SISSA

Advanced Analysis - A

Academic year 2019-2020

Proposed problems

1. Let X be a separable Banach space and Y a subspace of X. Show that Y, endowed with the induced norm, is separable.

2. Let X be a Banach space and Y a finite-dimensional subspace of X. Show that Y is closed.

3. Let (M, d) be a compact metric space. Show that M is complete and separable.

4. Let (M, d) be a complete metric space and $\{A_n, n \in \mathbb{N}\}$ a countable family of open and dense subsets of M. Show that the set

$$A \doteq \bigcap_{n\in\mathbb{N}} A_n$$

is dense in M.

5. Let H be a real Hilbert space and $a \in H$ a nonzero vector. Show that for every $x \in H$, we have

$$\text{dist}(x, \{a\}^{\perp}) = \frac{|(x, a)|}{\|a\|}.$$

6. Consider the Hilbert space ℓ^∞ with its usual norm $\|\cdot\|_{\ell^\infty}$ and the sets $c_0 \doteq \{(a_n) \in \ell^\infty : a_n \to 0\}$ and $c \doteq \{(a_n) \in \ell^\infty : a_n \to a \in \mathbb{R}\}$. Show that c_0 and c are closed separable subspaces of ℓ^∞.

Answer. Let $c_c(\mathbb{N}, \mathbb{R})$ be the elements with compact support (recall, we are dealing with functions $\mathbb{N} \to \mathbb{R}$). Then it is easy to see that $c_0 = \overline{c_c(\mathbb{N}, \mathbb{R})}$ and that $c_c(\mathbb{N}, \mathbb{Q})$ is a countable dense subspace of $c_c(\mathbb{N}, \mathbb{R})$, and so also of c_0. Since $c = \overline{\bigcup_{q\in\mathbb{Q}}(c_0 + q)}$ we obtain that also c is separable.

7. Consider the Hilbert space ℓ^2 and a real sequence (a_n) such that $a_n > 0$ for every $n \in \mathbb{N}$ and $a_n \to +\infty$. Show that the set

$$A \doteq \left\{ u \in \ell^2 : \sum_{n\in\mathbb{N}} a_n |u_n|^2 \leq 1 \right\}$$

is a precompact subset of ℓ^2.

Answer. Let $T : D(T) \to \ell^2$ be the map $x \longrightarrow y = ax$ where $x, y, a : \mathbb{N} \to \mathbb{R}$ and

$$D(T) \doteq \left\{ u \in \ell^2 : \sum_{n\in\mathbb{N}} a_n |u_n|^2 < \infty \right\}.$$

Now, it is easy to see that the above map is invertible, and that $\ell^2 \ni y \longrightarrow x = T^{-1}y := \frac{y}{a} \in D(T) \subset \ell^2$ is in fact a compact operator. Then, since $A = T^{-1}\overline{D_{\ell^2}(0, 1)}$ we conclude that A is relatively compact.

8. Let H be a Hilbert space and C_1, C_2 two nonempty, closed and convex subsets such that $C_1 \subset C_2$. Given $x \in H$, call $P_{C_i}x$ the projection of x on C_i and $d(x, C_i)$ the distance of x from C_i $(i = 1, 2)$. Show that

$$\|P_{C_1}x - P_{C_2}x\|^2 \leq 2\left(d(x, C_1)^2 - d(x, C_2)^2\right), \quad \forall x \in H.$$

9. Let H be a complex Hilbert space and $T \in \mathcal{L}(H)$ an operator such that $\|T\| \leq 1$. Show that

(a) $Tx = x$ if and only if $(Tx, x) = \|x\|^2$;
(b) $ker(I - T) = ker(I - T^*)$.

Answer. In (a) the implication $\Rightarrow$ is obvious. So let us consider $\Leftarrow$ and let us consider a nonzero $x \in H$ s.t. $(Tx, x) = \|x\|^2$. Then $\|x\|^2 \leq \|Tx\|\|x\| \leq \|x\|^2$, where we used $\|T\| \leq 1$, implies $\|Tx\| = \|x\|$. Now, for $\widehat{x} = x/\|x\|$ consider the orthogonal decomposition

$$Tx = (Tx, \widehat{x})\widehat{x} + (Tx - (Tx, \widehat{x})\widehat{x}) = x + (Tx - (Tx, \widehat{x})\widehat{x}).$$

Then, by $\|Tx\| = \|x\|$ and

$$\|Tx\|^2 = \|x\|^2 + \|Tx - (Tx, \widehat{x})\widehat{x}\|^2,$$

we conclude that $Tx - (Tx, \widehat{x})\widehat{x} = 0$, and so $Tx = x$.

10. Find a Banach space X and a subset $S \subseteq X$ such that S is strongly closed but not weakly closed.

Answer. In infinite dimension, take $S = \{x \in X : \|x\| = 1\}$. In general, if $f \in C^0(X, \mathbb{R})$ is a continuous convex function with $\lim_{x\to\infty} f(x) = +\infty$, then for $S_r = \{x \in X : f(x) = r\}$ is strongly closed but , by repeating the proof in the special case $f(x) := \|x\|$, the weak closure is $\{x \in X : f(x) \leq r\}$.

11. Find a Banach space X, a bounded closed subset $S \subseteq X$ and a continuous function $f : S \to \mathbb{R}$ such that

$$\sup_{x \in S} f(x) = +\infty.$$

Answer. Recall that one of the exercises of Cuccagna's notes states the following:

Let X be an infinite dimensional Banach space. Show that for any $r \in (0, 1/2)$ there exists a sequence $\{v_n\}$ in X such that $\|v_n\|_X = 1$ and the closed balls $\overline{D_X(v_n, r)}$ are pairwise disjoint. Show also that $\bigcup_{n=1}^{\infty} \overline{D_X(v_n, r)}$ is a closed set in X.

if we assume the above statement, let any $g \in C^0(X, \mathbb{R})$ with $g(0) \neq 0$ and $g(x) = 0$ for $\|x\| \geq 1/2$. Then consider

$$f(x) = \sum_{n=1}^{\infty} ng\left(\frac{x - v_n}{r}\right)$$

Then each term of the sum is zero outside $\overline{D_X(v_n, r/2)}$, so it is easy to conclude that $f \in C^0(X, \mathbb{R})$. Obviously we have $\bigcup_{n=1}^{\infty} \overline{D_X(v_n, r)} \subseteq \overline{D_X(0, r+1)} =: S$, so we have the desired result.

12. Let X be a Banach space and $K \subseteq X$ a compact subset. Show that any sequence in K which converges weakly, actually converges strongly.

Answer. If false, there would be an example of $x_n \rightharpoonup x$ which does not converge strongly to x. On the other hand, for any strongly convergence subsequence $\{x_{n_k}\}$, we have necessarily $x_{n_k} \xrightarrow{k\to+\infty} x$ (because of $x_{n_k} \rightharpoonup x$). But the fact that it is false that $x_n \xrightarrow{n\to+\infty} x$, implies that there exists an $\epsilon_0 >$ and a subsequence $\{x_{n_k}\}$ with $\|x_{n_k} - x\| \geq \epsilon_0$. By compactness, $\{x_{n_k}\}$ has a subsequence that converges strongly at a point $y \in K$ with $\|y - x\| \geq \epsilon_0$. But we have just discussed the fact that we must have $y = x$. So we get a contradiction.

