Computational Algebra

10. Improvements to Buchberger's Algorithm

Buchberger's Algorithm for the computation of Gröbner bases is, unfortunately, slow. Very slow. This is why a lot of research has been devoted to improving it. The bottleneck of the algorithm is constituted by those S-pairs that reduce to zero: in fact, we do not gain anything from performing a reduction to zero; it would be enough to know that they reduce to zero beforehand. We are now going to see that this is possible, sometimes.

Def: let $s, t \in \mathbb{T}^n$ be terms; we say that s and t are disjoint if they have no variable in common

Prop: (Buchberger's first criterion)
let $f, g \in k[x_1, \dots, x_n]$ and let $\leq$ be a term order; suppose that $lt_{\leq}(f)$ and $lt_{\leq}(g)$ are disjoint terms; then $S(f,g) \xrightarrow[\{f,g\}]{} 0$

Proof: let us write:
$$f = \sum_{i=1}^{m} a_i s_i \quad \text{and} \quad g = \sum_{j=1}^{l} b_j t_j$$

with $a_i, b_j \in k$, $a_i \neq 0$, $b_j \neq 0$ for all i, j, and $s_i, t_j \in \mathbb{T}$; we can assume that $s_1 > \dots > s_m$ and $t_1 > \dots > t_l$; by assumption we have $\gcd(s_1, t_1) = 1$, therefore $\operatorname{lcm}(s_1, t_1) = s_1 t_1$; therefore the S-polynomial of f and g looks as follows:

$$S(f,g) = \frac{s_1 t_1}{a_1 s_1} f - \frac{s_1 t_1}{b_1 t_1} g = \frac{1}{a_1} \cdot t_1 f_1 - \frac{1}{b_1} s_1 g$$

then $a_1 b_1 S(f,g)$ is

$$b_1 t_1 f - a_1 s_1 g = b_1 t_1 \cdot \sum_{i=2}^{m} a_i s_i - a_1 s_1 \sum_{j=2}^{l} b_j t_j$$

Claim: the two sums above have no terms in common

Proof: if $t_1 s_i = s_1 t_j$, for some $2 \leq i \leq m$ and $2 \leq j \leq l$, then $s_i t_1$ is a common multiple of s_1 and t_1, so it is a multiple of $\operatorname{lcm}(s_1, t_1) = s_1 t_1$; then $s_1 t_1 \leq s_i \cdot t_1$, thus $s_1 \leq s_i$, which contradicts our assumption.

Moreover, each term in the second sum is a multiple of $lt_<(f)$; hence we have a chain of reduction steps

$$a_1 b_1 S(f,g) \xrightarrow{f} a_1 b_1 S(f,g) + b_l t_l f \xrightarrow{f} a_1 b_1 S(f,g) + b_l t_l f + b_{l-1} t_{l-1} f$$

$$\xrightarrow{f} \dots \longrightarrow a_1 b_1 S(f,g) + b_l t_l f + \dots + b_2 t_2 f$$

notice that, at each step of this reduction, all the terms $t_{j-1} s_1, \dots, t_2 s_1$ will still be present because they are strictly bigger than any term in $b_l t_l f + \dots + b_j t_j f$, hence all the steps of the equation above are actual reduction steps; we then see that

$$a_1 b_1 S(f,g) \xrightarrow[\{f\}]{} b_1 t_1 \sum_{i=2}^{m} a_i s_i + \sum_{i=2}^{m} a_i s_i \sum_{j=2}^{l} b_j t_j =$$

(since reducing by f amounts to the substitution

$$a_1 s_1 \longmapsto - \sum_{i=2}^{m} a_i s_i \;)$$

$$= g \sum_{i=2}^{m} a_i s_i \xrightarrow[\{g\}]{} 0$$

Example: consider the 2×2 minors of the matrix

$$\begin{pmatrix} x & y & z \\ y & z & w \end{pmatrix}$$

with entries in $\mathbb{Q}[x,y,z,w]$ and let I be the ideal generated by them; fix the DegLex term order; let us compute a Gröbner basis for I: the generators of I are

$$\underbrace{xz - y^2}_{f_1}, \quad \underbrace{yw - z^2}_{f_2}, \quad \underbrace{xw - yz}_{f_3}$$

from Buchberger's criterion we see that we do not need to check the first S-polynomial, namely $S(f_1, f_2)$, since we know that it reduces to zero;

$$S(f_1, f_3) = -y^2 w + yz^2 \xrightarrow[f_2]{} 0$$

$$S(f_2,f_3) = -xz^2 + y^2z \xrightarrow[f_1]{} 0$$

so we see that $\{f_1,f_2,f_3\}$ are already a Gröbner basis with respect to DegLex.

