[3]R. E. Fischell, ARS J. **31**, 1210 (1961).
[4]M. D. Ripper and U. G. Endean, Proc. IEE **122**, 301 (1975).
[5]R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1969), Vol. 2.
[6]F. W. Sears, M. W. Zemansky, and H. D. Young, *University Physics* (Addison-Wesley, Reading, MA, 1976), 5th ed.
[7]M. A. Plonus, *Applied Electromagnetics* (McGraw-Hill, New York, 1978).
[8]P. Lorrain and D. R. Corson, *Electromagnetic Fields and Waves* (Freeman, San Francisco, CA, 1970), 2nd ed.
[9]J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), 2nd ed.
[10]R. K. Wangsness, *Electromagnetic Fields* (Wiley, New York, 1979), prob. 17–14 p. 318. The answer suggested in this reference is about 2.7 times larger than the measured value.
[11]J. R. Reitz, F. J. Milford, and R. W. Christy, *Foundations of Electromagnetic Theory* (Addison-Wesley, Reading, MA, 1979), 3rd ed. prob. 11-9, p. 247. Suggested answer is about 36% larger than measured value.
[12]W. R. Smythe, *Static and Dynamic Electricity* (McGraw-Hill, New York, 1968), 3rd ed.
[13]L. J. van der Pauw, Philips Res. Rep. **13**, 1 (1958).

Student difficulties in connecting graphs and physics: Examples from kinematics

Lillian C. McDermott, Mark L. Rosenquist,[a] and Emily H. van Zee
Department of Physics, University of Washington, Seattle, Washington 98195

(Received 21 February 1986; accepted for publication 21 May 1986)

Some common errors exhibited by students in interpreting graphs in physics are illustrated by examples from kinematics. These are taken from the results of a descriptive study extending over a period of several years and involving several hundred university students who were enrolled in a laboratory-based preparatory physics course. Subsequent testing indicated that the graphing errors made by this group of students are not idiosyncratic, but are found in different populations and across different levels of sophistication. This paper examines two categories of difficulty identified in the investigation: difficulty in connecting graphs to physical concepts and difficulty in connecting graphs to the real world. Specific difficulties in each category are discussed in terms of student performance on written problems and laboratory experiments. A few of the instructional strategies that have been designed to address some of these difficulties are described.

I. INTRODUCTION

Many undergraduates taking introductory physics seem to lack the ability to use graphs either for imparting or extracting information. As part of our research on student understanding in physics, the Physics Education Group at the University of Washington has examined some of the graphing errors made by students. Part of the motivation for undertaking this study has been a conviction that facility in drawing and interpreting graphs is of critical importance for developing an understanding of many topics in physics. We have been especially interested in exploring whether some of the difficulties with the kinematical concepts that we identified in an earlier study might be effectively addressed through an increased emphasis on graphical representations.[1,2]

The problems students have with graphing cannot be simply attributed to inadequate preparation in mathematics. Frequently students who have no trouble plotting points and computing slopes cannot apply what they have learned about graphs from their study of mathematics to physics. Therefore there must be other factors, distinct from mathematical background, that are responsible. The analysis of graphing errors identified in this study indicates that many are a direct consequence of an inability to make connections between a graphical representation and the subject matter it represents. In this paper, we describe two categories of student difficulty that we have investigated: difficulty in connecting graphs to physical concepts and difficulty in connecting graphs to the real world. Specific difficulties in each category are identified and discussed in terms of student performance on written problems and laboratory experiments. All of the examples used as illustrations are from kinematics, although our study also included other topics in physics and physical science.

Most of the work reported here was carried out over a period of several years in the context of a year-long preparatory physics course for undergraduates intending to enroll in either algebra- or calculus-based physics.[3] We have supplemented the information obtained from this group by extending the study to include students enrolled in our special physics courses for prospective and practicing precollege teachers and in the standard introductory physics courses at the University of Washington. We have also examined responses by high school physics and physical science students to some of the same questions that we administered to the college students in the study. Although there were differences in severity, the nature of the difficulties was the same across all of these populations. Our findings have also been consistent with reports by other investigators of graphing errors made by students ranging widely in age and educational background.[4,5] Similar difficulties with graphing have been identified even among students in the honors section of a calculus-based university physics course.[6]

II. DIFFICULTIES IN CONNECTING GRAPHS TO PHYSICAL CONCEPTS

We have identified a number of specific difficulties encountered by students in connecting graphs to physical concepts. In this section, we discuss a few of the most prevalent by examining some representative student responses to five problems in kinematics. By the time these questions are presented on course examinations in the preparatory physics course, the students generally have demonstrated a fairly good command of the kinematical concepts by performance on other problems that do not involve graphs. Thus most of the errors made by these students can be primarily ascribed to inability to interpret graphs rather than to inadequate experience with the concepts.

A. Discriminating between the slope and height of a graph

When interpreting a graph in physics, a student must be able to determine which features of a graph correspond to particular physical concepts. On a straight-line graph, for example, information may be contained in the coordinates of a point, the difference in the coordinates of two points (the rise or run), or the slope of a line. Many students seem to need assistance in learning how to choose which of these features to "read" in answering questions about the topic represented in the graph.

We have found that students frequently do not know whether to extract the desired information from the slope or the height of a graph. In the problem below, the students are asked to compare the two uniform motions represented on the position versus time graph shown in Fig. 1.

Problem 1: Fig. 1 shows a position versus time graph for the motions of two objects A and B that are moving along the same meter stick.

(a) At the instant $t = 2$ s, is the speed of object A greater than, less than, or equal to the speed of object B? Explain your reasoning.

(b) Do objects A and B ever have the same speed? If so, at what times? Explain your reasoning.

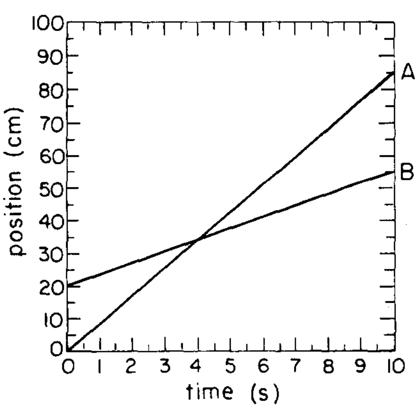

Fig. 1. Position versus time graph for problem 1. Students must decide whether the slope or the height of the graph gives the required information.

To answer part (a), it is only necessary to recognize that the slopes of the lines represent the velocities of the balls and that line A rises more steeply than line B. Since the slope of line A is obviously greater than that of line B, the speed of object A is greater than that of object B. Many students, however, do not give the correct response. Most incorrect answers seem to be due to failure to realize that information about the velocity cannot be extracted from the height. At $t = 2$ s, line B lies above line A, and many students focus on this difference in height, rather than on the difference in slope, to determine which object has the greater speed.