13. Let (X, d) be a metric space. Given two subsets $A, B \subseteq X$, set

$$\operatorname{dist}(A, B) \doteq \inf\{d(x, y) : x \in A, y \in B\}.$$

a) Given $x \in X$ and positive numbers $0 < \rho < r$, show that there exists $\delta > 0$ such that

$$\operatorname{dist}(B(x, \rho), B(x, r)^c) \geq \delta.$$

b) Given a proper, nonempty, closed subset $C \subseteq X$, show that there exists a ball $B(x, r)$ in X such that $\operatorname{dist}(B(x, r), C) > 0$.

14. Let X, Y be Banach spaces and $T \in \mathcal{L}(X, Y)$ a compact operator. Let (x_n) be a sequence in X weakly converging to x in X. Show that the sequence (Tx_n) converges strongly to Tx in Y.

Answer. We know that there is $C > 0$ such that $\|x_n\| < C$ for all n. Then (Tx_n) is a sequence in $K := \overline{TD_X(0, C)}$, which is a compact subspace of Y. Since T is continuous for the weak topologies, we know $Tx_n \rightharpoonup Tx$. Then use the result in Exercise 12.

15. Let $\alpha > 0$ and consider the sequence of functions given by

$$u_n(x) \doteq \min\{1, |x|^{-\alpha}\}\chi_{B(0,n)}(x), \quad n \in \mathbb{N},\ x \in \mathbb{R}^d.$$

Study the convergence of (u_n) in the strong and weak (weak* if $p = \infty$) topology of $L^p(\mathbb{R}^d)$ for $p \in [1, \infty]$.

Answer. If we set $u(x) := \min\{1, |x|^{-\alpha}\}$, we have $u \in L^p(\mathbb{R}^d)$ for $ap > d$. Now $u(x) - u_n(x) = \chi_{\mathbb{R}\setminus B(0,n)}(x)u(x)$ which converges monotonically to 0 for any $x \in \mathbb{R}^d$. For $d/a < p < \infty$, by dominated convergence, we obtain $u_n \xrightarrow{n\to\infty} u$ in $L^p(\mathbb{R}^d)$. If $p = \infty$, we have

$$0 \leq u(x) - u_n(x) = \chi_{\mathbb{R}\setminus B(0,n)}(x)u(x) \leq |n|^{-\alpha},$$

and so we get $u_n \xrightarrow{n\to\infty} u$ in $L^\infty(\mathbb{R}^d)$. Obviously, the above implies also weak convergence. For $p \leq d/a$, it is easy to check that $\|u_n\|_{L^p(\mathbb{R}^d)} \xrightarrow{n\to\infty} \infty$, and so the sequence in not weakly convergent.

16. Let H be a Hilbert space, $T \in \mathcal{L}(H)$ and T^* the adjoint of T.

(a) Show that $\|T^*T\| = \|TT^*\| = \|T\|^2$.

(b) Show that T^*T and TT^* are selfadjoint operators.

17. Let H be a Hilbert space and $\{M_k, k \in \mathbb{N}\}$ a countable collection of finite-dimensional subspaces of H. Call P_k the orthogonal projector on M_k ($k \in \mathbb{N}$) and set
$$P \doteq \sum_{k=1}^{\infty} 2^{-k} P_k.$$
Show that P is a compact operator in $\mathcal{L}(H)$.

Answer. First of all, we have $\|P_k\| \leq 1$ for all k. Then
$$\|P - S_N\| \leq \sum_{k=N+1}^{\infty} 2^{-k} = 2^{-N} \xrightarrow{N\to+\infty} 0, \text{ where } S_N := \sum_{k=1}^{N} 2^{-k} P_k.$$
Since S_N is finite rank for any N, we get that P is compact.

18. Consider the sequence of functions given by
$$u_n(x,y) = \left(\cos\left(\frac{x}{n}\right) + \sin\left(\frac{x}{n}\right)\right)(1+e^{-ny^2}), \quad (x,y) \in I \doteq [-1,1]\times[-1,1], \quad n \in \mathbb{N}.$$
Study the convergence of (u_n) in the strong and weak topology of $L^p(I)$ (weak* if $p = \infty$).

Answer. First of all, we have
$$u_n(x,y) - v_n(x,y) = \left(\cos\left(\frac{x}{n}\right) + \sin\left(\frac{x}{n}\right)\right) e^{-ny^2} \text{ for}$$
$$v_n(x,y) := \cos\left(\frac{x}{n}\right) + \sin\left(\frac{x}{n}\right).$$
It is straightforward that for $p < \infty$ we have
$$\|u_n - v_n\|_{L^p(I)} \leq 2\|e^{-ny^2}\|_{L^p(0,1)} \leq 2n^{-\frac{1}{2p}} \|e^{-y^2}\|_{L^p(\mathbb{R})} \xrightarrow{n\to\infty} 0.$$
Since
$$\|u_n - v_n\|_{L^\infty(I)} \geq \|v_n(\cdot,0)\|_{L^\infty((0,1))} \geq |v_n(0,0)| = 1,$$
we obviously do not have $u_n - v_n \xrightarrow{n\to\infty} 0$ strongly in $L^\infty(I)$. However, for $f \in L^1(I)$ we have
$$|\langle u_n - v_n, f\rangle| \leq 2\int_I e^{-ny^2}|f(x,y)|dxdy \xrightarrow{n\to\infty} 0$$
by dominated convergence, and so $u_n - v_n \rightharpoonup 0$ weakly in $L^\infty(I)$. Since (use bounds of errors for alternating series)*
$$|v_n(x,y) - 1| \leq \left|\cos\left(\frac{x}{n}\right) - 1\right| + |\sin\left(\frac{x}{n}\right)| \leq \frac{x^2}{2n^2} + \frac{x}{n} \leq \frac{3}{2n}$$
we have $v_n \xrightarrow{n\to\infty} 1$ in $C^0(I)$, and so in particular also for all $L^p(I)$. So, summing up, $u_n \xrightarrow{n\to\infty} 1$ strongly in $L^p(I)$ for $p < \infty$ and $u_n \rightharpoonup 1$ weakly, but not strongly, if $p = \infty$.*

19. Let H be a complex Hilber space, $T \in \mathcal{L}(H)$ and (x_n) a sequence in H weakly converging to $x \in H$. Show that the sequence (Tx_n) converges weakly to Tx.
Answer. We have
$$(Tx_n, y) = (x_n, T^*y) \xrightarrow{n\to+\infty} (x, T^*y) = (Tx, y) \text{ for all } y \in H \Longrightarrow Tx_n \rightharpoonup Tx.$$

20. Given $x \in \mathbb{R}$, let $B(x,1)$ be the open unit ball of center x in $\mathbb{R}$. Consider a sequence (x_n) in $\mathbb{R}$ and define the sequence of functions $u_n \doteq \chi_{B(x_n,1)}$, where χ denotes the characteristic function. Study the strong and weak convergence of the sequence (u_n) in the space $L^2(\mathbb{R})$ (that is to say establish if the sequence is converging in such topologies and, in affirmative case, find the limit), in the following cases:

(a) $x_n \to 0$;
(b) $|x_n| \to +\infty$.

