Towards a PCK of Physics and Mathematics interplay

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Abstract

The present study examined the intertwining of physics and mathematics within the context of physics teaching (Phys-Math interplay) through interviews conducted with experienced high school physics teachers from two countries. The teachers were asked about their views with regard to the importance of the topic at hand and to provide examples of how they address it in their teaching. The examples were categorized and fitted to an adopted theoretical PCK framework. Implications with regard to physics teaching were suggested.

Keywords:

Secondary education: upper (ages about 15-19), Physics and Mathematics Interplay, Physics Teachers, Teaching.

Introduction

The intertwining of physics and mathematics has a long history and was studied from the perspectives of history of science and its philosophy. Physics and mathematics are also heavily interwoven in learning physics at many levels. However, research indicates that learners, at different ages and levels, lack the ability to construct the mathematical models of physical processes or to describe the physical meaning of mathematical constructs. Clement et al. (1981) reported on the pitfalls freshman engineering majors encounter when asked to construct equations to match situations described in words. Bagno et al. (2007) carried out diagnostic studies showing that high- school students face difficulties describing the physical meaning of formulas. Cohen et al. (1983) found that both high school students and their teachers often fail in qualitative reasoning on DC circuits despite the fact that they are able to apply correctly the relevant mathematical algorithms. Rebmann and Viennot (1994) discuss the difficulty of many university physics students in applying and interpreting algebraic sign conventions consistently in a variety of topics. In the past, mathematics within the physics education context was mainly examined within the context of problem solving (Bagno et. al., 2007; Redish & Smith, 2008). Some researchers pointed out that there is a blending of conceptual and formal mathematical reasoning during the mathematical processing stage (Kuo et. al., 2013, Hull et. al., 2013).

With regard to teachers' mathematical competency, Baumert et al. (2010) have shown that teachers' mathematical knowledge highly reflects on the quality of their explanations of physical phenomena. Recently, a broader view has been suggested, according to which the context of physics teaching invites interplay between physics and mathematics (Eylon et. al., 2010).

We report here on a bi-national study in which we have studied expert high school physics teachers' views with regard to the "Phys-Math" interplay and the measure they take to implement it.

We have attempted to reveal from our data the teaching treks employed by the instructors when travelling between physics and mathematics, and their strategies in doing so. The treks, referred by us as teaching patterns, were then fitted to a general PCK framework. The existence of such patterns was suggested earlier (Lehavi et. al, 2013). We have further inspected our data to reveal how the teaching patterns were manifested by different teaching sequences.

In the present study we adopted the PCK framework suggested earlier (Magnusson et al., 1999). (figure 1):

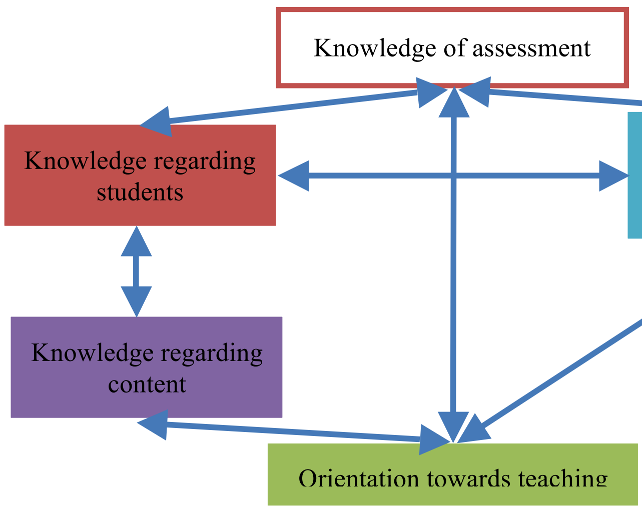


Figure 1: The adopted PCK framework

Magnusson mentioned several orientations toward science teaching: Help students develop the ‘science process skills’; Represent a particular body of knowledge; Transmit the facts of science; Facilitate the development of scientific knowledge by confronting students with contexts to explain that challenge their naïve concepts; Involve students in investigating solutions to problems; Represent science as inquiry; Constitute a community of learners whose members share responsibility for understanding the physical world, particularly with respect to using tools for science.
We limited our analysis of the interviews to the following components of the PCK framework: teachers' orientations towards teaching the Phys-Math interplay as it is related to knowledge regarding the content of this interplay and their knowledge of successful teaching strategies.