Students who respond incorrectly to part (b) do not realize that the two objects never have the same speed because the slopes of lines A and B are never the same. Rather, they choose $t = 4$ s, the point of intersection at which the lines have the same height, as a time when the speeds are the same. Again it seems that many students concentrate on the wrong feature of the graph in arriving at their answers. (It is also possible that some students correctly interpret the crossing of the graphs at $t = 4$ s as a time when the objects have the same position, but then incorrectly infer that the objects have the same speed as well.[1])

B. Interpreting changes in height and changes in slope

As might be expected, students find it more difficult to interpret curved graphs than straight-line graphs. Curved graphs involve changes in slope as well as changes in height. Changes in slope are not as perceptually obvious as changes in height and require more careful examination before information can be extracted from them. Some of the additional complications in the interpretation of curved graphs are illustrated in the following problem.

Problem 2: At which of the lettered points on the graph in Fig. 2:

(a) is the motion slowest?
(b) is the object speeding up?
(c) is the object slowing down?
(d) is the object turning around?

The answers can be determined by inspecting the heights and slopes and by considering the way these quantities are changing at each of the labeled points. The motion is slowest at both points B and F, where the magnitude of the slope is smallest, i.e., 0. The increasing magnitude of the

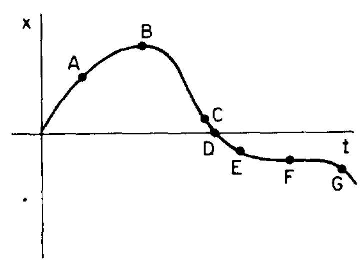

Fig. 2. Position versus time graph for problem 2. To answer questions about the velocity, the student must decide whether the height, slope, or changes in these quantities give the desired information.

slope at point G indicates that the object is speeding up. The object is slowing down at points A, C, D, and E, where the magnitude of the slope is decreasing. At point B, where the slope changes sign, the object turns around. In their written explanations, the students often reveal vestiges of the slope–height difficulties encountered with straight-line graphs. Incorrect answers for the slowest motion generally include point D, where the height is 0 but the slope is not. A typical error in part (b) is to claim that the object is speeding up at point A "because the graph is increasing." Of course, although the height is increasing at point A, the decreasing slope indicates a slowing down rather than a speeding up. Instead of looking for changes in slope, many students focus on the more perceptually obvious changes in height. In part (c), some students include point G as a point where the object is slowing down because "the slope is negative." These students base their responses on the sign of the slope rather than on changes in its magnitude. For the turnaround point in part (d), many students select point D because "the position is going from plus to minus." Instead of looking for a point where the slope changes sign, they identify a point where the height changes sign.

C. Relating one type of graph to another

In addition to difficulty in relating the various features of a graph to particular physical concepts, students often cannot relate one type of graph to another. Many are unable to translate back and forth from a position versus time (x vs t) graph to a velocity versus time (v vs t) graph. In the example below, students frequently do not realize that they should use the slope of an x vs t graph as the height of a v vs t graph. It is even more difficult for them to envision an increasing slope on an x vs t graph as an increasing height on a v vs t graph.

Problem 3: Several interesting times are labeled A, B, C, etc., on the position versus time graph in Fig. 3(a). Sketch a velocity versus time graph for this motion. Label times A–F on your time axis.

The correct response is shown in Fig. 3(b). In the sample incorrect response shown in Fig. 3(c), it is clear that the student did try to obtain a new graph by manipulating the information in the given graph. The result, however, is only an inversion of the original graph. The student was apparently focusing on the height of the position versus time graph rather than on the slope. In trying to construct one graph from another, students often seem unable to ignore the shape of the original graph.

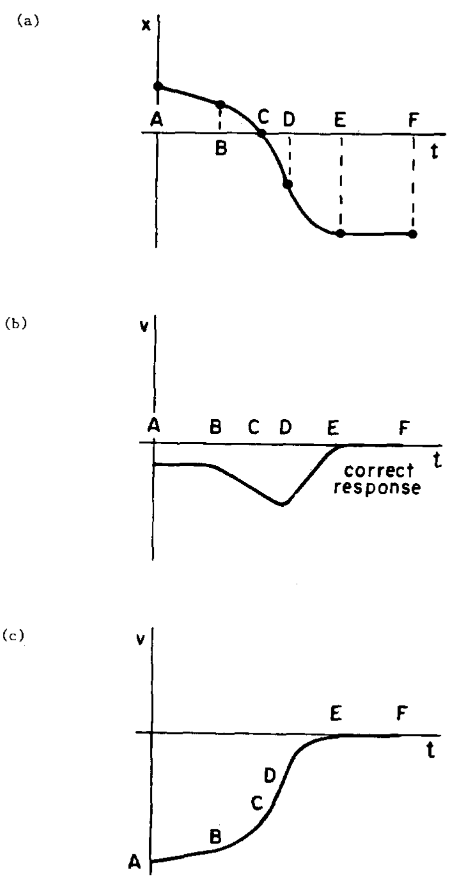

Fig. 3. Motion graphs for problem 3. (a) A position versus time graph for which the student must produce a velocity versus time graph. (b) Correct velocity versus time graph. (c) Incorrect velocity versus time graph. Shape resembles inversion of position versus time graph.

D. Matching narrative information with relevant features of a graph

The task of matching the information in a narrative passage to a graphical representation is difficult for many students. To answer the questions in the next problem, the students must refer both to the graph in Fig. 4 and to the problem statement.

Problem 4: A spaceship has three different rocket engines, each of which gives the ship a uniform acceleration when it is turned on. In the graph in Fig. 4, point P represents the velocity of the rocket at a particular time. At point P, the captain turns on the #1 engine. At point Q, the #1 engine is turned off and the #3 engine turned on. At point R, the #3 engine is turned off and the #2 engine is turned on. We lose all information about the ship after point R.

Find the acceleration produced by each engine listed below, if this information is represented on the graph. Explain your reasoning.

(a) #1 engine.
(b) #2 engine.
(c) #3 engine.

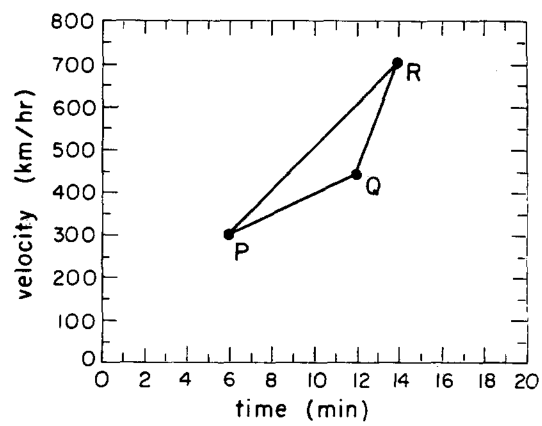

Fig. 4. Velocity versus time graph for problem 4. To answer questions about the acceleration of various engines, the student must associate the slope of a particular line with the acceleration of a particular engine.