Answer. Notice that $u_n(x) = \chi_{B(0,1)}(x - x_n)$ *and that in Cuccagna's notes it is shown that if* $x_n \xrightarrow{n\to\infty} \infty$ *then* $\chi_{B(0,1)}(\cdot - x_n) \rightharpoonup 0$ *in* $L^2(\mathbb{R})$. *In the case* $x_n \xrightarrow{n\to\infty} 0$ *we have, instead,* $\chi_{B(0,1)}(\cdot - x_n) \xrightarrow{n\to\infty} \chi_{B(0,1)}$ *strongly in* $L^2(\mathbb{R})$, *by the fact that the group* $\mathbb{R}^d$ *is strongly continuous (but not continuous in the operator norm) in* $L^2(\mathbb{R})$.

21. Let H be a Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$ and D a subset of H such that $\operatorname{lsp}(D)$ is dense in H. Show that, given a bounded sequence (x_n) in H, such that $\langle x_n, y\rangle \to \langle x, y\rangle$ for any $y \in D$, then $x_n \rightharpoonup x$.

22. Let $I = [0,1] \subseteq \mathbb{R}$ and consider the Hilbert space $X = L^2(I, \mathbb{R})$. Set

$$(Tu)(x) \doteq \int_0^x u(t)\, dt.$$

Show that $T \in \mathcal{L}(X)$ and find the adjoint T^* of T. *Answer. Continuity is very simple, since, in fact,* T *is a bounded operator from* $L^2(I,\mathbb{R})$ *into* $C^{1/2}(I)$, *by, for* $x < y$,

$$|Tu(x) - Tu(y)| \le \int_x^y |u(t)|dt \le |x-y|^{\frac{1}{2}} \|u\|_{L^2(I)}$$

Notice that this implies that $TD_{L^2(I)}(0,1)$ *is relatively compact in* $C^0(I)$, *and so also in* $L^2(I)$, *which implies that* T *is a compact operator. For the adjoint*

$$\int_0^1 Tu(x)v(x)dx = \int_0^1 dx v(x) \int_0^x u(t)dt = \int_0^1 dt\ u(t) \int_t^1 v(x)dx \Longrightarrow T^*v(x) = \int_x^1 v(t)dt.$$

23. Consider the set $E \doteq \{e^n, n \in \mathbb{N}\}$ in ℓ^2 defined by

$$e^n(k) = \delta_{n,k}.$$

Show that E is a Hilbert basis in ℓ^2.

24. Let $\mathcal{U}$ be a bounded family in $L^1(\mathbb{R})$ and $\rho \in C_c^\infty(\mathbb{R})$. Show that the family $\{\rho \star u, u \in \mathcal{U}\}$ is equicontinuous.

Answer. Since $\rho \in C_c^\infty(\mathbb{R})$, ρ *is uniformly continuous, and so*

$$\forall\ \epsilon > 0 \quad \exists\ \delta_\epsilon > 0 \ s.t.\ |x_1 - x_2| < \delta_\epsilon \Longrightarrow |\rho(x_1) - \rho(x_2)| < \ \epsilon$$

Since there exists $C > 0$ *s.t.* $\|u\|_{L^1(\mathbb{R})} < C$ *for all* $u \in \mathcal{U}$, *for* $|x_1 - x_2| < \delta_\epsilon$ *we have*

$$|\rho \star u(x_1) - \rho \star (x_2)| \le \int |\rho(x_1 - y) - \rho(x_2 - y)|\ |u(y)|dy < C\epsilon \text{ for all } u \in \mathcal{U}.$$

This implies that $\{\rho \star u, u \in \mathcal{U}\}$ is equicontinuous.

25. Let H be a Hilbert space and $T \in \mathcal{L}(H)$. Show that T is compact if and only if the adjoint T^* is compact.

Answer. T is compact if and only if for any $\epsilon > 0$ there exists a finite rank T_ϵ s.t. $\|T - T_\epsilon\| < \epsilon$. Notice now, that $\|T^* - T_\epsilon^*\| = \|T - T_\epsilon\| < \epsilon$. So, if T_ϵ^* is finite rank, then we conclude the exercise. So let S be finite rank. Then

$$S = \sum_{j=1}^{n} (\cdot, f_j) g_j.$$

In the special case $S = (\cdot, f_1) g_1$

$$(Su, v) = ((u, f_1)g_1, v) = (u, f_1)(g_1, v) = (u, \overline{(g_1, v)} f_1) = (u, S^* v).$$

So

$$S^* v = \overline{(g_1, v)} f_1 = (v, g_1) f_1.$$

So, more generally

$$S^* = \sum_{j=1}^{n} (\cdot, g_j) f_j,$$

which implies that T_ϵ^* is finite rank.

26. Let H be a Hilbert space on $\mathbb{C}$, $\{e_k, k \in \mathbb{N}\}$ an orthonormal system in H and (λ_k) an element of $\ell^1(\mathbb{C})$. Set

$$Tx \doteq \sum_{k=1}^{\infty} \lambda_k (x, e_k) e_k.$$

Show that T is a compact operator in $\mathcal{L}(H)$.

27. Consider the Hilbert space $E \doteq L^2(\mathbb{R}^n, \mathbb{C})$ and let $K \in L^2(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{C})$. Define

$$(T_K u)(x) \doteq \int_{\mathbb{R}^n} K(x, y) u(y)\, dy.$$

Show that $T_K \in \mathcal{L}(E)$ and that T_K is selfadjoint if and only if $K(x, y) = \overline{K(y, x)}$ for any pair $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$.

Answer. $u, v \in C_c^0(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} (T_K u)(x) \overline{v(x)} dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x, y) u(y)\, dy \overline{v(x)} dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x, y) \overline{v(x)} dx u(y)\, dy$$
$$= \int_{\mathbb{R}^n} \overline{\int_{\mathbb{R}^n} \overline{K(y, x)} v(y) dy} u(x)\, dx = \int_{\mathbb{R}^n} u(x) \overline{T_S v(x)} dx$$

with $S(x, y) = \overline{K(y, x)}$. If the operator is selfadjoint, we conclude that

$$\int_{\mathbb{R}^n} u(x) \overline{T_S v(x)} dx = \int_{\mathbb{R}^n \times \mathbb{R}^n} u(x) \overline{v(y)} K(y, x) dx dy = \int_{\mathbb{R}^n} u(x) \overline{T_K v(x)} dx$$
$$= \int_{\mathbb{R}^n \times \mathbb{R}^n} u(x) \overline{v(y) K(x, y)} dx dy \text{ for all } u, v \in C_c^0(\mathbb{R}^n).$$

Since $C_c^0(\mathbb{R}^n)\bigotimes C_c^0(\mathbb{R}^n)$ *generates all* $L^2(\mathbb{R}^n\times\mathbb{R}^n,\mathbb{C})$*, from the above we conclude*
$\int_{\mathbb{R}^n\times\mathbb{R}^n} f(x,y)K(y,x)dxdy = \int_{\mathbb{R}^n\times\mathbb{R}^n} f(x,y)\overline{K(x,y)}dxdy$ *for all* $f\in L^2(\mathbb{R}^n\times\mathbb{R}^n,\mathbb{C})$.
This implies $K(x,y)=\overline{K(y,x)}$ *for a.a.* $(x,y)\in\mathbb{R}^n\times\mathbb{R}^n$.