The research method

In pursuing the goal of finding the characteristics of the teachers' orientations towards teaching the Phys-Math interplay, we have interviewed highly experienced high school physics teachers (N = 9) from Israel (N = 8) and Germany (N = 1). Some of the teachers in our sample are considered to be master teachers.
We employed open interviews in which we asked the teachers how they construct the Phys-Math interrelation in their classroom and how they use it to enhance students' understanding of physics. Our interviewees were asked to address such questions as:
How do you construct the Phys-Math interrelation within your teaching? What mathematical insights do you use in developing insights in physics? What insights in physics are impossible to be developed without the aid of mathematics? Are there some important aspects of the interrelation between math and physics that you do not succeed in bringing them into your teaching?
The teachers provided us with examples from their own experience and also pointed out at challenges that arise in constructing this interplay.

Findings and first interpretation

Our findings reveal that teachers practice the use of phys-math interplay in order to foster different teaching goals. They employ physics to construct mathematical tools and descriptions and to simplify mathematical representations. Mathematics is often used to explore the behavior of a physical system, to solve a physical problem, and to study the general context of a physical problem. In analyzing the interviews we found that the examples provided by the teachers can be grouped and arranged according to such goals.
Group A. The examples in this group are descriptions of how our interviewees demonstrate the use of mathematical exploration in examining possible behavior of a physical system. The examples cover such aspects as examination of borders of validity, borders of approximation, extreme cases and the physical ramifications of changing the value of certain parameters.
Examples
A.1. "In Optics: the student will examine Snell's law in the format: $\sin\theta_2 = (n_1/n_2)\cdot\sin\theta_1$. No problem arises when light passes from a low refractive index medium to a medium of greater refractive index. But in the opposite direction a real problem appears. Here mathematics gently suggests to us that we may be facing a fascinating new phenomenon"
This example illustrates that the teacher find it important to demonstrate to the students how mathematical knowledge (the behavior of the sine function) can be used to study the borders of validity of a physical law. Furthermore, the teacher equips the students with a powerful physical insight: whenever the mathematical investigation of a physical law indicates a mathematical difficulty, direct your attention back at nature.
A.2. "Consider the case of the minimum force required to move the object on a table (see figure 2).

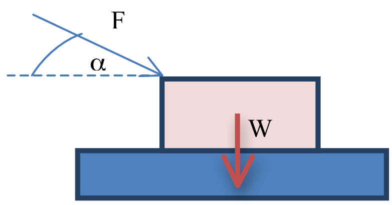


Figure 2: The minimal force required to set an object in motion

The answer is: $\boldsymbol{F_{min}} = \frac{\mu\cdot W}{\cos\alpha - \mu\sin\alpha}$. I ask the students to look at the denominator (where all the demons are…) and figure out what happens when it goes to zero, or even worse, gets negative."
Here the teacher demonstrates to the students what can be gained from studying mathematical extremities of an expression that was derived for a physical problem. In the next example we can see a demonstration of how one can ask questions on the behavior of a physical system, based on exploring the parameters of a certain mathematical representation:

A.3. "There is the equation of the “oscillating circuit”. I tell the children that this [equation] has been found; what has to be changed in order for the frequency to increase? Maxwell said I need a very high frequency in order to generate electromagnetic waves. I have the theoretical prediction of Maxwell.... Then I can look with the help of the equation. What has to be changed in the circuit? The area of the capacitor has to be smaller; the number of turns has to be increased."

The teachers also stressed the importance of a good mathematical knowledge in order to be able to examine physical cases, by referring to the consequences of a lack of such knowledge.

A.4. "... We have no point-like objects in the world. [But] we find ourselves saying: do not think about a human being, you must not think about your experiences or on a swing or a car. Anyone who begins to think about a car gets lost: what do the wheels do, the engine, whatever. No, no, no. Only point-like objects."

A.5. "The whole algebra course, that I agree should be done, is a course which actually conceals from the eyes of the students that Newton’s Second Law is a law of continuity - a differential equation. [It tells you:] go to cases with constant F then constant acceleration then all is a quadratic equation, and that’s where it ends."

The teachers feels that a partial mathematical knowledge restricts the integration of math and physics and thus limits the possibility to examine borders of approximation (of point-like objects) or the true nature of physical laws with the students.