In problem 4, the line segments PQ and QR give part of the velocity versus time history of motion of the rocket. The line segment PR has no such interpretation, but is drawn on the graph so that all three segments are displayed identically. In order to answer the questions correctly, the students must identify the slope of PQ with the acceleration produced by the #1 engine, identify the slope of QR with the acceleration produced by the #3 engine, and realize that the acceleration produced by the #2 engine cannot be determined. Thus the students must refer to the slopes of two of the line segments and ignore the third.

Several kinds of errors on this problem are common. For example, in finding the acceleration produced by the #1 engine, some students do not recognize that they must calculate the ratio of $\Delta v/\Delta t$ during the time interval that the #1 engine is firing, i.e., they must find the slope of PQ. Instead, some students divide the coordinates of point P, while others divide the coordinates of point Q. These students do not recognize the difference between the ratios of v/t and $\Delta v/\Delta t$. Although most refer to the need to calculate a change in velocity divided by a time interval, they do not associate this ratio with the slope of PQ.

Among those students who do compute the accelerations using slopes, many are unable to match the correct slope with the correct engine. By far the most common error, however, is the failure to realize that the acceleration of the #2 engine cannot be determined. Most students who make this error calculate the slope of PR for the acceleration of the #2 engine. This kind of error indicates a lack of attention to the details of the written description and may indicate the use of a memorized algorithm for finding acceleration as "the slope of the v vs t graph."

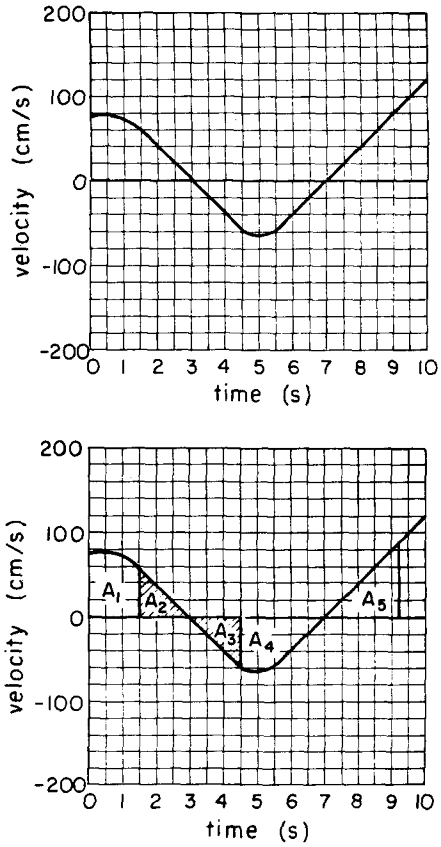

Fig. 5. (a) Velocity versus time graph for problem 5. (b) Cross-hatched areas show motion analyzed into series of displacements.

E. Interpreting the area under a graph

Interpreting the area under a graph is a new idea for many of the students in the preparatory physics course. The process of finding displacements by counting the number of squares under a velocity versus time graph requires interpreting areas as lengths. Students often find it difficult to envision a quantity that they associate with square units as representing a quantity with linear units. In the following example, the students must interpret a velocity versus time graph to determine when an object is located at a particular position.

Problem 5: Fig. 5(a) shows a velocity versus time graph for an object that is located at $x = 0$ when $t = 0$. When is the object located at $x = 110$ cm?

Few of the students are initially able to obtain a qualitative overview of the motion by reading the graph in Fig. 5(a). They do not interpret the alternating positive and negative areas above and below the $v = 0$ axis as representing alternating positive and negative displacements. Hence they do not form a mental picture of an object that is oscillating back and forth. Without an image of an oscillating motion, the students are unaware that they need to find more than one time when the object is located at $x = 110$ cm.

To answer the question, it is first necessary to note that the area of one square corresponds to a displacement of 10 cm. From a quick inspection of the graph and a rough count of squares, the following information is readily extracted: The object reaches its first maximum positive displacement (about 150 cm) and turns around ($v = 0$) at $t = 3.0$ s. It then moves in the negative direction until it reaches its second turnaround at $t = 7.0$ s. The approximate equality of the first positive area and the negative area indicates that the object has returned approximately to its starting point ($x = 0$) when it reverses direction at $t = 7.0$ s. The larger size of the second positive area reveals that the object eventually moves in the positive direction beyond its first maximum.

The analysis above indicates that the object passes $x = 110$ cm three times. By marking off the first 11 squares, we can find that $x = 110$ cm (A_1) for the first time at $t = 1.5$ s. To find the other times, it is convenient to subdivide the area under the curve into five subareas, thereby simplifying the process of balancing positive and negative areas. Areas A_{1-5} are shown in Fig. 5(b). If they have a mental picture of the path in mind, the students can determine that the object continues toward its maximum positive displacement (A_{1-2}) at $t = 3.0$ s, turns around, then returns to $x = 110$ cm (A_{1-3}) at $t = 4.5$ s. It then continues to move in a negative direction until it reaches $x = 0$ (A_{1-4}) at $t = 7.0$ s, where it turns around again and moves in a positive direction. The object passes $x = 110$ cm (A_{1-5}) for the third time at $t = 9.3$ s.

Most of the difficulties the students have with this problem are directly related to an inability to visualize the motion that is depicted in the velocity versus time graph. However, they also make a variety of other errors in trying to extract information about the displacement from the v vs t graph. At the most rudimentary level, students may calculate the displacement represented by the area of one square, but not know what to do with this number. Students who know that they need to multiply the area of one square by the total number of squares to find a total displacement often cannot determine which squares they should count. For example, in counting the squares "under the curve," some students include all the squares between the curve and the bottom line of the grid, where the horizontal scale is labeled. By ignoring the $v = 0$ axis, they fail to perceive its role in defining the positive and negative areas. They do not associate a positive area on a velocity versus time graph with a displacement in the positive direction or a negative area with a displacement in the negative direction. At a more sophisticated level of difficulty, students often do not realize that to respond to a question about the position at a particular time, they need to refer to information that is not provided on the v vs t graph. In this case, the students must make use of the initial position ($x = 0$) given in the statement of the problem.

Commentary on examples

A common characteristic of the examples discussed above is that to answer the questions correctly, a student must do more than simply remember a procedure, such as calculating the slope of a position versus time graph to find the velocity. We have found that problems in which no more than simple recall is needed usually have presented little difficulty to most of the students in the preparatory physics course or in the courses for precollege teachers. However, when a question requires detailed interpretation of a graph—matching a written narrative to an accompanying graph or comparing two motions represented on the same graph—memory alone does not suffice. The practical application of graphical skills in any field usually involves more in the way of interpretation than remembering what the slope of a particular kind of graph represents. A realistic assessment of student ability to extract information from a graph must therefore involve elements of interpretation similar to those required in the above examples.