28. Let H be a Hilbert space and (u_n) an orthonormal sequence in H. Show that (u_n) converges weakly to zero. *Answer. For any* $f\in H$ *we have*

$$f=\sum_{n=1}^{\infty}(f,u_n)u_n \text{ with } \|f\|^2=\sum_{n=1}^{\infty}|(f,u_n)|^2.$$

Obviously the above implies $(u_n,f)\xrightarrow{n\to+\infty}0$ *for any* $f\in H$. *This is equivalent to* $u_n\rightharpoonup 0$.

29. Let $p\in[1,\infty[$ and $f\in L^p(\mathbb{R})$. Show that for every $\delta>0$ we have

$$\operatorname{meas}(\{x:|f(x)|>\delta\})\le\delta^{-p}\|f\|_p^p.$$

Answer. This is the well known, and simple to prove, Chebyshev's inequality.

30. Let $E\subseteq\mathbb{R}$ be a measurable set with finite measure, $p\in\,]1,\infty]$, (u_n) a sequence in $L^p(E)$ and $u\in L^p(E)$ such that $u_n\rightharpoonup u$ for $p<\infty$ or $\overset{*}{\rightharpoonup}$ if $p=\infty$. Prove that the sequence (u_n) is equiintegrable.

31. Let H be a real Hilbert space, $M\subseteq X$ a closed subspace and P the orthogonal projector on M. Show that P is selfadjoint. *Answer. Recall that* $Px\in M$ *with*

$$(Px-x,y)=0 \text{ for all } x\in H \text{ and } y\in M.$$

Notice that $H=M\oplus M^{\perp}$. *Then, since*

$$(x,(P^*-1)y)=0 \text{ for all } x\in H \text{ and } y\in M \implies P^*y=y \text{ for all } y\in M.$$

So $P^*=P$ *in* M. *Next,*

$$0=(Px,y)=(x,P^*y) \text{ for all } x\in H \text{ and } y\in M^{\perp}.$$

So $P^*y=0$ *for* $y\in M^{\perp}$. *Since we also have* $Py=0$ *for* $y\in M^{\perp}$, *we conclude* $P^*=P$ *everywhere in* H.

32. Consider the space $X=C_0(\mathbb{R}^d)\doteq\overline{C_c(\mathbb{R}^d)}^{\|\cdot\|_\infty}$. Given $S\in X'$ define

$$U\doteq\left\{O\subseteq\mathbb{R}^d \text{ open: } \langle S,u\rangle_{X',X}=0\ \forall u\in X \text{ with } \operatorname{supp}u\subseteq O\right\}.$$

Then introduce

$$N\doteq\bigcup_{O\in U}O \text{ (domain of nullity of } S); \qquad \operatorname{supp}S\doteq\mathbb{R}^d\setminus N \text{ (support of } S).$$

(i) Given $a\in\mathbb{R}^d$, set

$$T_a(u)=u(a).$$

Show that $T_a\in X'$ and find its norm and support.

(ii) Let (a_n) be a sequence in $\mathbb{R}^d$, consider the sequence $(S_n) = (T_{a_n})$ and the series

$$S = \sum_{n=1}^{\infty} 3^{-n} S_n.$$

(a) Show that $S \in X'$ and find its norm and support.
(b) Show that there exists a subsequence (S_{n_k}) weakly* converging in X'.

Answer. We have $\|T_a\| = 1$ *as can be seen that more generally considering*

$$\sigma_N f = \sum_{n=1}^{N} 3^{-n} f(a_n) \text{ for } \sigma_N := \sum_{n=1}^{N} 3^{-n} S_n$$

and, since we can always find $f \in C_c(\mathbb{R}^d)$ *of norm 1 and with* $f(a_n) = 1$ *for all* $n = 1, ..., N$*, we get*

$$\|\sigma_N\| = \sum_{n=1}^{N} 3^{-n} = 3^{-1} \frac{1 - 3^{-N-1}}{1 - 3^{-1}} = 2^{-1}(1 - 3^{-N-1})$$

The implies that for all $N < M$ *we have*

$$\|\sigma_N - \sigma_M\| = \sum_{n=N+1}^{M} 3^{-n} < \sum_{n=N+1}^{\infty} 3^{-n} = 3^{-N-1}\frac{3}{2}.$$

Hence, by completeness of X'*, there exists* S *limit of the above sum, with* $\|\sigma_N\| \xrightarrow{N\to+\infty} 2^{-1} = \|S\|$. *I skip the discussion of the support and go to the last question. Given any sequence* (a_n) *in* $\mathbb{R}^d$ *there always exists a subsequence* (a_{n_k}) *which either a limit* $a \in \mathbb{R}^d$ *or which goes to infinity. In the first case, for any* $f \in C_0(\mathbb{R}^d)$*, we have*

$$T_{a_{n_k}} f = f(a_{n_k}) \xrightarrow{k\to+\infty} f(a) = T_a f$$

while in the second case we have

$$T_{a_{n_k}} f = f(a_{n_k}) \xrightarrow{k\to+\infty} 0,$$

since it is easy to see that $C_0(\mathbb{R}^d) = \{f \in C^0(\mathbb{R}^d) : \lim_{x\to\infty} f(x) = 0\}$. *In the first case* $T_{a_{n_k}} \rightharpoonup T_a$ *while in the second* $T_{a_{n_k}} \rightharpoonup 0$.

33. Consider the sequence of functions given by $u_n(t) = \sin(nt)$, with $n \in \mathbb{N}$ and $t \in I \doteq [0, 2\pi]$. Study the convergence of (u_n) in the uniform topology of $C(I)$, in the strong topology of $L^\infty(I)$ and in the weak* topology of $L^\infty(I)$ (that is to say establish if the sequence is converging in such topologies and, in affirmative case, find the limit). *Answer. Clearly* $(\sin(nt))$ *is not convergent strongly, while for any* $f \in L^1(I)$ *we have*

$$\int_I \sin(nt) f(t) dt \xrightarrow{n\to+\infty} 0$$

by the Riemann Lebesgue lemma, and so $\sin(nt) \rightharpoonup 0$ *in the weak* topology of* $L^\infty(I)$.

34. Let $X = C_0(\mathbb{R}^2, \mathbb{R})$, endowed with the uniform norm, and (a_n) a sequence in $\mathbb{R}^+$. Set

$$\langle f_n, u\rangle \doteq \int_0^{2\pi} u(a_n \cos\theta, a_n \sin\theta)\, d\theta, \quad \forall n \in \mathbb{N},\ \forall u \in X.$$

Show that $f_n \in X'$ for every $n \in \mathbb{N}$ and find its norm and support.

Suppose $a_n \to 0+$ and study the convergence of the sequence (f_n) in the strong and weak* topology of X' (that is to say establish if the sequence converges in such topologies and, in the affirmative case, find the limit).