Another possibility of speeding up Buchberger's Algorithm is to notice that calling the algorithm with certain term orders tends to yield faster computations than the ones for other term orders. Heuristically, one notices that in general DegRevLex performs better than other term orders, and in particular better than the lexicographical ones. For certain purposes, for example for establishing ideal membership, it is irrelevant which term order we use, since all Gröbner bases allow one to solve the membership problem. However, this is not always the case: if we want to compute an elimination ideal, we should use an elimination order, which tends to be slow. A possibility then, in the zero-dimensional case, is to "convert" a Gröbner basis with respect to one term order to a Gröbner basis with respect to another one. This is not always possible, but it works when the final term order is the lexicographic one. Hence what one can do is to compute a Gröbner basis of a zero-dimensional ideal with a "fast" term order, as for example DegRevLex, and then convert it to Lex.

To understand how we can achieve this, we start with a little detour into how to compute linear algebra in a quotient $k[x_1, \ldots, x_n]/I$, which will lead to an algorithm for computing univariate polynomials in a zero-dimensional ideal without computing elimination ideals, which in turn will open the door to the possibility of a conversion algorithm.

Let us start with an algorithm that computes the reduced terms with respect to an ideal $I \subseteq k[x_1, \ldots, x_n]$.

<u>Algorithm</u> Reduced Terms

<u>Input</u>: a Gröbner basis G of a proper ideal $I \subseteq k[x_1, \ldots, x_n]$

natural numbers $k_1, \ldots, k_n \in \mathbb{N}$

<u>Output</u>: the set $R \subseteq \mathbb{T}$ of terms that are reduced with respect to I and such that $\deg_{x_i}(t) \leq k_i$ for all $i \in \{1, \ldots, n\}$ and for all $t \in R$.

1. Set $R := \{1\}$
2. For $i \in \{1, \ldots, n\}$:
3. Set $T := R$
4. While $T \neq \emptyset$:
5. Pick $t \in T$
6. Remove t from T
7. For $l \in \{1, \ldots, k_i\}$:

8. Set $t := t \cdot x_i$

9. If t is reduced with respect to G,

10. Append t to R

11. Return R.

Now, suppose that $I \subseteq k[x_1, \ldots, x_n]$ is zero-dimensional. Then we know already that for every $i \in \{1, \ldots, n\}$, there exists $f_i \in I \cap k[x_i]$. This polynomial can be computed via eliminations, but can be also computed by linear algebra: consider, in fact, the set

$$C_{i,m} := \{[1]_I, [x_i]_I, [x_i^2]_I, \ldots, [x_i^m]_I\} \subseteq k[x_1, \ldots x_n]/I$$

Then the polynomials $f_i \in I \cap k[x_i]$ correspond to non-trivial dependencies among the elements of $C_{i,m}$, and these can be computed once we can compute a basis of $k[x_1, \ldots x_n]/I$, which is the case due to the previous algorithm, which we can instantiate with the values $\{k_i\}$ obtained from any Gröbner basis. There is also another way of performing this computation.

Let $I \subseteq k[x_1, \ldots x_n]$ be zero-dimensional, let G be a Gröbner basis of I, let $t, s_1, \ldots, s_m \in \mathbb{T}$ be terms and let $h, h_1, \ldots, h_m$ be their normal remainders with respect to G. Let $y_1, \ldots, y_m$ be new variables, and set

$$p := h + \sum_{j=1}^{m} y_j \cdot h_j \in k[\underline{y}][\underline{x}]$$

Let now C be the set of all coefficients (elements of $k[\underline{y}]$) of p seen as a polynomial in $(k[y])[x]$. Then each $f \in C$ is linear in each variable y_j, so we can consider the following set of solutions of a linear system:

$$S = \{ a \in k^m : f(a) = 0 \text{ for all } f \in C \}$$

<u>Lemma</u>: in the situation above, $a \in S$ if and only if the polynomial

$$g = t + \sum_{j=1}^{m} a_j s_j$$

lies in I.