A.6. "I would want someone else to do the work [teach the required math] for me. … In Grade 10 Snell’s Law uses up a lot of time. We used to eat gravel before we were successful in teaching them what sine is. And suddenly they learn in mathematics about sine earlier. Life became simpler."

This example provides a different perspective on the challenge encountered when students arrive to class with limited knowledge of math. The teacher finds herself required, against her own will, to teach math in order to be able to teach physics. According to her view of the phys-math interplay, the mathematical knowledge is a necessary tool for the physics teacher that should be drawn from a previously prepared "mathematical drawer".

Group B. Here we grouped examples of the means by which the teachers construct and develop mathematical representations for a physical system. They were found to do so either from experimental data or from first principles.

Examples

B.1. "Yes, let's look at what the object actually does in the lab as problem solving.… I will teach from the behavior, or characteristics of behavior [of a system], the mathematical properties of that system. Right.

B.2. "[With regard to refraction] I begin with experimental data and challenge them to arrive at a general relationship [between the angles]. They always try to draw a graph of one angle with respect to the other, and fit this graph with a linear function. This doesn't come out very well. So I draw for them a representation of the phenomena (see figure 3a) and ask them how can we compare the distance of the incidence ray and the refracting ray from the normal. In some cases they come up with the idea to draw a circle around the point of incident and sometimes I suggest this idea to them phenomena (figure 3b). Then they measure the distances and find out that their ratio is constant. Usually I am lucky and they draw different circles and we discuss what the mathematical meaning of this is. We then arrive from here to the sine representation of Snell's law."

Figure 3: A geometrical construction of Snell's law

B.3. . "If I want to analyze more exactly the magnetic field dependence on the number of turns [of a coil], then I have to conduct appropriate experiments, then I can develop a formula and then test this formula."

These examples demonstrate how a mathematical representation of a physical law can be constructed from empirical findings and then further elaborated within the mathematical playground and tested within the domain of physics.

Group C. Here we grouped examples that demonstrate how teachers employ mathematics to provide their students with a bird's-eye view on a specific physical problem. These examples cover the use of general laws of physics, symmetries, similarities and analogies in order to simplify the solution of a problem or to reveal how it might be related to wider aspects and contexts of physics. We included in this group also examples of a deductive derivation of physical statements.

Examples

C.1. “…A body slides with friction on another body which is placed on a frictionless surface (figure 4). I begin with Newton's laws. I have a very long board. I start at the top corner, draw the forces on each body

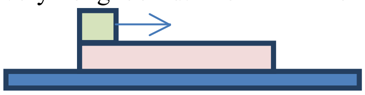


Figure 4: Analyzing the sliding of an object from different physical perspectives lead to different mathematical analysis

individually, write the equations for this axis and that axis, find the acceleration and from kinematics I get the speed versus time,… I then say, mmm…. can we do it in two rows? We know that there are no external forces so let’s use momentum conservation… I hear voices and then I say: but two rows are too much... The CM velocity does not change, so I have V…In one row! *...At that instance they cannot hold back their enthusiasm and applaud… they cannot hold back their pure pleasure”*

The teacher here demonstrates how one can use physical considerations - fundamental laws and symmetry - in order to simplify the required mathematics. He further depicts that the students are not indifferent with regard to such a thread of argumentations.

C.2. "Symmetries. OK, so it is always about *symmetrical aesthetics*… For example, connecting springs and connecting resistors. I can switch between them…[or] two objects interacting through [gravitational] force, how the expression should look like so it would be symmetric as required by the third law? Addition or duplication? … Certainly not a minus. Now, where did this requirement of symmetry comes from? *I do it explicitly, always.*

C.3. The third law, I mean, it entails mutual interaction. We therefore expect mathematics to be symmetrical. Finally, I can say [to the students] that symmetry is related to conservation."

C.4. According to Newton's second law, under a force proportional to the mass of the object on which it acts [i.e.: F = Km], all bodies have the same acceleration (a = K). We know about four such forces: (1) A force acting on a body resting in a linearly accelerated system; (2) Centrifugal force (3) Coriolis force and (4) Gravity. The first three forces are called "imaginary forces"… Maybe the same mathematical pattern suggests that there is something in common between all four forces. And maybe, God forbid, the force of gravity is also an imaginary force? And here we find ourselves in the delivery room of general relativity ...”