Answer. We have $f_n = \dfrac{1}{a_n} dx_{|\partial D(0,a_n)}$, *where* $dx_{|\partial D(0,a_n)}$ *is the restriction of the Lebesgue measure of* $\mathbb{R}^2$ *on the circle* $\partial D(0, a_n)$ *and with* $\|f_n\|_{(C_0)'} = 2\pi$. *It is elementary that for any* u *we have*

$$\int_0^{2\pi} u(a_n \cos\theta, a_n \sin\theta)\, d\theta \xrightarrow{n\to+\infty} u(0),$$

and so $f_n \rightharpoonup \delta(x)$ **–weakly . But it is easy to show that* $\|f_n - f_m\|_{(C_0)'} \geq 2\pi$ *for* $a_n \neq a_m$.

35. Given $\alpha \in \mathbb{R}$ and $R > 0$, consider the function u defined on $\mathbb{R}^d$ by

$$u(x) \;=\; \begin{cases} |x|^\alpha, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Establish for which $p \in [1, +\infty]$ we have $u \in L^p(B_{\mathbb{R}^d}(0, R))$.

Answer. We need $p\alpha > -d$. *So, for example, if* $p = \infty$, *then* $\alpha \geq 0$.

36. Let $E \subseteq \mathbb{R}^d$ be a measurable set, $p, q \in [1, \infty[$ and $u \in L^p(E) \cap L^q(E)$. Given $\alpha \in [0, 1]$, set

$$\frac{1}{r} \doteq \frac{1-\alpha}{p} + \frac{\alpha}{q}.$$

Show that $u \in L^r(E)$ and that

$$\|u\|_{L^r(E)} \leq \|u\|^{1-\alpha}_{L^p(E)} \cdot \|u\|^{\alpha}_{L^q(E)}.$$

Answer. We have

$$\|u\|_{L^r(E)} = \|\ |u|\ \|_{L^r(E)} = \|\ |u|^{1-\alpha}|u|^\alpha\ \|_{L^r(E)} \leq \|\ |u|^{1-\alpha}\ \|_{L^{\frac{p}{1-\alpha}}(E)} \||u|^\alpha\ \|_{L^{\frac{p}{\alpha}}(E)} \text{ by Hölder}$$
$$= \|u\|^{1-\alpha}_{L^p(E)} \cdot \|u\|^{\alpha}_{L^q(E)},$$

where we use the identity $\||f|^a\|_{L^q} = \|f\|^a_{L^{aq}}$.

37. Let $I \doteq [0, 1]$ and consider the sequence of functions given by

$$u_n(t) = e^{-nt}, \quad t \in I, \quad n \in \mathbb{N}.$$

Study the convergence of the sequence (u_n) in the following spaces:

(*i*) $C^0(I)$ endowed with the uniform topology;
(*ii*) $L^1(I)$ endowed with the strong topology;
(*iii*) $L^1(I)$ endowed with the weak topology;
(*iv*) $L^\infty(I)$ endowed with the strong topology;

(v) $L^\infty(I)$ endowed with the weak* topology.

Answer. *We have* $\lim_{n\to+\infty} e^{-nt} = 0$ *if* $t > 0$ *and 1 if* $t = 0$. *So, clearly the sequence is not convergent uniformly in* $[0,1]$ *(since the limit function is not continuous). This also excludes convergence in* $L^\infty(I)$ *endowed with the strong topology. We have*

$$0 < \int_0^1 e^{-nt}dt = n^{-1}\int_0^n e^{-t}dt < n^{-1} \xrightarrow{n\to+\infty} 0,$$

which implies that $e^{-nt} \xrightarrow{n\to+\infty} 0$ *strongly, and so also weakly, in* $L^1(I)$. *Finally, if* $f \in L^1(I)$

$$\int_0^1 e^{-nt}f(t)dt \xrightarrow{n\to+\infty} 0,$$

by dominated convergence, and shows $e^{-nt} \rightharpoonup 0$ *in* $L^\infty(I)$ *endowed with the weak* topology.*

38. Let $E \subseteq \mathbb{R}$ be a measurable set with finite measure and let $m \in L^\infty(E)$. Set

$$(Tu)(x) \doteq m(x)\cdot u(x) \quad \text{for a.e. } x \in E.$$

Given $p, q \in [1,\infty[$, with $p \geq q$, show that $T \in \mathcal{L}(L^p(E), L^q(E))$ and provide an estimate of its norm.

Answer. We have

$$\|mu\|_{L^q(E)} \leq \|m\|_{L^\infty(E)}\|u\|_{L^q(E)} = \|m\|_{L^\infty(E)}\|1_E u\|_{L^q(E)} \leq \|m\|_{L^\infty(E)}|E|^{\frac{1}{r}}\|u\|_{L^p(E)},$$

where $|E|$ *is the measure of* E *and* $\frac{1}{p} = \frac{1}{r} + \frac{1}{q}$.

39. Let $X = C_c(\mathbb{R})$ and $T : X \to X$ be a linear application such that

$$\|Tu\|_{L^1} \leq \|u\|_{L^1}; \qquad \|Tu\|_{L^2} \leq \|u\|_{L^1} \quad \forall u \in X.$$

Given $r \in [1,2]$, show that there exists $\tilde{T} \in \mathcal{L}(L^1, L^r)$ such that $\|\tilde{T}\|_{\mathcal{L}(L^1,L^r)} \leq 1$ and $\tilde{T}|_X = T$.

Answer. This is part of a more general result, called the Riestz interpolation Theorem. Here is just a consequence ot Hölder's inequality, because for $u \in C_c(\mathbb{R})$, *by exercise 36, using* $\dfrac{1}{r} = \dfrac{1-\frac{2}{r'}}{1} + \dfrac{\frac{2}{r'}}{2}$

$$\|Tu\|_{L^r} \leq \|Tu\|_{L^1}^{1-\frac{2}{r'}}\|Tu\|_{L^2}^{\frac{2}{r'}} \leq \|u\|_{L^1}^{1-\frac{2}{r'}}\|u\|_{L^1}^{\frac{2}{r'}} = \|u\|_{L^1}.$$

Then, by the density of $C_c(\mathbb{R})$ *in* $L^1(\mathbb{R})$, *we get the unique extension* $\tilde{T} \in \mathcal{L}(L^1, L^r)$ *with* $\|\tilde{T}\|_{\mathcal{L}(L^1,L^r)} \leq 1$ *and* $\tilde{T}|_X = T$.

40. Consider the sequence of functions given by

$$u_n(x,y) = \cos(nx)e^{-ny}, \quad (x,y) \in I \doteq [0,2\pi]\times[0,2\pi], \quad n \in \mathbb{N}.$$

(a) Study the equicontinuity of (u_n) on I.
(b) Study the convergence of (u_n) in the uniform topology of $C(I)$, in the strong topology of $L^\infty(I)$ and in the weak* topology of $L^\infty(I)$.

