Computational Algebra

8. Exercises

Exercise: consider $I := (x^2 + xy + 1,\ y^3 - 1) \subseteq \mathbb{Q}[x,y]$; compute a $\mathbb{Q}$-basis of $\mathbb{Q}[x,y]/I$

Sol: we compute a Gröbner basis; for example we fix the DegLex term order with $x > y$, so we have:

$$I = (f_1, f_2) \quad f_1 = x^2 + xy + 1, \quad f_2 = y^3 - 1$$

$$S(f_1, f_2) = y^3 f_1 - x^2 f_2 = xy^4 + y^3 + x^2 \xrightarrow{f_2} xy + y^3 + x^2 \xrightarrow{f_2}$$

$$\longrightarrow xy + 1 + x^2 \xrightarrow{f_1} 0$$

so $G = \{f_1, f_2\}$ is a Gröbner basis for I; to compute a $\mathbb{Q}$-basis, we select those terms that are irreducible with respect to G, namely:

$$1,\ x,\ y,\ y^2,\ xy,\ xy^2$$

Exercise: let $I := (x^2 - y^2,\ x^3 - y^3)$, compute the reduced Gröbner basis of I with respect to the Lex term order with $x > y$

Sol: $I = (f_1, f_2)$ with $f_1 = x^2 - y^2 \quad f_2 = x^3 - y^3$

$$S(f_1, f_2) = x f_1 - f_2 = -xy^2 + y^3$$

we set $f_3 := -xy^2 + y^3$

we have to compute $S(f_1, f_3)$ and $S(f_2, f_3)$

$S(f_1,f_3) = y^2 f_1 + x f_3 = -y^4 + xy^3 \xrightarrow{f_3} 0$

$S(f_2,f_3) = y^2 f_2 + x^2 f_3 = -y^5 + x^2y^3 \xrightarrow{f_3} 0$

hence $G = \{f_1, f_2, f_3\}$ is a Gröbner basis for I, it is not minimal, since the leading coefficient of f_3 is -1; hence $\{f_1, f_2, -f_3\}$ is a minimal Gröbner basis, and it is reduced

Exercise: let $I = (t^2-x, tx-y) \subseteq \mathbb{Q}[x,y,t]$; compute $I \cap \mathbb{Q}[x,y]$; prove that $t^3 - y \in I$ and find two polynomials $a, b \in \mathbb{Q}[x,y,t]$ such that $t^3 - y = a(t^2-x) + b(tx-y)$

Sol: we fix an elimination order for t, for example Lex with $t > x > y$, then we compute a Gröbner basis for I.

$I = (f_1, f_2)$, $f_1 = t^2 - x$ $\quad f_2 = tx - y$

$S(f_1,f_2) = xf_1 - tf_2 = -x^2 + ty \qquad f_3 := ty - x^2$

$S(f_1,f_3) = yf_1 - tf_3 = -xy + tx^2 \xrightarrow{f_2} 0$

$S(f_2,f_3) = yf_2 - xf_3 = -y^2 + x^2 \qquad f_4 := x^2 - y^2$

$S(f_1,f_4) = x^2 f_1 - t^2 f_4 = -x^3 + t^2y^2 \xrightarrow{f_4} t^2y^2 + xy^2 \xrightarrow{f_3}$

$\longrightarrow xy^2 - tyx^2 \xrightarrow{f_2} 0$

$S(f_2,f_4) = xf_2 - tf_4 = -xy + ty^2 \xrightarrow{f_3} -xy + x^2y \xrightarrow{f_4}$

$\longrightarrow y^3 - xy \qquad f_5 := -xy + y^3$

$S(f_3,f_4) = x^2 f_3 - ty f_4 = -x^4 + ty^3 \xrightarrow{f_3} -x^4 + y^2x^2 \xrightarrow{f_4} 0$

$$S(f_1, f_5) = xyf_1 + t^2 f_5 = -x^2y + t^2y^3 \xrightarrow{f_4} -x^2y + xy^3 \xrightarrow{f_5} 0$$

$$S(f_2, f_5) = yf_2 + tf_5 = -y^2 + ty^3 \xrightarrow{f_3} -y^2 + x^2y^2 \longrightarrow y^4 - y^2$$

$$f_6 := y^4 - y^2$$

$$S(f_3, f_5) = xf_3 + tf_5 = -x^3 + ty^3 \xrightarrow{f_2} -xy^2 + ty^3 \xrightarrow{f_3}$$

$$\longrightarrow -xy^2 + x^2y^2 \longrightarrow -xy^2 + y^4 \longrightarrow -xy^2 + y^2$$

$$\longrightarrow -y^4 + y^2 \longrightarrow 0$$

$$S(f_4, f_5) = yf_4 + xf_5 = -y^3 + xy^3 \longrightarrow -y^3 + y^5 \longrightarrow 0$$

$$S(f_1, f_6) \longrightarrow 0$$

$$S(f_2, f_6) \longrightarrow 0$$

$$S(f_3, f_6) = y^3 f_3 - tf_6 = -x^2y^3 + ty^2 \longrightarrow -x^2y^3 + yx^2 \longrightarrow 0$$

$$S(f_4, f_6) \longrightarrow 0$$

$$S(f_5, f_6) = y^3 f_5 + xf_6 = y^6 - xy^2 \longrightarrow y^4 - xy^2 \longrightarrow$$

$$\longrightarrow y^2 - xy^2 \longrightarrow 0$$

so we found a Gröbner basis!

$$G = \{f_1, f_2, f_3, f_4, f_5, f_6\}$$

hence, $I \cap \mathbb{Q}[x, y] = (f_4, f_5, f_6)$, now

$$t^3 - y \longrightarrow tx - y \longrightarrow 0 \quad \text{hence } t^3 - y \in I$$

and we have $t^3 - y = (t^2 - x)(t) + (tx - y)$

Exercise: let $I \subsetneq k[x_1, \dots x_n]$ be an ideal, let G be the reduced Gröbner basis of I for a given term order; prove that if $x_i \in I$, then $x_i \in G$

Sol: if $x_i \in I$, then there exists $g_i \in G$ such that $LT_\leq(g_i)$ divides x_i, namely $LT_\leq(g_i) = x_i$ since I is not the whole ring; now consider the set $\tilde{G} := G \setminus \{g_i\} \cup \{x_i\}$; by construction, $\tilde{G}$ is also a reduced Gröbner basis of I for the same term order; from uniqueness, $g_i = x_i$.

Exercise: let $I := (xz-1, yz^3-1) \subseteq \mathbb{Q}[x,y,z]$ and consider Lex with $x > y > z$; compute $LT_\leq(I)$

Sol, let us compute a Gröbner basis:

$$I = (f_1, f_2), \quad f_1 = xz-1, \quad f_2 = yz^3 - 1$$

$$S(f_1, f_2) = yz^2 f_1 - x f_2 = -yz^2 + x \qquad f_3 = x - yz^2$$

notice that $I = (f_2, f_3)$, so we can restart

$$S(f_2, f_3) = x f_2 - yz^3 f_3 = -x + yz^5 \xrightarrow{f_2} -x + yz^2 \xrightarrow{f_3} 0$$

so $\{f_2, f_3\}$ is a Gröbner basis, which means

$$LT_\leq(I) = (x, yz^3)$$