III. DIFFICULTIES IN CONNECTING GRAPHS TO THE REAL WORLD

In Sec. II, we examined some of the errors made by students in relating various features of a graph to the physical concepts the graph represents. The errors were classified into five groups of difficulties, each of which was discussed in terms of student responses to a written graphing problem. Other difficulties, both of a similar and different nature, arise when students attempt to relate a graph to a particular object or event in the real world.

In a series of experiments in the preparatory physics course, the students construct graphs for the motion of a steel ball that is released from a starting ramp and allowed to roll along various combinations of straight aluminum tracks. The two track arrangements in Figs. 6(a) and 6(b) are used to produce a uniform motion and an accelerated motion, respectively. The students are asked to draw position versus time and velocity versus time graphs. Stop clocks and a meter stick are available, but no specific directions are given about which quantities should be measured or calculated. Under these conditions, we have found that the task of representing an observed motion on a graph is very difficult for many students. The track arrangement in Fig. 6(c) is similar to that in Fig. 6(b), but has an additional inclined segment. The students do not make any measurements for the motion produced in this case, but sketch qualitative graphs for velocity versus time and acceleration versus time.

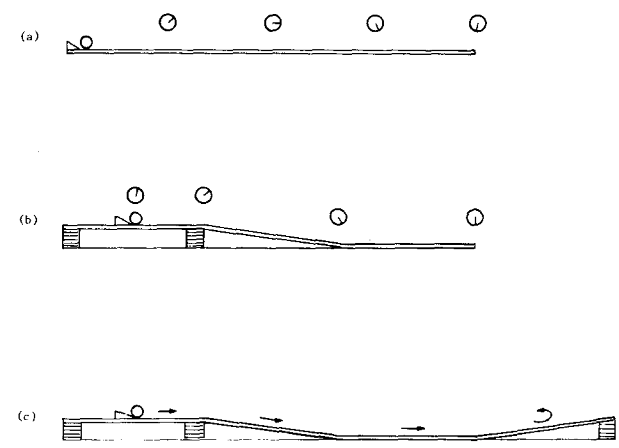

Fig. 6. (a) Experiment in which ball rolls along straight level track. (b) Experiment in which ball rolls along track with level and inclined segments. (c) Experiment in which ball rolls along track with level and inclined segments, with turnaround on last segment.

For the simplest case, that of uniform motion, the steel ball rolls on a long level track with four stop clocks placed at equal distances from one another and from the starting ramp, as shown in Fig. 6(a). To correct for frictional effects, the "level" track is actually slightly inclined. The clocks are started synchronously when the motion begins. Each clock is stopped as the ball passes its location. A correct x vs t graph is shown in Fig. 7(a). In Figs. 7(b)–7(d) are student graphs.

In the motion shown in Fig. 6(b), the ball rolls along a level track at a low speed, accelerates down an incline, and then rolls along a lower level track at a higher uniform speed. There is one stop clock at the foot of the starting ramp and one at each of the intersections of the track segments. The four clocks are started synchronously before the motion begins. Each clock is stopped as the ball passes

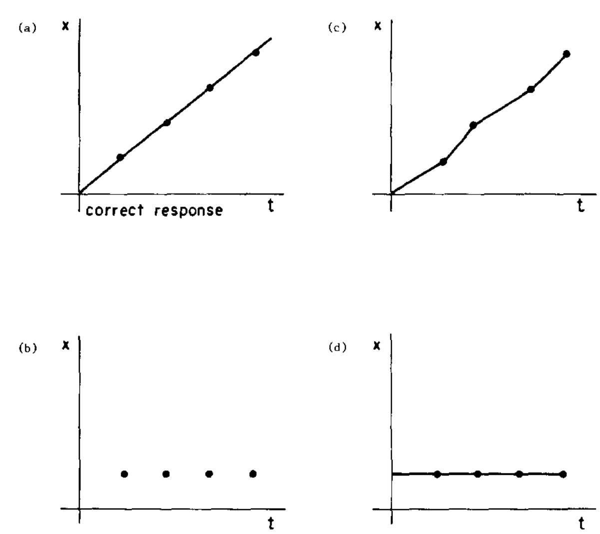

Fig. 7. (a) Correct position versus time graph for motion in Fig. 6(a). (b) Successive displacements are plotted as unconnected points. (c) Lines are drawn between data points. (d) Shape of x vs t graph resembles path of motion.

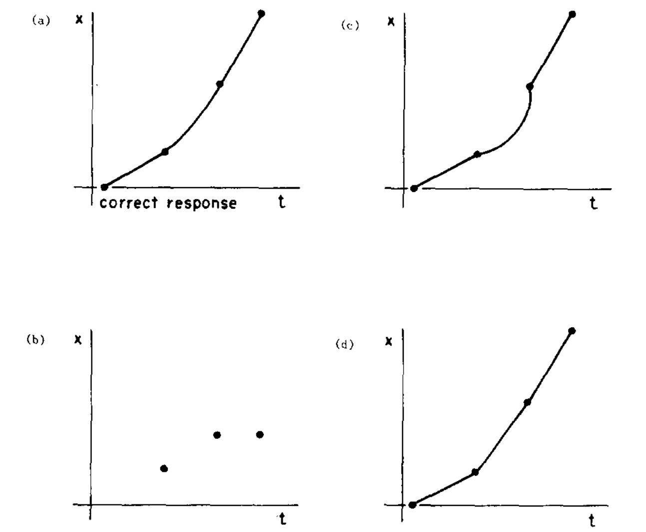

Fig. 8. (a) Correct position versus time graph for motion in Fig. 6(b). (b) Length of each segment is plotted at end of corresponding time interval. Although motion is continuous, points are not connected. (c) Graph segments are connected by kinks instead of smooth curves. (d) Shape of x vs t graph is straight like the track, instead of curved, during accelerated part of the motion in which ball rolls down straight track.

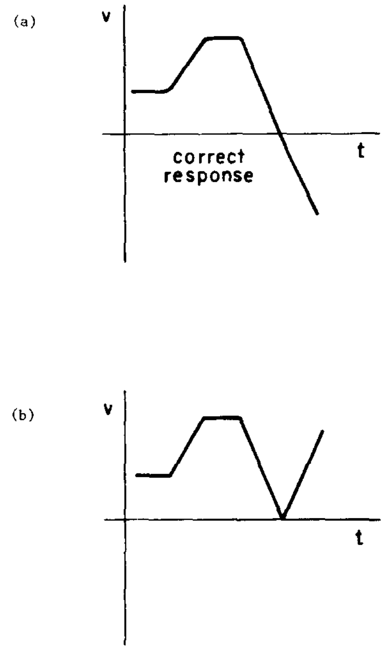

Fig. 10. (a) Correct velocity versus time graph for motion in Fig. 6(c). (b) v vs t graph has "V" that corresponds to reversal along the path of motion.

its location. Correct x vs t and v vs t graphs are shown in Figs. 8(a) and 9(a), respectively. Student graphs appear in Figs. 8(b)–8(d) and 9(b)–9(d).