Answer. It is not equicontinuous. Indeed, for $y = 0$ and for fixed m and $n = 2mk$ we have

$$\cos(0) - \cos\left(2mk\frac{\pi}{2m}\right) = 1 - (-1)^k.$$

Since here $\frac{\pi}{2m}$ is arbitrarily close to 0 for $m \gg 1$, we see that equicontinuity in the origin is false. Since $u_n(x, y) \xrightarrow{n\to+\infty} 0$ pointwise for $y > 0$ and $u_n(0, 0) \equiv 0$, there can be no uniform convergence in $C(I)$, nor in the strong topology of $L^\infty(I)$ (which would imply uniform convergence in $C(I)$). But there is weak convergence in $L^\infty(I)$ to 0, like in exercise 37.*

41. Let $X = C_0(\mathbb{R}^2, \mathbb{R})$ endowed with the uniform topology and consider the family of subsets of $\mathbb{R}^2$ given by

$$A_\alpha \doteq \left\{(x, y) \in \mathbb{R}^2 : y > \alpha|x|,\ x^2 + y^2 < \alpha^{-2}\right\}, \quad \alpha > 0.$$

Set

$$T_\alpha u \doteq \int_{A_\alpha} u(x, y)\, dxdy, \quad \alpha > 0.$$

(a) Show that $T_\alpha \in X'$ for every $\alpha > 0$ and find its norm and support.
(b) Study the convergence of the family $(T_\alpha)_{\alpha>0}$ in the strong and weak* topology of X' when $\alpha \to 0+$ and when $\alpha \to +\infty$.

Answer. We have

$$|T_\alpha u| \leq \int_{A_\alpha} |u(x, y)| dxdy \leq \int_{A_\alpha} dxdy \|u\|_{L^\infty} = |A_\alpha| \|u\|_{L^\infty}.$$

So $\|T_\alpha\| \leq |A_\alpha|$ and it is easy to show that there is equality. Finally,

$$|A_\alpha| = 2\left(\frac{\pi}{2} - \arctan\alpha\right)\frac{1}{\alpha^2}.$$

We obviously have $T_\alpha \xrightarrow{\alpha\to+\infty} 0$ strongly, while we cannot have weak convergence for $\alpha \searrow 0$ because $\|T_\alpha\| \xrightarrow{\alpha\to0^+} +\infty$.*

42. Let $I = [0, 1]$, $X = C(I, \mathbb{R})$ and $Y = L^2(I)$. Set

$$(Tu)(x) \doteq \int_{x^2}^{x} u(t)\, dt.$$

(a) Show that $T \in \mathcal{L}(X)$ and establish if $T(B_1^X)$ is relatively compact in X.
(b) Show that $T \in \mathcal{L}(Y)$ and establish if $T(B_1^Y)$ is relatively compact in Y.

Answer. One can write $T = A - B$ with

$$Au(x) = \int_0^x u(t)\, dt$$

$$Bu(x) = \int_0^{x^2} u(t)\, dt$$

Using Hölder inequality, for $x_1 < x_2$,

$$|Au(x_1) - Au(x_2)| \le \int_{x_1}^{x_2} |u(t)|dt \le |x_1 - x_2|^{\frac{1}{2}} \|u\|_{L^2(I)}$$

$$|Bu(x_1) - Bu(x_2)| \le \int_{x_1^2}^{x_2^2} |u(t)|dt \le |x_1^2 - x_2^2|^{\frac{1}{2}} \|u\|_{L^2(I)} \le \sqrt{2}|x_1 - x_2|^{\frac{1}{2}} \|u\|_{L^2(I)}.$$

Then $T : Y \to C^{\frac{1}{2}}(I)$ is bounded, and hence $T(B_1^Y)$ is bounded in X and equicontinuous. This means that $T : Y \to X$ is compact and, a fortiori, also the other maps $T : Y \to Y$ and $T : X \to X$.

43. Consider the sequence of functions given by

$$u_n(x, y) = \sin\left(\frac{n^2 x}{n+1}\right) e^{y/n}, \quad (x, y) \in I \doteq [0, 2\pi] \times [0, 2\pi], \quad n \in \mathbb{N}.$$

(a) Study the equicontinuity of (u_n) on I.
(b) Study the convergence of (u_n) in the uniform topology of $C(I)$; in the strong and in the weak* topology of $L^\infty(I)$.

Answer. Using

$$\frac{1}{1 + \frac{1}{n}} = 1 - \frac{1}{n} + O\left(\frac{1}{n^2}\right)$$

we have

$$u_n(\pi/2, 0) = \sin\left(n\frac{\frac{\pi}{2}}{1 + \frac{1}{n}}\right) = \sin\left(n\frac{\pi}{2} - \frac{\pi}{2} + O\left(\frac{1}{n}\right)\right)$$
$$= -\cos\left(n\frac{\pi}{2} + O\left(\frac{1}{n}\right)\right) \sin\left(\frac{\pi}{2}\right) = -\cos\left(n\frac{\pi}{2} + O\left(\frac{1}{n}\right)\right)$$
$$= -\cos\left(n\frac{\pi}{2}\right) \cos\left(O\left(\frac{1}{n}\right)\right) + \sin\left(n\frac{\pi}{2}\right) \left(O\left(\frac{1}{n}\right)\right) = -\cos\left(n\frac{\pi}{2}\right) + O\left(\frac{1}{n}\right)$$

and

$$u_n(\pi/2 - \pi/(2n), 0) = \sin\left(n\frac{\left(\frac{\pi}{2} - \frac{\pi}{2n}\right)}{1 + \frac{1}{n}}\right) = \sin\left(n\frac{\pi}{2} - \pi + O\left(\frac{1}{n}\right)\right)$$
$$= -\sin\left(n\frac{\pi}{2} + O\left(\frac{1}{n}\right)\right).$$

For $n = 2k$ we have

$$u_n(\pi/2 - \pi/(2n), 0) - u_n(\pi/2, 0) = \cos(k\pi) - \sin\left(k\pi + O\left(\frac{1}{k}\right)\right) + O\left(\frac{1}{k}\right)$$
$$= \cos(k\pi) - \sin(k\pi) + O\left(\frac{1}{k}\right) = (-1)^k + O\left(\frac{1}{k}\right).$$

This excludes equicontinuity in $(\pi/2, 0)$, which would require that for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left|x - \frac{\pi}{2}\right| < \delta \Longrightarrow |u_n(x, 0) - u_n(\pi/2, 0)| < \epsilon \text{ for all } n.$$

There is no strong convergence since otherwise we would have equicontinuity, which we have just excluded. On the other hand, for any $f \in C^0(I)$ *we have*

$$\int_I u_n(x,y)f(x,y)dxdy = \int_0^{2\pi} dy e^{y/n} \int_0^{2\pi} \sin\left(\frac{n^2 x}{n+1}\right) f(x,y)dx$$
$$= \int_0^{2\pi} dy e^{y/n} \int_0^{2\pi} \sin\left(nx\left(1-\frac{1}{n}+O\left(\frac{1}{n^2}\right)\right)\right) f(x,y)dx$$
$$= \int_0^{2\pi} dy e^{y/n} \int_0^{2\pi} \sin\left(nx - x + O\left(\frac{1}{n}\right)\right) f(x,y)dx$$
$$= \int_0^{2\pi} dy e^{y/n} \int_0^{2\pi} \sin(nx)\cos\left(x+O\left(\frac{1}{n}\right)\right) f(x,y)dx$$
$$- \int_0^{2\pi} dy e^{y/n} \int_0^{2\pi} \cos(nx)\sin\left(x+O\left(\frac{1}{n}\right)\right) f(x,y)dx =: I_n + II_n.$$

We claim that $I_n \xrightarrow{n\to+\infty} 0$ *and* $II_n \xrightarrow{n\to+\infty} 0$. *We will prove only the first limit, since the second is similar. We have*

$$I_n = \int_0^{2\pi} dy e^{y/n} \int_0^{2\pi} \sin(nx)\cos(x) f(x,y)dx + O\left(\frac{1}{n}\right).$$

Then, for $f(x,y) = a(x)b(y)$

$$I_n \le \int_0^{2\pi} dy e^{2\pi} |b(y)| \left|\int_0^{2\pi} \sin(nx)\cos(x)\, a(x)dx\right| + O\left(\frac{1}{n}\right).$$

Since by Riemann-Lebesgue

$$\int_0^{2\pi} \sin(nx)\cos(x)\, a(x)dx \xrightarrow{n\to+\infty} 0$$

we get $I_n \xrightarrow{n\to+\infty} 0$ *by dominated convergence. This extends by linearity for all* $f \in L^1(0,2\pi) \bigotimes L^1(0,2\pi)$ *and, since the* u_n *are uniformly bounded in* n, *by density, to all* $f \in L^1(I)$.