C.5. I use mathematics as a tool to address various phenomena that are essentially identical. I did this for harmonic motions and afterwards when we spoke about the formation of electromagnetic waves…. In other words, I take the mathematical tool as, in fact, an *organizer of knowledge* in order to see the similarity between the phenomena even if it seems that they are completely unrelated.

Physicists often look for mathematical similarities as implying similarities in nature. This example demonstrates how teaching can direct the students' attention to such mode of physical thinking. Moreover, the teacher suggests that such awareness on behalf of the students, renders introducing deep and advanced physical ideas possible. Such a teaching approach may assist students in developing a general view of physics and reduce their tendency to consider separately different topics in a textbook.

The next example demonstrates how a teacher refers to the historical evolution of physics in order to highlight the importance of a deductive derivation of physical statements:

C.6. "At some places you can arrive at [physical] insights you would not have seen in the reality, only with the aid of equations. Maxwell has thought, from his equations, that there have to be electromagnetic waves in space, and only 20 years later Hertz found these electromagnetic waves with his inductor. First was the theoretical physics, then one has tried to highlight this practically; sometimes first is the practice and then you try to mathematize it. Both aspects I try to show the children…. If I have an equation I can take many things out of it and imagine what is practically behind."

Group D. The examples in this group demonstrate how our interviewees use mathematics as a tool to solve physical problems and arrive at a better physical understanding. For some of the teachers this was the most obvious and clear demonstration of the Phys-Math interplay.

Examples

D.1. "… The [mathematical] capability will allow the students…, to acquire the tools that will enable them to deal with solving problems in physics."

D.2. "[With regard to] motion at constant acceleration, the student must be able to answer the question at what time will a body be at some point X. He will arrive [mathematically] at two solutions. He must understand that there are two physical states corresponding to the two solutions. In other words, if only one solution results, he must explain what that solution is and why there is no other solution to the equation that he wrote, for the specific problem that he addressed… he has to understand that… if mathematics is a tool, any solution that he gets by mathematics must apply to the specific case that he is investigating and that they are not two separate, unconnected drawers."

Phys-Math "patterns"

The interviews revealed that the teachers employ phys-math interplay as part of their practice. They use it to foster a better understanding of physics as it enables analyzing extreme cases, examining solutions and creating functional relations between physical entities. For some teachers the interplay is central in organizing and structuring the knowledge of physics: it creates webs of concepts and relationships and reveals similarities between different phenomena. Some teachers emphasized that the interplay is manifested in problem solving as it enables working with various mathematical representations.

We observed that the teachers' strategies for introducing the Phys-Math interplay follow different patterns. By patterns we do not mean here teaching sequences but rather different treks from physics to mathematics and within each of the two domains. All patterns begin with a physical description of a phenomenon, continue with mathematical manipulations and end in seeking new physical insights. However, the patterns differ by the number of steps going back and forth between the domains of physics and mathematics and within each domain and by the nature of these steps.

The patterns are listed below:

- An exploration pattern (group A): Exploring within math ramifications for the physical system: borders (of validity, of approximation), extreme cases, etc.

The trek characterizing this pattern begins with a certain physical phenomenon or system, then a mathematical representation is derived and studied purely mathematically. Then the ramifications of the mathematical analysis for the case in hand are discussed with new physical insights.

- A construction pattern (group B): Constructing and developing (from experiments or from first principles) mathematical tools to describe and analyze physical phenomena.

This trek begins either from an empirical data or from a description of a physical phenomenon using basic physical laws. Then a mathematical representation (graph, formula) is constructed. The mathematical construct is then applied to the initial physical case to provide new physical insights.

- A broadening pattern (group C): Adopting a bird's-eye view and employing general laws of physics, symmetries, similarities, analogies.

Here again the trail begins from a phenomenon or a physical case, employs an already known mathematical representation, broadens them and then broadens the physical scope and seeks for new insights.

- An application pattern (group D): Employing already known laws and mathematical representations in problem solving.

The steps characterizing this pattern go from the physical case to its already known mathematical representation, conduct mathematical manipulations and arrive at a mathematical solution which is then tested against the case in hand.