In Fig. 6(c), the additional segment added to the track in Fig. 6(b) allows the ball to roll up a second incline, slow down, turn around, and accelerate back down again. Qualitatively correct v vs t and a vs t graphs are shown in Figs. 10(a) and 11(a), respectively. Figures 10(b) and 11(b) contain student graphs.

Below we examine some of the problems that students encounter as they attempt to construct graphs from their observations of the three motions described above. To facilitate the discussion, the errors made by the students on the different types of motion graphs have been classified into five groups of difficulties.

A. Representing continuous motion by a continuous line

Before enrolling in the physics courses in which this study was primarily conducted, nearly all the students had encountered position versus time graphs in mathematics

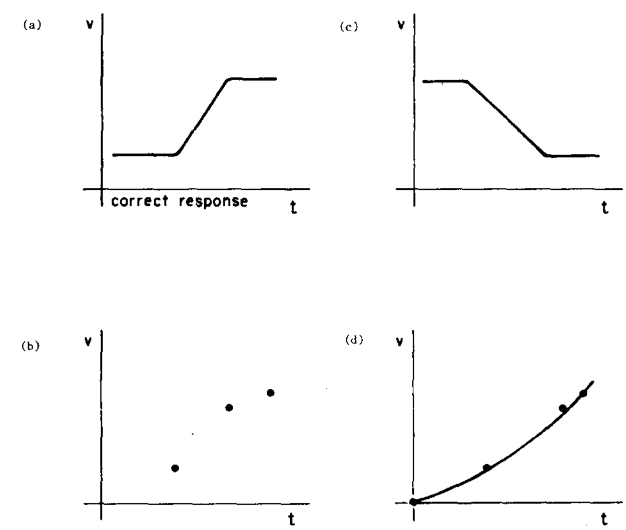

Fig. 9. (a) Correct velocity versus time graph for motion in Fig. 6(b). (b) Velocity for each segment is plotted at end of corresponding time interval. Although motion is continuous, points are not connected. (c) Shape of v vs t graph resembles path of motion. (d) Shape of v vs t graph resembles shape of x vs t graph.

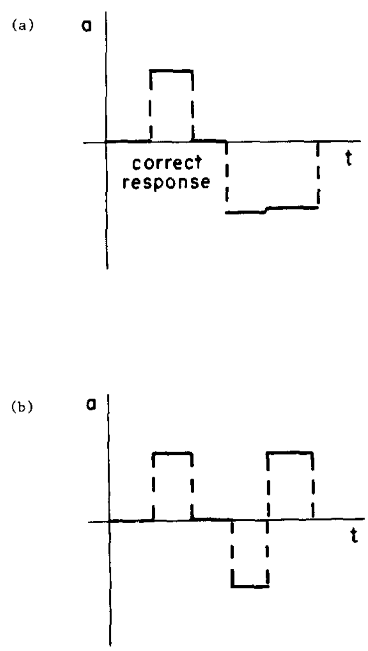

Fig. 11. (a) Correct acceleration versus time for motion in Fig. 6(c). (b) a vs t graph has positive acceleration when velocity is increasing and negative acceleration when velocity is decreasing.

classes. In spite of this preparation, many did not know how to begin constructing the graph of a uniform motion observed in the laboratory. The students had difficulty deciding which data to take and how to make appropriate use of their measurements.

To plot an x vs t graph for the motion in Fig. 6(a), it is necessary to choose an origin, establish a coordinate system, and assign position numbers along the track. The positions of the object are then plotted against corresponding times. Instead of following this procedure, however, some students simply measure the length of the segments of track and then plot these lengths versus the clock readings as the ball passes the end of each segment. Figure 7(b) illustrates such a graph. These students fail to distinguish between the position of the ball (x) at a particular instant and its displacement (Δx) during a time interval. They do not recognize that the motion of the ball should be represented by a continuous line instead of a series of separate points. Similarly, when asked to draw an x vs t graph for a stationary object, some students draw only a point (usually at the origin) rather than a horizontal line. Thus their graphs fail to indicate that there is no change in position over a period of time.

Failure to represent the continuity of motion persists for some students who produce graphs for accelerated motion that consist of discrete points. In the x vs t graph in Fig. 8(b), the student has plotted the length of each segment of the motion as the position of the ball at the end of the time interval. Although the motion is continuous, the points are not connected. As with uniform motion, part of the problem may be due to a confusion between position at an instant and displacement during a time interval.

Similar errors in representing continuous motion occur on velocity versus time graphs. In Fig. 9(b), the student does not represent the velocity as a continuously varying quantity, but rather as three distinct points, each obtained by dividing the length of track by the time taken to travel along it. Students who draw this kind of graph seem to associate a single velocity with each segment of the track, whether level or inclined, and usually plot this velocity only for the instant that the ball is at the end of that section of track. For the level sections, these students do not seem to realize that the calculated velocity extends over a period of time and should therefore be represented by a horizontal line. For the inclined section, they fail to associate the average velocity they have calculated with the instantaneous velocity at the middle of the time interval during which the ball rolls down the incline. To plot a correct velocity versus time graph, the students must recognize that a point on a v vs t graph represents the velocity at a single instant and that a line or curve represents the variation in velocity over a period of time.

Some students who plot values of position and time correctly may not join the points in a smooth curve. Instead they make point-to-point connections that form a disjointed line, as in the position versus time graph in Fig. 7(c). For the corresponding v vs t graph, instead of drawing a horizontal line, these students often calculate velocities from their measurements of position and time, plot the points, and connect them in a zigzag line. They seem unaware that measured values are only approximations and that the observed regularity in a continuous motion should be represented by fitting a smooth curve to the data points.

Errors at a more sophisticated level are made by some students who draw a continuous straight line or curved line for each segment of a motion but do not link the segments properly. For example, the student who drew the position versus time graph in Fig. 8(c) drew kinks instead of smooth curves to connect the segments. A kink or cusp indicates an abrupt change in speed that does not occur in the actual motion. When confronted with this contradiction, even good students often show little concern that the slope changes abruptly on the x vs t graph they have drawn and do not know how their graph should be modified to eliminate this problem.