<u>Proof</u>: by construction $t - h \in I$ and $s_j - h_j \in I$ for all $j \in \{1, \dots, m\}$, then

$$\left(t + \sum_{j=1}^{m} a_j s_j \right) - \underbrace{\left(h + \sum_{j=1}^{m} a_j h_j \right)}_{} \in I$$

for every $a \in k^m$; notice that the polynomial above is nothing but $p(a, x)$; hence we conclude that $t + \sum_{j=1}^{m} a_j s_j \in I$ if and only if $p(a, x) \in I$; however, from the definition of p, we see that all the terms in $p(a, x)$ are reduced with respect to I, which in turn implies that the only possibility for it to be in I is that all its coefficients as a polynomial in x are zero, which in turn implies that $f(a) = 0$ for all $f \in C$, i.e. $a \in S$; notice then that all these implications are "if and only if".

This lemma provides the theoretical justification for the following algorithm.

<u>Algorithm</u> UnivariatePolynomial

<u>Input</u>: a Gröbner basis G of a zero-dimensional ideal I

a natural number $i \in \{1, \dots, n\}$

Output: the unique monic polynomial of minimal degree in $I \cap k[x_i]$

1. Set $N := \min\{\nu : x_i^\nu \in LT_\leq(G)\}$
2. Set $q := y_N x_i^N + \dots + y_1 x_i + y_0$
3. Set $t := x_i^{N+1}$
4. While True:
5. Set h to be the normal remainder of t with respect to G
6. Set $p := h + q$
7. Set C to be the list of coefficients of p once we consider it as a polynomial in $(k[\underline{y}])[\underline{x}]$
8. Set $S := \{a \in k^{N+1} : f(a) = 0$ for all $f \in C\}$
9. If $S \neq \emptyset$:
10. Pick an element $a \in S$
11. Set $f := x_i^{N+1} + \sum_{j=0}^{N} a_j x_i^j$
12. Return f
13. Else:
14. $N := N+1$, $t := t \cdot x_i$, $q = y_N h + q$

Now we are ready to explore the conversion algorithm between Gröbner bases of zero-dimensional ideals, which uses ideas similar to the ones of Univariate Polynomial. The idea is to check for bigger and bigger terms t whether there is a polynomial in the new Gröbner basis that has t as leading term. The algorithm skips all terms that are multiples of a leading term of an element which is already in the Gröbner basis. Therefore, we will require that the following property holds for the new term order:

(*) given $t \in \mathbb{T}$ and $S \subseteq \mathbb{T}$, one can decide whether

$$N := \{ u \in \mathbb{T};\ t < u \text{ and } s \nmid u \text{ for all } s \in S \}$$

is empty or not, and in the latter case compute an element of N that is minimal with respect to the term order.

For such a term order, we denote by $\text{MinimalTerm}_{\leq}$ the algorithm that takes t and S as inputs, and returns False if the set N in (*) is empty, or it returns (True, u) if N in (*) is not empty and $u \in N$ is the minimal element with respect to $\leq$. We can now state the conversion algorithm:

<u>Algorithm</u> ConvertGröbner

<u>Input</u>: a Gröbner basis with respect to some term order $\leq'$, say G', of a zero-dimensional ideal

<u>Output</u>: the Gröbner basis G with respect to a term order $\leq$ that satisfies (*) of (G') and the set R of reduced terms with respect to $\leq$.

1. Set $G = \emptyset$, $H = \emptyset$, $t = 1$, $R = \{1\}$, $\underline{y} = \{y_1\}$, $q = y_1$
2. While $\text{MinimalTerm}_{\leq}(t, H) \neq \text{False}$:
3. $\quad t = \text{MinimalTerm}_{\leq}(t, H)[2]$
4. $\quad h =$ the normal remainder of t with respect to G'
5. $\quad p = h + q$
6. $\quad C = \{\text{coefficients of } p \text{ as a polynomial in } (k[\underline{y}])[\underline{x}]\}$

$S = \{\underline{a} \in k^R : f(\underline{a}) = 0 \ \forall f \in C\}$

If $S \neq \emptyset$:

  Pick $\underline{a} \in S$

  Set $g = t + \sum_{r \in R} a_r \cdot r$

  Set $H = H \cup \{t\}$

  Set $G = G \cup \{g\}$

Else:

  $\underline{y} = \underline{y} \cup \{y_t\}$ (new indeterminate)

  $q = q + y_t \cdot h$

Return (G, R)

__Propo__: Algorithm ConvertGröbner terminates in finite time and is correct.

Proof: let us prove that the algorithm terminates and let us assume for a contradiction, that it does not; then the if condition or the else condition is satisfied infinitely many times; suppose first, that the if condition is satisfied infinitely many times, then we get that H has more and more elements; let us denote by $\{t_s\}_{s \in \mathbb{N}}$ the sequence of elements that get added to H, in order; then, the specification of MinimalTerm$_<$ implies that it holds $t_i \nmid t_j$ if $i < j$, but this contradicts Dixon's lemma; let us now assume that the else condition is satisfied infinitely many time; notice that through the while loop the quantity $p = \sum_{r \in R} y_r h_r$ (where h_r is the normal remainder of r modulo G'); then let $\{\Xi_u\}_{u \in \mathbb{N}}$ be

the sequence of terms that are added to R in their order;
since the if-statement condition is not satisfied, then the set S of solutions of the linear equations is empty, which means, in light of the previous lemma, that (G') does not contain any polynomial of the form

$$f_u = r_u + \sum_{i=1}^{u-1} a_i r_i \qquad \text{with } a_i \in k$$

from this we get infinitely many linear independent elements in $k[x_1, \ldots, x_n]/(G')$, thus contradicting the hypothesis that (G') is zero-dimensional.

(we do not prove correctness).

So far, we discussed how to convert a Gröbner basis of an ideal with respect to a given term order to a Gröbner basis of the same ideal with respect to another term order, provided that:

1. the ideal is zero-dimensional
2. the new term order allows one to compute, given a term $t \in \mathbb{T}^n$ and a set of terms S, the minimal element of the set

$$\{ u \in \mathbb{T}^n : \ t < u \ \text{and} \ s \nmid u \ \ \forall s \in S \}.$$

when this set is non-empty.

Key point: we reduce the computation of a Gröbner basis to the solution of a linear system.

This algorithm is usually known as FGLM.

To deal with positive-dimensional ideals, we need to introduce a new framework and a new set of ideas

Theorem: let $I \subseteq k[x_1, \dots, x_n]$ be an ideal, then there exist finitely many reduced Gröbner bases of I (with respect to any possible term order).

Theorem: let $I \subseteq k[x_1, \dots, x_n]$ be an ideal, then there exist finitely many leading term ideals of I.

Proof: by contradiction, suppose that I admits infinitely many leading term ideals; then $I \neq (0)$; then, there exists $f_1 \in I$, with $f_1 \neq 0$; now, f_1 has finitely many terms and every leading term ideal contains one of these finitely many terms; it follows that there exists a term t_1 of f_1 such that infinitely many leading term ideals of I contain t_1; let Σ_1 be the set of those leading term ideals; set $J_1 := (t_1)$; since Σ_1 is infinite, there exists $\overline{M} \in \Sigma_1$ that strictly contains J_1; recall that the terms in $k[x_1, \dots, x_n]$

that are not contained in a leading term ideal of I determine a basis for $k[x_1, \dots, x_n]/I$ over k; since we have $J_1 \subsetneq \overline{M}$, the terms not contained in J_1 cannot form a basis of $k[x_1, \dots x_n]/I$ once we take their classes; then their classes mod I are linearly dependent over k; a linear dependence between these classes of terms determines a polynomial $f_2 \in I$ with the property that no term in f_2 belongs to J_1; now we can repeat with f_2 what we have done with f_1, and so we can find a term t_2 of f_2 and a set Σ_2 of leading term ideals with the property

$$\Sigma_2 = \{M \in \Sigma_1 : t_2 \in M\} \quad \text{and} \quad |\Sigma_2| = +\infty$$

now we set $J_2 = J_1 + (t_2)$ and in this way we obtain $J_1 \subsetneq J_2 \subsetneq \dots$ a strictly increasing chain of monomial ideals, which contradicts Dickson's lemma.