44. Let $X = C_0(\mathbb{R}^2, \mathbb{R})$ endowed with the uniform norm and consider the family of subsets of $\mathbb{R}^2$ given by

$$A_\alpha \doteq \left\{(x,y) \in \mathbb{R}^2 : x > 0,\ y > \alpha|x|,\ x^2+y^2 < \alpha^2\right\}, \quad \alpha > 0.$$

Set

$$T_\alpha u \doteq \frac{1}{\alpha^2} \int_{A_\alpha} u(x,y)\, dxdy, \quad \alpha > 0.$$

(a) Show that $T_\alpha \in X'$ for any $\alpha > 0$ and find its norm and support.
(b) Establish if the family $(T_\alpha)_{\alpha>0}$ converges in the strong and weak* topology of X' when $\alpha \to 0+$ and, in affirmative case, determine the limit T_0.
(c) Find norm and support of T_0.

45. Let $I = [0,1]$, $X = C(I, \mathbb{R})$ and $\alpha(x) \doteq \min\{1, 2x\}$. Set

$$(Tu)(x) \doteq \int_0^{\alpha(x)} |u(t)|^2\, dt.$$

Establish if $T \in \mathcal{L}(X)$ and if $T(B_1^X)$ is relatively compact in X.

46. Consider the following family of Cauchy problems:

$$\begin{cases} y' = \frac{1}{1+ty} & t > 0 \\ y(0) = 1 + \frac{1}{n} & n \in \mathbb{N}. \end{cases}$$

(a) Show that for every $n \in \mathbb{N}$ there exists a solution $y_n(\cdot)$ defined on the whole $\mathbb{R}^+$.

(b) Show that the sequence (y_n) amdits a subsequence uniformly converging on each compact subinterval of $\mathbb{R}^+$.

Answer. First of all, for $y(0) > 0$ the corresponding solution is positive and strictly growing. Suppose that one of these solutions, with initial value $y_0 > 0$, has maximum forward time of existence $[0, t_1)$. We need to show that $t_0 = +\infty$. If not and $t_1 < +\infty$, then, by monotonicity, the limit $\lim_{t \to t_1^-} y(t)$ exists. If this limit is finite and equals to y_1, by $0 < y_0 < y_1$ we have $y_1 \in \mathbb{R}_+$. but then, considering

$$\begin{cases} y' = \frac{1}{1+ty} & t > 0 \\ y(t_1) = y_1 \end{cases}$$

it is easy to see that we can extend the previous equation beyond $[0, t_1)$. So we get a contradiction, and $\lim_{t \to t_1^-} y(t) = +\infty$. Let us show that also this is impossible. Let $0 < t_2 < t_1$ with $y(t_2) \geq 1$. Then, for $t_2 < t < t_1$ we have $y(t) > y(t_2)$

$$y(t) - y(t_2) = \int_{t_2}^{t} y'(s)ds = \int_{t_2}^{t} \frac{1}{1+sy(s)} ds \leq \int_{t_2}^{t} \frac{1}{1+t_2} dt = \frac{t-t_2}{1+t_2} \xrightarrow{t \to t_1^-} \frac{t_1 - t_2}{1+t_2}.$$

Hence $+\infty = \lim_{t \to t_1^-} (y(t) - y(t_2)) \leq \dfrac{t_1 - t_2}{1+t_2}$, which obviously is a contradiction. So we conclude that if $y(0) = y_0 > 0$, the corresponding solution is defined in $[0, +\infty)$. The uniform convergence on compact intervals, is a consequence of the "well posedness" of solutions of the Cauchy problem for ODE's, which is a very general fact. Suppose that (y_{0n}) is a sequence in $\mathbb{R}_+$ with $y_{0n} \xrightarrow{n \to +\infty} y_0$ with $y_0 \in \mathbb{R}_+$. Then we can prove that the $y_n \xrightarrow{n \to +\infty} y$ in $C^0([0, T])$ for any $T > 0$, where $y_n(t) = y_{0n}$ and $y(0) = y_0$ using the Growall inequality. From

$$y' - y'_n = f(t, y) - f(t, y_n)$$

we get, after some elementary computations,

$$|y(t) - y_n(t)| \leq \left(1 + \int_0^t A_n(s)ds\right) |y_0 - y_{0n}| + \int_0^t A_n(s)|y(s) - y_n(s)|ds$$

$$\text{with } A_n(s) := \frac{|f(s, y(s)) - f(s, y_n(s))|}{|y(s) - y_n(s)|}.$$

Notice now that

$$A_n(s) \leq \int_0^1 |\partial_y f(s, y_n(s) + \tau(y(s) - y_n(s)))| d\tau \leq \sup \{|\partial_y f(s, y)| : s \in [0, T] \text{ and } y \geq 0\} \leq T$$

So we get

$$|y(t) - y_n(t)| \leq \left(1 + T^2\right) |y_0 - y_{0n}| + T \int_0^t |y(s) - y_n(s)|ds.$$

Now, Gronwall's inequality yields

$$|y(t) - y_n(t)| \le e^{Tt}\left(1 + T^2\right)|y_0 - y_{0n}|$$

which yields $y_n \xrightarrow{n\to+\infty} y$ *in* $C^0([0,T])$.

47. Consider the sequence of functions given by

$$u_n(x,y) = \sin\left(\frac{nx}{n+1}\right)(1 + e^{-n|y|}), \quad (x,y) \in I \doteq [-1,1] \times [-1,1], \quad n \in \mathbb{N}.$$

(a) Study the equicontinuity of (u_n) on I.

(b) Study the convergence of (u_n) in the uniform topology of $C(I)$, in the strong topology of $L^\infty(I)$ and in the weak* topology of $L^\infty(I)$.

48. Let $I = [0,1]$, $X = C^0(I)$ and $m \in X$. Set

$$(T_m u)(x) \doteq m(x)u(x), \quad u \in X, \ x \in I.$$

Show that $T_m \in \mathcal{L}(X)$ and that it is compact if and only if $m(x) = 0$ for every $x \in I$.

Answer. We have $\sigma(T_m) = m(I)$. *We must have* $0 \in m(I)$ *and* $m(I)\setminus\{0\}$ *must be discrete and* $m(I)$ *must be connected. So, summing up,* $m(I) = \{0\}$, *which implies* $m \equiv 0$ *and* $T_m = 0$.