B. Separating the shape of a graph from the path of the motion

Among the errors that the students make in trying to construct a graph of the uniform motion depicted in Fig. 6(a) are attempts to reproduce the spatial appearance of the motion. Some students seem to have difficulty in accepting the idea that uniform motion on a level track can be represented by an x vs t graph that is sharply inclined, as shown in Fig. 7(a). They seem to expect that the shape of the graph should resemble the shape of the track and thus draw a horizontal line, as in the graph in Fig. 7(d). Similar expectations have been observed among younger students as well.[5]

Disassociating the shape of a motion graph from the shape of the track seems to be particularly difficult for motions that involve inclined segments, such as the track shown in Fig. 6(b). The correct x vs t graph, with x measured along the track, is shown in Fig. 8(a). The graph consists of a straight line of small slope for the first level track, followed by a section curving upward for the inclined track and a straight section with a steep slope for the final level track. Many students do not readily associate the two uniform horizontal velocities on each of the level sections of track with two inclined lines of different, but constant, slope on a position versus time graph. Nor do they envision that the motion along the inclined section of the track will be represented by a curve on the graph. Even when students plot the positions and clock readings correctly, they may have difficulty deciding whether to connect the points with straight lines or curves. Some students seem reluctant to draw curves connecting the points. Instead they try to make the x vs t graph look like the track by drawing a graph that consists of three straight-line segments, as shown in Fig. 8(d).

When students are asked to sketch graphs without taking measurements, they are even more likely to include features that mirror the shape of the track. For example, when presented with a diagram of the three tracks shown in Fig. 6(b), but with the ball placed at the beginning of the first track, students often produce motion graphs in which the time intervals for the three segments are of equal length. Few seem to take into account the fact that because of its increasing speed the ball spends successively less time on each segment. It is also not uncommon for students to draw the first and third segments of the x vs t graph parallel to each other, just as the first and third segments of the track are parallel. Sometimes, if asked to sketch a v vs t graph for this motion, students produce a graph like the one in Fig. 9(c), in which the graph resembles the physical path of the moving object rather than the variation of velocity with time.

We have found that students also have trouble separating the shape of a graph from the path of the actual motion in the converse situation of going from a graphical representation to a laboratory situation. When instead of being asked to construct a graph for an observed motion, students are directed to produce a motion represented on a graph, they will often try to arrange the tracks so that they look like the x vs t or v vs t graph they are trying to interpret.

C. Representing a negative velocity on a v vs t graph

When a negative velocity is involved, as must be the case when a moving object reverses direction, students often cannot translate the actual physical event into a correct representation on a velocity versus time graph. In the motion in Fig. 6(c), the ball rolls up the second inclined track, turns around, and then rolls down. The correct graph in Fig. 10(a) shows an initial uniform speed on the first level track, a steadily increasing speed as the ball rolls down the first incline, a uniform speed on the second level section, a decreasing speed as the ball rolls up the second incline, and an increasing speed as the ball rolls down. The reversal in direction on the second incline is marked only by the crossing of the horizontal axis. The bend in the line at this point reflects the difference in acceleration due to friction as the ball rolls up and down the incline. Since no measurements are made in this case, an acceptable sketch by a student might not show a bent line.

In drawing a graph of the motion in Fig. 6(c), students often fail to represent the motion of the ball on the second incline by a line that crosses the $v = 0$ axis. Instead, they may produce a graph, like that in Fig. 10(b), which has a "V" with a vertex marking the instant of turnaround. The change in direction of the graph appears to be an attempt to represent on paper the reversal in direction of the actual motion in space.

D. Representing constant acceleration on an a vs t graph

Students also have difficulty in drawing an a vs t graph that is qualitatively correct for the motion of an object that slows down, turns around, and speeds up in the opposite direction. The situation is even more complicated when not only the velocity, but also the acceleration changes direction, as occurs for the motion in Fig. 6(c). Unlike the sign of the position or velocity of the ball, the sign of the acceleration is not immediately perceptible but must be inferred.

A correct a vs t graph is shown in Fig. 11(a). Drawing such a graph requires consideration of how the velocity and acceleration are related on each section of the track. There are two intervals of zero acceleration corresponding to the level sections of the track. The acceleration of the ball as it rolls down the first incline is indicated by a horizontal line above the $a = 0$ axis. The acceleration on the second incline is negative, consistent with a decreasing positive velocity as the ball rolls up and an increasing negative velocity as it rolls down. The difference in the magnitude of the acceleration for motion up and down the incline reflects the different effect of friction in each case. We would not expect this kind of detail in the qualitative graphs drawn by the students.

Rather than drawing an a vs t graph with the characteristics described, the students produce a variety of different graphs. There seems to be little difficulty in graphing the acceleration for the first three segments of the motion. However, the second incline presents problems. A feature common to many of the incorrect graphs for this portion of the motion is the representation of the acceleration on the incline by two separate horizontal lines, one positive and one negative. The order may vary, but in both cases the acceleration shown is sharply discontinuous and not consistent with the actual motion.

The incorrect graph in Fig. 11(b) shows a positive acceleration for time intervals when the object is speeding up and a negative acceleration for time intervals when the object is slowing down. This type of error is common and seems to reflect the association of a negative acceleration with an object that is decelerating. We have found that this particular conceptual difficulty is persistent and impedes the progress of many students who fail to realize that an object with a negative acceleration may be either speeding up (if the velocity is also negative) or slowing down (if the velocity is positive). Students who make the reverse error of drawing a positive acceleration for motion up the second incline and a negative acceleration for motion down the incline seem to link the direction of the acceleration with the direction of motion. Many students seem unaware that one cannot tell from an acceleration versus time graph whether an object is speeding up or slowing down or in what direction it is traveling.

E. Distinguishing among different types of motion graphs

When students are asked to sketch x vs t, v vs t, and a vs t graphs for a motion demonstrated in the laboratory, they often draw three graphs that have basically the same shape. Even when they make measurements of a motion and obtain data to plot, we have found that students often try to make the shapes of the graphs match one another. For example, the student who drew the v vs t graph shown in Fig. 9(d) to represent the motion in Fig. 6(b) plotted the data points properly, but then also added a point at $v = 0$. This student then connected all the points on his v vs t graph in a way that mimicked the x vs t graph he had just completed, a correct graph similar to the one in Fig. 8(a).

We have also observed that students make similar errors in drawing an a vs t graph for a motion for which they have drawn a correct v vs t graph. Instead of horizontal lines to represent constant accelerations on the a vs t graph, students sometimes draw straight inclined lines that parallel the accelerated portions of the v vs t graph. Some students seem to find it very difficult to accept the idea that the same motion can be represented by graphs of very different shape.

Commentary on examples

Many of the difficulties illustrated in this section do not surface in the course of traditional instruction. Most of the students in the preparatory physics course and in the courses for precollege teachers have had the necessary skills to draw motion graphs for data that can be plotted directly or that can be transformed almost mechanically through an algorithmic procedure. We have found that those students who lack the requisite skills on entry into the course can usually develop this degree of proficiency through specific instruction. However, when a problem requires an analysis for which the student may not have a pattern—graphing a motion that is qualitatively different from motions encountered before or finding information that must be inferred from a graph rather than directly read—memorized procedures are not sufficient. A deeper knowledge is required. The examples used as illustrations above involve different kinds of questions from those usually asked about laboratory experiments and lecture demonstrations. Each of the examples demands that students recognize explicitly the relation between an actual motion and one or more of its graphical representations. The ability to reverse one's thinking in translating a laboratory situation into a graph and a graph into a laboratory situation demands a level of understanding beyond that which is ordinarily assessed in most physics courses.