Hence, in particular, given an ideal $I \subseteq k[x_1, \dots x_n]$, we can join all its finitely many reduced Gröbner bases and obtain a set of polynomials that is a Gröbner basis for I with respect to any term order. This is called a universal Gröbner basis.

The goal now is to put this finite set of leading term ideals into a geometric setting. More precisely, we want to introduce a polyhedral fan such that its cones of maximal dimension correspond to leading term ideals.

To understand these words, we make a detour into polyhedral geometry.

Def: a set $U \subseteq \mathbb{R}^n$ is called convex if $\forall u, v \in U$, $\forall \lambda \in [0,1]$

we have $\lambda u + (1-\lambda) v \in U$

Def: the convex hull of a set $V \subseteq \mathbb{R}^n$ is the intersection of all convex sets containing V.

Def: a convex polyhedron is the intersection of finitely many half-spaces, i.e., it is a set of the form

$$\{ x \in \mathbb{R}^n ; \; Ax \leq b \}$$

where A is an $m \times n$ matrix with coefficients in $\mathbb{R}$ and $b \in \mathbb{R}^m$

Rmk: the formulation "$Ax \leq b$" allows one to encode also equations:

$$x_1 + x_2 = 2 \iff \begin{cases} x_1 + x_2 \leq 2 \\ -x_1 - x_2 \leq 2 \end{cases}$$

Def: a bounded convex polyhedron is called a polytope.

Theorem: a subset of $\mathbb{R}^n$ is a polytope if and only if it is the convex hull of finitely many points.

In this sense we say that polytopes have:

- H-representations (intersections of half spaces)
- V-representations (convex hulls of points)

Def: the dimension of a convex polytope is the dimension of the smallest affine space containing it.

Rmk: H-representations are good for

- membership problems
- intersections

V-representations are good for

- computing the dimension.

Def: a convex polyhedron of the form

$$\{x \in \mathbb{R}^n : Ax \leq 0\}$$

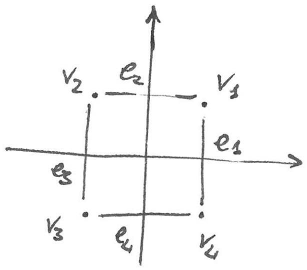
is called a convex cone. (polyhedral cone)

Rmk: if C is a convex polyhedral cone and $u \in C$ and $\lambda \in \mathbb{R}_{\geq 0}$, then $\lambda \cdot u \in C$.

Def: a face of a polyhedron P is either the empty set or the subset of P maximizing a linear functional; i.e., if $\omega \in \mathbb{R}^n$,

we set $\quad \mathrm{face}_\omega(P) = \{ p \in P : \langle \omega, p \rangle = \max_{q \in P} \langle \omega, q \rangle \}$.

Example: consider $P = \text{convex hull}(\{(\pm 1, \pm 1)\}) \subset \mathbb{R}^2$



