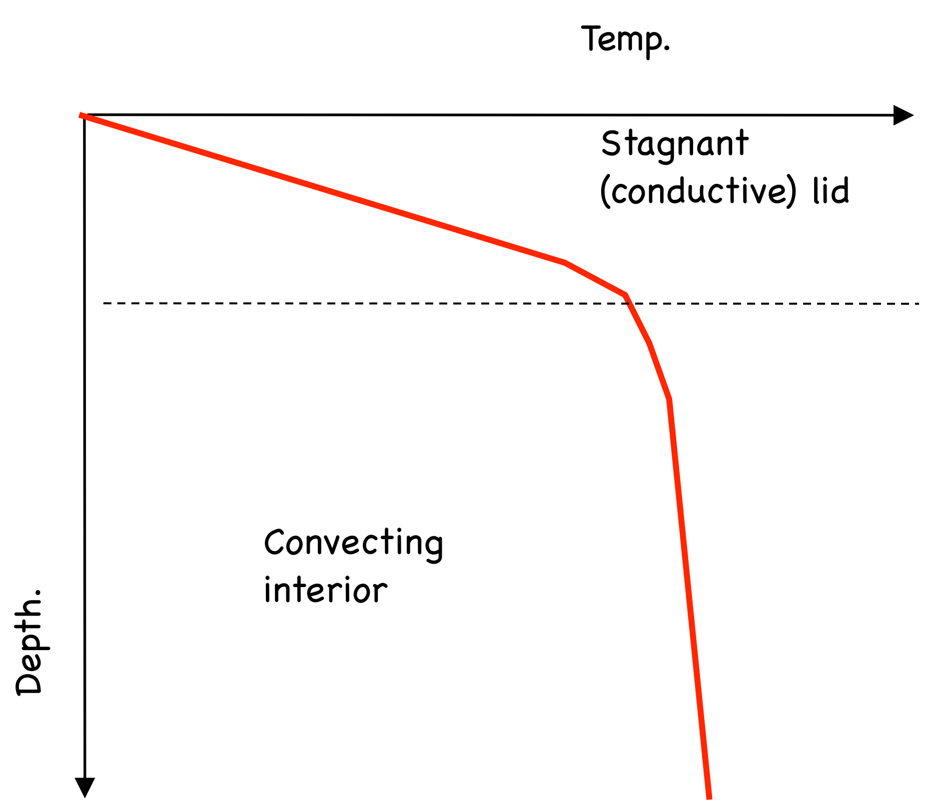
$$t_{cond} \sim \frac{L^2}{\alpha} \qquad t_{adv} \sim \frac{L}{u} \qquad \Rightarrow \qquad Pe \sim \frac{uL}{\alpha}$$

- The ratio of these two timescales is called a dimensionless number called the Peclet number Pe and tells us whether advection is important
- High Pe means advection dominates diffusion, and v.v.*
- E.g. lava flow, $u \sim 1$ m/s, $L \sim 10$ m, $Pe \sim 10^7$ $\therefore$ advection is important

* Often we can't ignore diffusion even for large Pe due to stagnant boundary layers

Cooling a planet (cont'd)

- Planets which are small or cold will lose heat entirely by conduction
- For planets which are large or warm, the interior (mantle) will be convecting beneath a (conductive) stagnant lid (also known as the lithosphere)



Rayleigh number

When the mass density difference is caused by temperature difference, Ra is, by definition, the ratio of:

- the time scale for diffusive thermal transport to
- the time scale for convective thermal transport at speed $u \sim \Delta\rho l^2 g/\eta$

$$Ra = \frac{l^2/\alpha}{\eta/\Delta\rho l g} = \frac{\Delta\rho l^3 g}{\eta\alpha} = \frac{\rho\beta\Delta T l^3 g}{\eta\alpha}$$

Here ρ is density, g is gravity, β is thermal expansivity, ΔT is the temperature contrast, l is the layer thickness, α is the thermal diffusivity and η is the viscosity. Note that η is strongly temperature-dependent.