IV. IMPLICATIONS FOR INSTRUCTION

We have discussed above several errors in drawing and interpreting graphs that reflect difficulty in making connections between graphs and specific subject matter. The illustrations used were culled over a period of several years from observations of several hundred students as they studied kinematics. The errors we identified did not seem to be idiosyncratic to any particular group but were evident among different populations and across different levels of sophistication. In this section, we consider some instructional implications.

The results of our investigation have guided us in the design of an instructional module on kinematics.[7] Initially developed, tested, and revised in a special course to prepare minority students to succeed in physics,[3,8] these materials have since been used with other populations, including prospective and practicing teachers, business students, and liberal arts students. We present below a few examples of instructional strategies that we have found effective in helping students develop facility in connecting graphs to physical concepts and in connecting graphs to the real world.

Throughout the entire span of instruction, we give the students a great deal of practice in selecting the appropriate feature of a graph to obtain the information required. One example of how we try to develop this capability is given by an examination question based on three different, identically shaped motion graphs. The students are presented with the x vs t, v vs t, and a vs t graphs shown in Fig. 12. They are told that the initial velocity in each case is 10 m/s and are asked to find the velocity at $t = 9$ s.

Being confronted with all three types of motion graphs at the same time helps impress upon the students the difference in the ways that the same information is conveyed in each graph. Although the graphs are identical in shape, the motions represented are very different. Information about the velocity must be extracted from a different feature of each graph. The students find the velocity from the x vs t graph by calculating the slope ($-$ 10 m/s) and the velocity from the v vs t graph by reading the vertical coordinate ($-$ 15 m/s). For the a vs t graph, they must examine the area under the curve between $t = 0$ and $t = 9$ s. Because the positive and negative areas cancel, the total change in velocity is 0. Therefore the velocity at $t = 9$ s is the same as the initial velocity (10 m/s). The students find that they cannot obtain the initial value of the velocity from the a vs t graph but must refer to the problem statement. Thus in addition to providing practice in extracting information from motion graphs, this type of problem directs attention to what may and may not be learned from a particular graphical representation.

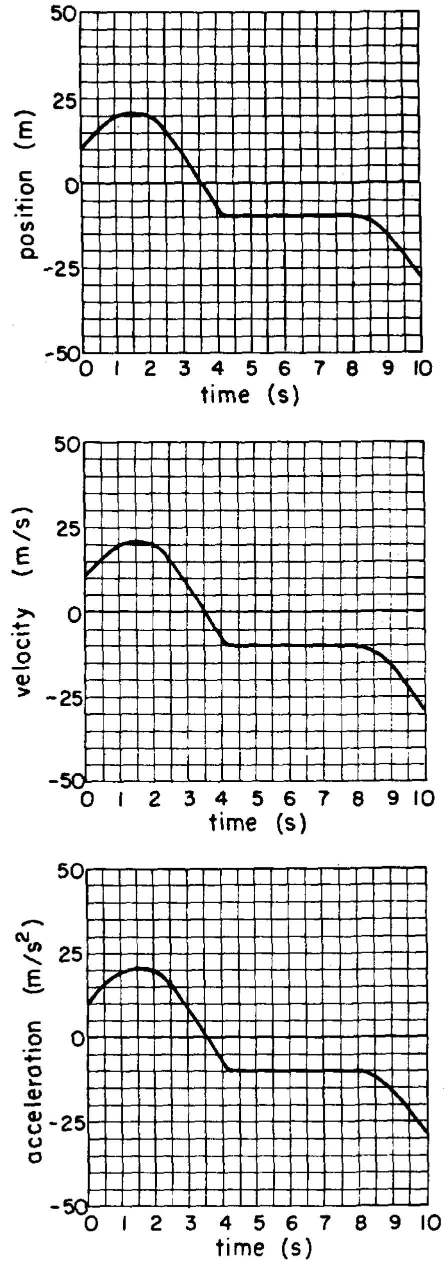

Fig. 12. Three motion graphs of the same shape: position versus time, velocity versus time, and acceleration versus time. Students must select the appropriate feature for finding the velocity at $t = 9$ s.

We also provide the students with experience in interpreting graphs in contexts other than kinematics. Similar types of graph interpretation questions are presented in various topics covered in the preparatory physics course. Some examples are mass versus volume, solute versus solvent, and heat transferred versus temperature rise. For each type of graph, the students are asked to identify the physical quantities represented by the coordinates of a point, by the difference between the coordinates of two points, by the slope of a line at a point, etc. They are asked a series of questions for which they must decide whether the information needed can be extracted from the slope or height of a graph. In addition to reinforcing graphing skills, practice in interpreting similar graphs in different contexts also helps deepen an understanding of the associated concepts such as density, concentration, and heat capacity.[9]

Some of the graphing exercises that we present to the students relate to material that is not within the scope of the course. For example, the students may be given a graph that shows the variation of temperature with depth of the ocean, changes in elevation with distance from a particular location, or consumption of oil over long periods of time. There are two purposes for extending practice in interpreting graphs beyond physics: (i) To develop a general ability to work with graphs that may be useful to students long after they may have forgotten much of their physics, and (ii) to take advantage of the increased depth of understanding that comes from using the same procedures and reasoning in several contexts. From careful monitoring of student performance on examination questions, we have found that the ability to choose correctly whether to extract information from the height, slope, or area of a graph develops slowly but steadily as students gain experience with graphs in different contexts.

An important component of understanding the connection between a graphical representation and the reality it represents is the ability to translate back and forth in both directions. A special set of exercises in our module on kinematics is designed to help students learn to translate from graphical representations to physical reality and vice versa. In some of these exercises, the students observe various motions of balls rolling on tracks and then construct graphs that represent the motions. Equally important, however, are experiments in which students are given position or velocity versus time graphs and are asked to use balls and tracks to produce the motions that are represented on the graphs. Below we present as examples an exercise and an experiment in which students are required to proceed alternately in the two directions.

One of the motions that the students are asked to observe, measure, and graph is illustrated in Fig. 13(a). The ball starts from rest, increasing in speed as it rolls down the first inclined track. It then rolls with uniform motion along the first level track, slows down as it rolls up the second incline, turns around, and rolls down. A switch at point A allows insertion of an additional section of track so that the ball rolls along a longer inclined track on the way down than on the way up. The motion terminates on a second section of level track.

Producing a correct velocity versus time graph for this motion has proved to be quite challenging for most of the students. To construct the v vs t graph in Fig. 13(b), the students must be able to relate the actual motion of the ball to different values of the velocity, associate these values with corresponding times, and represent this information on a graph. The students must first observe that the velocity is equal to 0 at two instants (when the ball is released and when it turns around), that the velocity is uniform along the level sections of track, and that the velocity changes along the inclines. The students must then translate these observations and their measurements into points and lines that they draw on a v vs t graph.