$P = \mathrm{face}_0(P)$

$\mathrm{face}_{(1,1)}(P)$ is the set

$$\{ (a,b) \in P : a+b = \max_{(c,d) \in P} c+d \}$$

$$= \{ (1,1) \}$$

similarly we get

$$\mathrm{face}_{(\pm 1, \pm 1)}(P) = \{ (\pm 1, \pm 1) \}$$

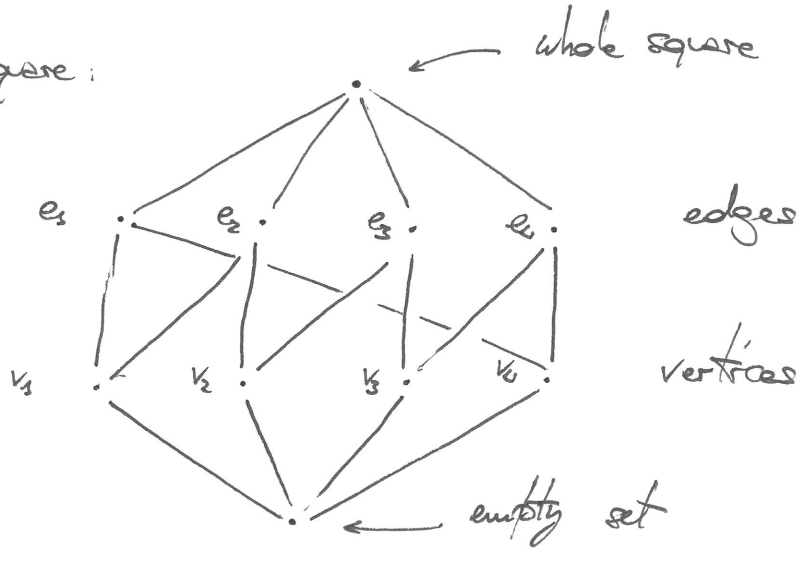
moreover

$$\mathrm{face}_{(1,0)}(P) = \{ (a,1) : -1 \le a \le 1 \} =$$

$$= \text{convex hull}(\{(1,1), (1,-1)\})$$

Faces come with a structure of a poset (partially ordered set) with respect to inclusion.

Example: for the square:



Rmk: a face of a face is a face.

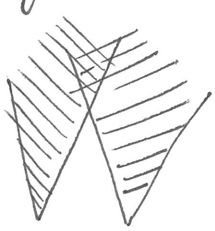
Def: a facet is a face of codimension 1.

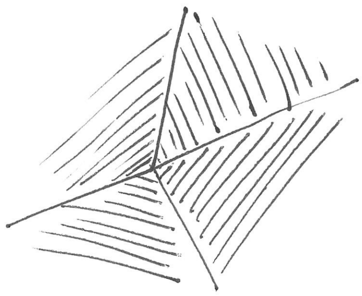
Def: a collection $\mathcal{E}$ of polyhedra is a polyhedral complex if:

- every face of every polyhedron in $\mathcal{E}$ is in $\mathcal{E}$, i.e., $\mathcal{E}$ is closed under taking faces.
- for every $P_1, P_2 \in \mathcal{E}$, the intersection $P_1 \cap P_2$ is a face of both P_1 and P_2

the support of $\mathcal{E}$ is the union of the polyhedra in $\mathcal{E}$;

if all polyhedra in $\mathcal{E}$ are cones, then $\mathcal{E}$ is called a fan.

Example:  this is not a fan

 this is a fan

Rmk: notice that the same set can be the support of different polyhedral fans.

Now we discuss how polyhedral geometry enters Gröbner bases theory, and it does so via weight vectors.

Def: let $w \in \mathbb{R}^n$ and $t \in \mathbb{T}^n$ be a term; then the degree of t

with <u>respect to ω</u>, or its <u>ω-degree</u>, is defined as

$$\langle \log(t), \omega \rangle \in \mathbb{R}$$

(notice that the standard total degree is the ω-degree where $\omega = (1, 1, \ldots, 1)$, while the degree with respect to a variable x_i is the ω-degree with $\omega = (0, \ldots, \overset{(i)}{1}, \ldots, 0)$)

if $f \in k[x_1, \ldots, x_n]$ is a polynomial, then we say that f is <u>ω-homogeneous</u> if all terms in the support of f have the same ω-degree; an ideal $I \subseteq k[x_1, \ldots, x_n]$ is called <u>ω-homogeneous</u> if it is generated by ω-homogeneous polynomials;

moreover, given ω and f as above we define the <u>initial form of f with respect to ω</u> as follows:

$$in_\omega(f) = \sum_{\{t \in \mathrm{supp}(f) : \langle \log(t), \omega \rangle \text{ is maximal}\}} c_t \cdot t$$

coefficient of t in f

<u>Example</u>: consider $f \in \mathbb{Q}[x, y, z]$

$$f = xyz + x^2y + xy^2 + yz^2$$

- take $\omega = (1, 2, 7)$, then $in_\omega(f) = yz^2$
- take $\omega = (5, 1, 4)$, then $in_\omega(f) = x^2y$
- take $\omega = (5, 2, 4)$, then $in_\omega(f) = x^2y$
- take $\omega = (1, 1, 2)$, then $in_\omega(f) = yz^2$

- take $\omega = (2,1,2)$, then $in_\omega(f) = x^2y + yz^2$

so the initial form needs not to be a monomial

- take $\omega = (1,1,1)$, then $in_\omega(f) = f$

question: can we choose ω so that $in_\omega(f) = LT_\leq(f)$ for some term order $\leq$? for example, take $\leq = Lex$ with $x > y > z$; then

$$LT_\leq(f) = x^2y$$

so for $\omega = (6,1,1)$ we have $in_\omega(f) = LT_{Lex}(f)$ (this holds also for many other $\omega \in \mathbb{R}^3$)

we see, however, that the chosen ω works well for this particular polynomial f, but there exists no $\omega \in \mathbb{R}^3$ such that for all $f \in \mathbb{Q}[x,y,z]$ we have $in_\omega(f) = LT_{Lex}(f)$.