Continuity and Moment Equation

- Other than advecting momentum, the only other way to change the momentum in our representative volume is to exert forces on it. These forces come in two flavors: stress that acts on the surface of the volume (flux of force) and body forces (acting as a source of momentum):

$$\frac{\partial(\rho \boldsymbol{V})}{\partial t} + div(\rho \boldsymbol{V}\boldsymbol{V}) = div(\boldsymbol{\tau}) + grad(\rho\phi)$$

or

$$\rho\frac{\partial \boldsymbol{V}}{\partial t} + \rho(V \bullet grad)\boldsymbol{V} = div(\boldsymbol{\tau}) + \rho\mathbf{g}$$

Navier-Stokes & Transport equations

- Coupled description, necessary for studies of convection inside the Earth at long time scales:

- **advective inertial term**
- **diffusion like viscosity term**
- **buoyancy gravity term**

$$\rho \frac{\partial \boldsymbol{V}}{\partial t} + \rho\left(\boldsymbol{V} \bullet grad\right)\boldsymbol{V} = \eta \Delta \boldsymbol{V} - grad(P) - \rho g \alpha T$$

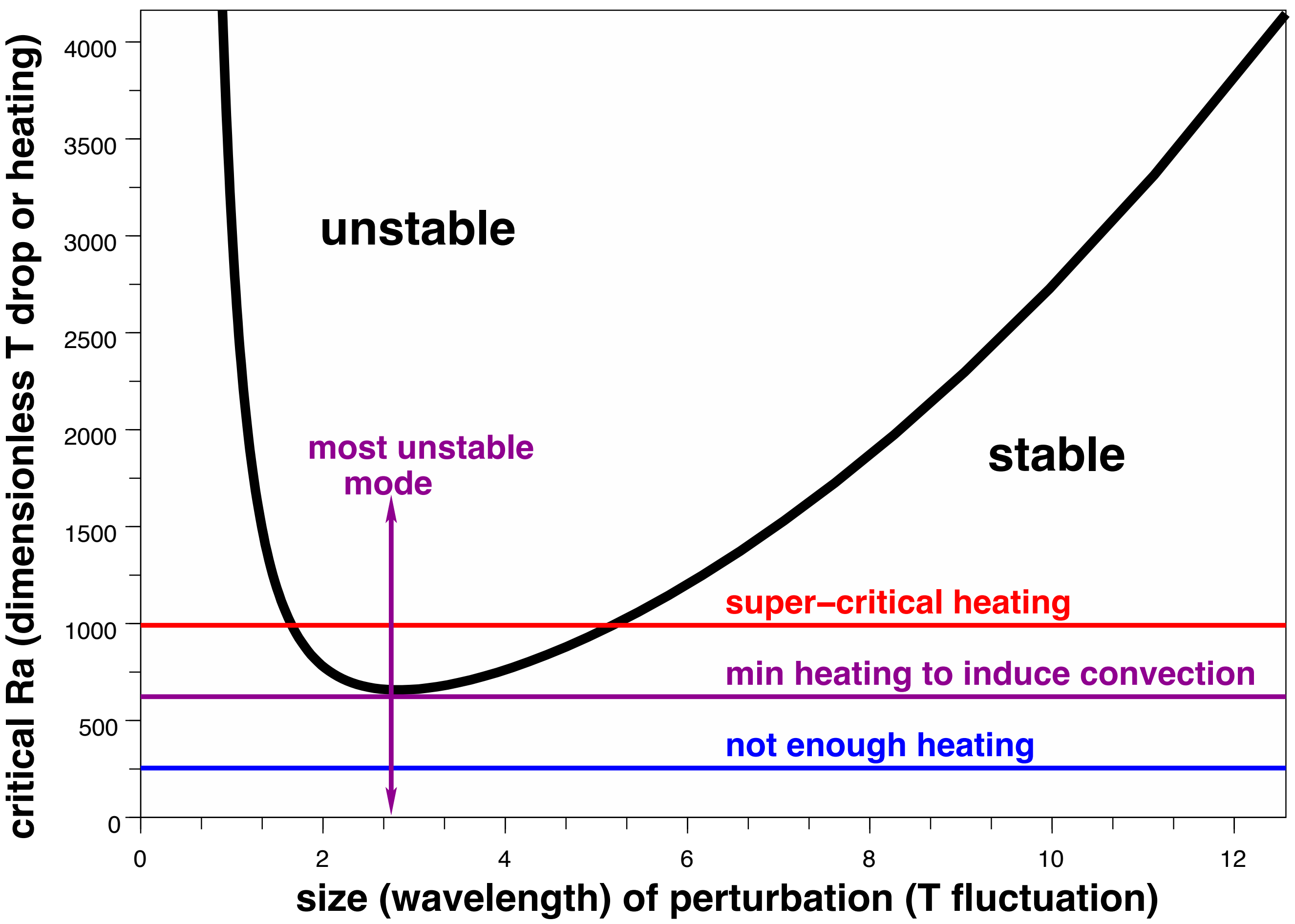
$$\frac{\partial T}{\partial t} = \alpha \Delta (T) - div(\boldsymbol{V}T) + \frac{H}{C_p}$$