Alternating with experiments in which a graph is constructed from a motion are several that require the students to translate in the reverse order. Using a set of aluminum tracks, the students must design track arrangements that can produce various motions which are depicted on graphs. Initially, most students have very little idea about how to proceed. To produce a given motion in the laboratory, it is necessary first to interpret the graph, segment by segment. Speeds for the various segments must then be compared and a determination made of the time invervals when the object is speeding up, slowing down, etc. Finally, the sections of track must be arranged sequentially and smooth connections made between them.

To produce the motion depicted in the graph in Fig. 14(a), the students must form a mental image in which the ball moves in the negative direction with a relatively slow uniform speed along a level track, slows down as it rolls up an incline, turns around, speeds up (at a slighly slower rate) as it rolls down the incline in the positive direction, and finally rolls with uniform speed along a level track. The students must notice that the ball reaches a higher speed after it turns around than before. Therefore a switch is necessary for inserting a second inclined section that will allow the ball to roll along a longer track on the way down than on the way up. The arrangement of tracks that will produce this motion is illustrated in Fig. 14(b).

We have found that many students need the type of labo-

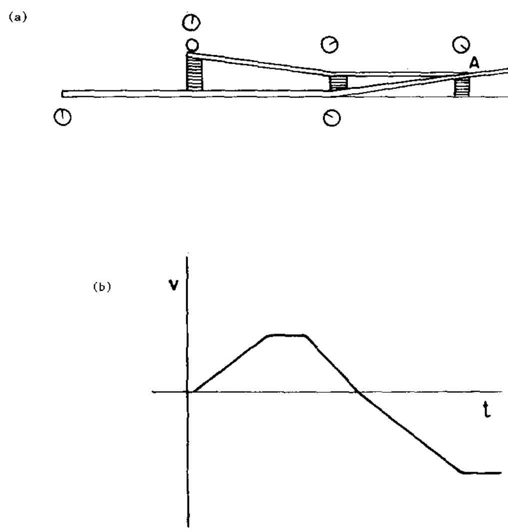

Fig. 13. (a) Track arrangement for which students must produce a v vs t graph. (b) Correct velocity versus time graph for the motion.

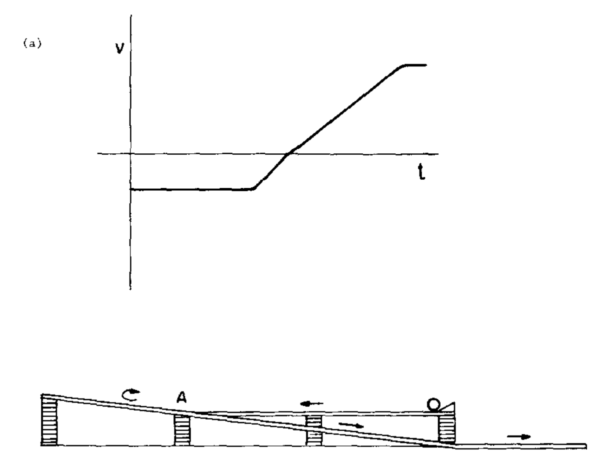

Fig. 14. (a) Velocity versus time graph for which students must produce a motion. (b) Track arrangement that produces motion depicted in graph.

ratory experience described in order to be able to make connections between an actual motion and its graphical representation. By the end of the portion of the preparatory physics course devoted to kinematics, most of the students have developed considerable skill in constructing a graph from an observed motion and in producing a motion depicted on a graph.

V. CONCLUSION

Among the many skills that can be developed in the study of physics, the ability to draw and interpret graphs is perhaps one of the most important. To be able to apply the powerful tool of graphical analysis to science, students must know how to interpret graphs in terms of the subject matter represented. They should be able to choose the feature of a graph that contains the required information and to recognize a relationship that may exist among different graphs. They should be able to represent real systems graphically and to visualize a system from its graphical representations. As the examples presented above indicate, however, many students have trouble in each of these areas. There are also other important aspects of graphing not treated in our investigation that are known to be difficult for students, e.g., the relation between algebraic relations and graphs.

As can be inferred from the discussion in this article, we believe that facility with graphing can play a critical role in helping students deepen their understanding of the kinematical concepts. The same is true for other topics in physics. However, the benefits to students of emphasizing graphs in a physics course extend beyond their application to the material covered. For most students taking physics, either in high school or in college, an ability to work with graphs is likely to be more useful in future academic work than knowledge acquired about any specific topic. It has been our experience that literacy in graphical representations often does not develop spontaneously and that intervention in the form of direct instruction is needed.

ACKNOWLEDGMENTS

The authors would like to thank Brian D. Popp for his substantive contributions. The research and curriculum development projects on which the paper is based have been partially supported by the National Science Foundation under grant nos. SED81-12924 and DPE84-70081.

[a] Present address: Shell Oil Company, Rocky Mountain Division, P. O. Box 831, Houston, TX 77001.

[1] D. E. Trowbridge and L. C. McDermott, Am. J. Phys. **48**, 1020 (1980); Am. J. Phys. **49**, 242 (1981).

[2] M. L. Rosenquist and L. C. McDermott, Am. J. Phys. **55**, 407 (1987).

[3] M. L. Rosenquist, Ph.D. thesis, University of Washington, Seattle, WA, 1982 (unpublished).

[4] Some of these studies have been discussed at recent meetings. See, for example, F. Goldberg and J. Anderson, AAPT Announ. **14** (4), 78 (1984); F. Goldberg, B. C. A'Hearn, and B. E. Churchill, AAPT Announ. **15** (4), 82 (1986).

[5] Written reports of studies of graphing difficulties include J. Clement, *Proceedings of the Ninth Conference of the International Group for the Psychology of Mathematics Education* (State University of Utrecht, The Netherlands, 1985); G. J. Hitch, M. C. Beveridge, S. E. Avons, and A. T. Hickman, in *Acquisition of Symbolic Skills*, edited by D. Rogers and J. A. Sloboda (Plenum, New York, 1983), pp. 551–560; C. Janvier, Ph.D. thesis (University of Nottingham, England, 1978) (unpublished); R. K. Thornton, 7th National Educational Computing Conference, San Diego, CA (1986).

[6] P. C. Peters, Am. J. Phys. **50**, 501 (1982).

[7] M. L. Rosenquist and L. C. McDermott, *Properties of Matter and Kinematics* (ASUW, University of Washington, Seattle, WA, 1984).

[8] L. C. McDermott, M. L. Rosenquist, and E. H. van Zee, New Direct. Teach. Learn. **16**, 59 (1983).

[9] A. B. Arons, Phys. Teach. **21**, 576 (1983). The importance of graphing in helping students develop conceptual understanding is also discussed in Phys. Teach. **22**, 88 (1984).