Theorem: (Robbiano, Mora)

let $\leq$ be a term order on $\mathbb{T}^n$; there exists a matrix $A_\leq$ with n rows and n columns, where $\tau_1, \dots, \tau_n$ are the rows and such that for all $u, v \in \mathbb{N}^n$ we have

$$x^u < x^v \iff \text{there exists } j \in \{1, \dots, n\} \text{ such that}$$

$$\langle \tau_j, u \rangle < \langle \tau_j, v \rangle$$

and for all $i < j$ we have

$$\langle \tau_i, u \rangle < \langle \tau_j, v \rangle$$

(notice that $A \in \mathbb{R}^{n \times n}$)

Def: given a term order $\leq$ on $\mathbb{T}^n$ and a vector $w \in \mathbb{R}^n_{\geq 0}$ we can define a new term order $\leq_w$ by saying

$$x^\alpha \leq_w x^\beta \iff \langle w, \alpha \rangle < \langle w, \beta \rangle \text{ or } \langle w, \alpha \rangle = \langle w, \beta \rangle \text{ and } x^\alpha \leq x^\beta$$

(this is a generalization of DegLex term orders)

Def. let $I \subseteq k[x_1, \dots, x_n]$ be an ideal and let $w \in \mathbb{R}^n_{\geq 0}$; we define the initial ideal of I with respect to w as

$$in_w(I) = (\{ in_w(f) : f \in I \})$$

this construction mimics the one of leading term ideals, but here we do not have monomial ideals anymore.

Def: given an ideal $I \subseteq k[x_1, \dots, x_n]$ we introduce an equivalence relation on $\mathbb{R}^n$:

$$u \sim v \iff in_u(I) = in_v(I)$$

we define

$$C_v(I) = \overline{\{ u \in \mathbb{R}^n : in_u(I) = in_v(I) \}}$$

closure with respect to the standard Euclidean topology.

moreover, we define analogous objects for leading terms:

$$C_{\leq}(I) = \overline{\{ u \in \mathbb{R}^n : in_u(I) = LT_{\leq}(I) \}}$$

We are going to establish that every $C_\leq(I)$ is of the form $C_v(I)$ for some $v \in \mathbb{R}^n$.

<u>Lemma</u>: let $\leq$ be a term order, let $I \subseteq k[x_1, \dots, x_n]$ be an ideal; let G be the reduced Gröbner basis of I with respect to $\leq$, let $u \in \mathbb{R}^n$, then

$$in_u(I) = LT_\leq(I) \leftrightarrow \forall g \in G, \quad in_u(g) = LT_\leq(g)$$

<u>Cor</u>: given an ideal $I \subseteq k[x_1, \dots, x_n]$, every $C_\leq(I)$ is of the form $C_w(I)$ for some $w \in \mathbb{R}^n$.

The next is the key result.

<u>Prop</u>: let $\leq$ be a term order and let $v \in C_\leq(I)$; for $u \in \mathbb{R}^n$:

$$in_u(I) = in_v(I) \leftrightarrow \forall g \in G, \quad in_u(g) = in_v(g)$$

where G is the reduced Gröbner basis of I with respect to $\leq$.

The previous proposition tells us that $C_\leq(I)$ is a polyhedron: in fact, once we fix v, the conditions on u such that $in_u(I) = in_v(I)$ are linear inequalities! More precisely, $C_\leq(I)$ is a polyhedral cone.

<u>Def</u>: the <u>Gröbner fan</u> of an ideal $I \subseteq k[x_1, \dots, x_n]$ is the set of cones $C_v(I)$ that intersect the positive orthant, together with their faces.

<u>Theorem</u>: the Gröbner fan is a polyhedral complex and a fan.