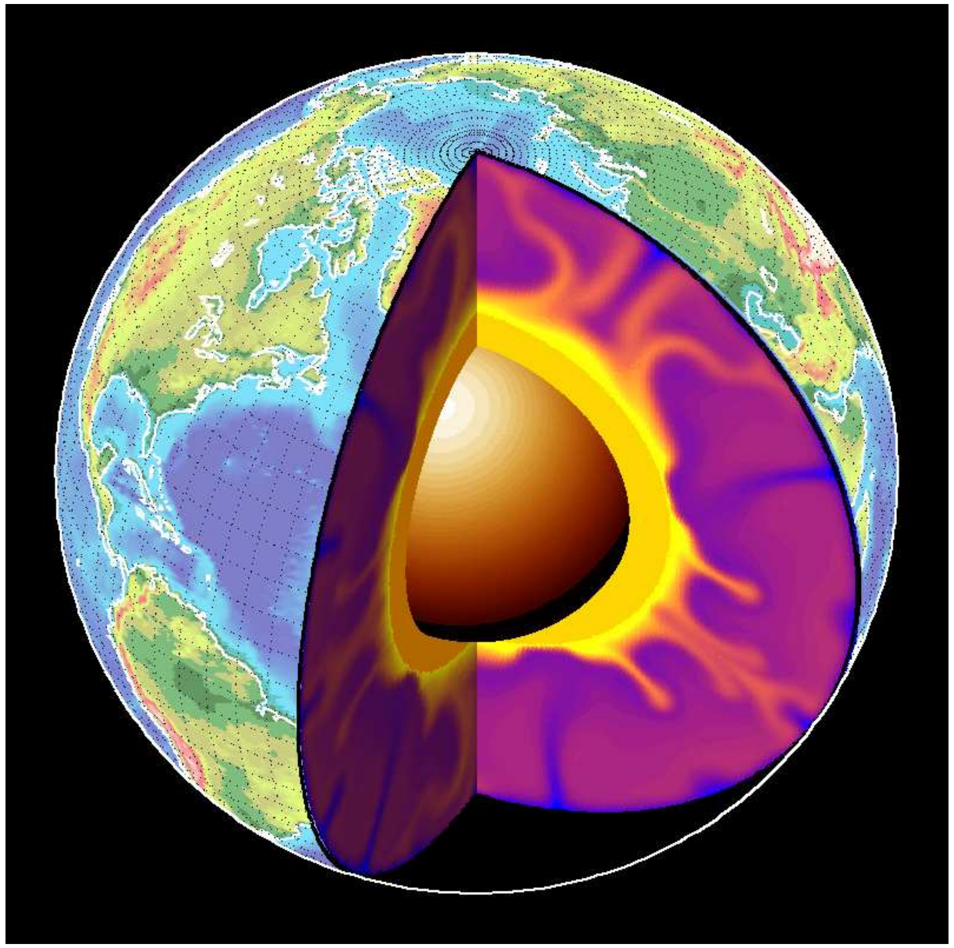
- **internal heating term**
- **conductive term**
- **advective term**

when the mass density difference is caused by temperature difference, Rayleigh number (Ra) is, the ratio of the time scale for diffusive thermal transport to the time scale for convective thermal transport

$$Ra = \frac{\Delta \rho l^3 g}{\eta \alpha}$$

Convection in the Mantle





Values of Ra above the Ra_c curve are associated with the conductive layer being convectively unstable (perturbations grow), while below the curve the layer is stable (perturbations decay). The minimum in the Ra_c curve occurs at the wavelength of the first perturbation to go unstable as heating and Ra is increased, often called the most unstable mode.

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