To summarize: Groups A-D might be represented by the following categories of the various aspects that are manifested in the phys-math interplay as practiced within physics teaching (table 1):

Table 1: Categories of the practices A-D

Category	The phys-math aspect	The teaching practices
A. Exploration	Mathematics is used to explore the behavior of physical systems.	Examination of: • Borders of validity • Borders of approximation • Extreme cases
B. Construction	Mathematical model can be constructed for physical systems.	Constructing mathematical model from: • Empirical data • First principles
C. Broadening	Mathematics can broaden the scope of a physical context.	Employing mathematics to seek for: • Similarities • Symmetries • Analogies
D. Application	Mathematics provides aid in problem solving.	Manipulating with mathematical representations in order to arrive at a solution for a given problem

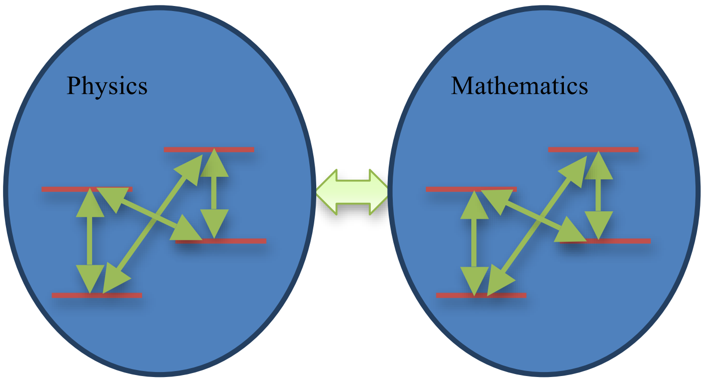
The considerations within each of the two disciplines are presented in table 2.

Table 2

Physical considerations	Mathematical considerations

• Description of a physical case • Examining applicability for the physical system • Scrutinizing laws of physics • Broadening the physical applicability • Seeking for new physical insights	• Applying an existing mathematical representation • Exploring the representation mathematically • Constructing simple mathematical representation • Sophisticating the mathematical tools • Manipulating with the mathematical representations

One may visualize how different steps which comprise a certain pattern are conducted between the two disciplines and within each of them:



Phys-math "patterns" and the teachers' PCK

Teaching orientations is highly important in teachers' PCK as they serve as ‘conceptual maps’ that guide a teacher’s instructional decisions concerning curricula, classroom activities, classroom materials, student assignments and the evaluation of students’ learning (Magnusson et al., 1999). The examples provided by the teachers clearly address most of these aspects of teaching physics. They mentioned the role of Phys-Math knowledge in deductive reasoning, in the relation of experiment and theory, in constructing students' broad view of physics and in problem solving.
In our research the teachers' orientations towards teaching the phys-math interplay was manifested in the patterns they have chosen as each pattern is employed to serve specific teaching goals. The teachers demonstrated to their students how each of the patterns: phys-math exploration, construction, broadening and application facilitate different aspects of physics understanding. Thus, these patterns may represent different answers to the question: what approach would foster the

Figure 5: Optional treks within each domain and between them. A specific trek represents a teaching pattern which addresses certain goals of the Phys-Math interplay.

understanding of a certain aspect of the Phys-Math interplay (see table 1) which is important for me (the teacher) to teach? Each choice served the teachers as a guide by which they designed their teaching.
Our study supports the claim that orientations play a critical role in distinguishing the quality of teaching (Abell, 2007). Most of the examples in groups A-C were provided to us by teachers who are considered to be master teachers. These teachers were very clear on rendering students *aware* of various aspects of the phys-math interplay. They also addressed in the interviews the deep relations between physics and mathematics in philosophical and historical perspectives.

Summary

The teachers in our study demonstrated knowledge about the interplay between physics and mathematics in different perspectives. They demonstrated different levels of awareness regarding the various patterns of the interplay and the teaching methods for each pattern. The level of awareness differentiated the master teachers from the other expert teachers.
In order to address the question what are the teachers' orientations towards teaching the phys-math interplay we first categorized their teaching practices in that respect.
Our interviewees also described various teaching strategies that they employ within the phys-math interplay. Those strategies might be related to another aspect of the PCK framework: Knowledge of successful teaching strategies (figure 1).
As we proceed with the analysis of our data, we shall address the question what do teachers regard as good teaching approaches to foster students' mastering the aspects of the Phys-Math interplay? We also plan to reach a wider range of teachers than our group of selective top teachers. We have also started working with a small group of teachers on

developing examples of math-physics interplay patterns and strategies to use and try in classroom. We will than study their effects on both teachers' instruction and students learning.

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