49. Let $\overline{B}$ the closed unit ball in $\mathbb{R}$, endowed with the euclidean norm $\|\cdot\|$. Define

$$u_n(x) \doteq |\sin(\|x\|)|^{\frac{1}{n}} \quad n \in \mathbb{N}.$$

Study the equicontinuity of the family $\{u_n, n \in \mathbb{N}\}$ on $\overline{B}$. *Answer. Recall that the condition of the Ascoli Arzela Theorem are both sufficient and necessary. Obviously the above sequence is is bounded in* $C^0(B)$. *Notice that pointwise we have*

$$|\sin(|x|)|^{\frac{1}{n}} \xrightarrow{n\to+\infty} \begin{cases} 1 \ \textit{if}\ x \neq 0 \\ 0 \ \textit{if}\ x = 0 \end{cases}$$

and this excludes the existence of a subsequence converging uniformly. So the sequence is not equicontinuous.

50. Consider the sequence of functions given by

$$u_n(x,y) = \frac{e^{-\frac{ny}{n+1}}}{(1 + e^{-nx^2})}, \quad (x,y) \in I \doteq [-1,1] \times [-1,1], \quad n \in \mathbb{N}.$$

Study the convergence of (u_n) in the uniform topology of $C(I)$, in the strong topology and in the weak* topology of $L^\infty(I)$.

51. Let $\varphi \in C_c(\mathbb{R})$ and (a_n) a sequence in $\mathbb{R}$. Define

$$u_n(x) \doteq \varphi(x - a_n), \qquad x \in \mathbb{R}, \quad n \in \mathbb{N}.$$

a) Show that $u_n \in L^p(\mathbb{R})$ for every $p \in [1,\infty]$.

b) Study the relative compactness of the sequence (u_n) in the strong and in the weak topology of L^p (weak* if $p = \infty$). That is to say: establish if and for which $p \in [1,\infty]$ there exists a converging subsequence in such topologies.

52. Let $I = [0,1] \subset \mathbb{R}$ and $B \doteq \{u \in C^1(I) : \|u'\|_{L^2(I)} \le 1\}$.

a) Show that B is an equicontinuous family.

b) Given a sequence (u_n) in $\{u \in B : u(0) = 0, u(1) = 1\}$, show that there exist $u \in C^0(I)$ and a subsequence (u_{n_k}) which converges uniformly to u.

c) Show by a counterexample that property **b)** does not hold in B.

53. Let $\overline{B}$ the closed unit ball in $\mathbb{R}^d$, endowed with the euclidean norm $\|\cdot\|$. Set

$$u_n(x) \doteq e^{-n\|x\|} \quad n \in \mathbb{N}.$$

Study the equicontinuity of the family $\{u_n, n \in \mathbb{N}\}$ on $\overline{B}$.

54. Consider the sequence of functions given by

$$u_n(x,y) = \min\left\{n, |x|^{-\frac{1}{2}}\right\} \sin\left(\frac{ny}{n+1}\right), \quad (x,y) \in I \doteq [-1,1] \times [-1,1], \quad n \in \mathbb{N}.$$

Study the convergence of (u_n) in the strong and weak topology (weak* if $p = \infty$) of $L^p(I)$.

55. Let $\varphi \in C_c(\mathbb{R})$ with $\operatorname{supp}\varphi \subseteq [-1,1]$, $\varphi \geq 0$ e $\int_{\mathbb{R}} \varphi\, dt = 1$. Consider the Dirac sequence given by

$$\rho_n(t) \doteq n\varphi(nt) \quad \forall t \in \mathbb{R} \quad \forall n \in \mathbb{N}$$

and let (a_n) be a sequence in $\mathbb{R}$. Set

$$u_n(t) \doteq \rho_n(x - a_n) \qquad \forall t \in \mathbb{R} \quad \forall n \in \mathbb{N}.$$

a) Show that $u_n \in L^p(\mathbb{R})$ for every $p \in [1,\infty]$ and for every $n \in \mathbb{N}$.

b) Considering the cases $a_n = n$ and $a_n = n^{-2}$, study the convergence of the sequence (u_n) in the strong and weak topology of L^p (weak* if $p = \infty$).

56. Let $I = [0,1] \subset \mathbb{R}$, $X = C^0(I)$ and $Y = L^1(I)$. Set

$$Tu(x) \doteq \int_0^x xyu(y)\, dy.$$

a) Show that $T \in \mathcal{L}(X)$ and $T \in \mathcal{L}(Y)$.

b) Establish if T is compact in $\mathcal{L}(X)$ and in $\mathcal{L}(Y)$, explaining the reasons.

57. Let $(\rho_n)_{n\in\mathbb{N}}$ be a regularizing sequence in $\mathbb{R}$. Study the equiintegrability of the following families:

a) $f_n = \rho_n, \quad n \in \mathbb{N};$

b) $g_n = \rho_n', \quad n \in \mathbb{N};$

c) $h_n \doteq \rho_1 \star \rho_n, \quad n \in \mathbb{N}.$

58. Let $\alpha > 0$ and consider the sequence of functions given by

$$u_n(x) \doteq \min\{1, |x|^{-\alpha}\}\chi_{B(0,n)}(x), \quad n \in \mathbb{N},\ x \in \mathbb{R}^d.$$

Study the strong and weak (weak* if $p = \infty$) convergence of (u_n) in the spaces $L^p(\mathbb{R}^d)$ for $p \in [1,\infty]$.

59. Let $Q \doteq [-1,1]^3 \subseteq \mathbb{R}^3$ and set

$$f(x_1,x_2,x_3) = \begin{cases} \left(x_1\, x_2^2\, x_3^3\right)^{-1}, & x_1\, x_2\, x_3 \neq 0 \\ 0, & x_1\, x_2\, x_3 = 0. \end{cases}$$

- Establish for which $p \in [1,\infty]$ we have $f \in L^p(\mathbb{R}^3)$;
- establish for which $p \in [1,\infty]$ we have $f \in L^p(Q)$;

- establish for which $p \in [1, \infty]$ we have $f \in L^p(\mathbb{R}^3 \setminus Q)$.

60. Let $E \subseteq \mathbb{R}$ be a measurable set, $p_i \in [1, \infty]$, $f_i \in L^{p_i}(E)$ for $i = 1, \dots, n$, and $r \in [1, \infty]$ given by

$$\frac{1}{r} \doteq \sum_{i=1}^{n} \frac{1}{p_i}.$$

Show that

$$\prod_{i=1}^{n} f_i \in L^r(E)$$

and that the following inequality holds:

$$\left\| \prod_{i=1}^{n} f_i \right\|_{L^r(E)} \leq \prod_{i=1}^{n} \|f_i\|_{L^{p_i}(E)}.$$

61. Let $(\rho_n)_{n \in \mathbb{N}}$ be a regularizing family in $\mathbb{R}$ and $f \in C^0(\mathbb{R})$. Set

$$f_n(x) \doteq (\rho_n \star f)(x), \quad x \in \mathbb{R}.$$

Show that the definition is well posed and that the sequence (f_n) converges uniformly to f on any compact subset $K \subseteq \mathbb{R}$.

62. Let $I = [-1, 1] \subseteq \mathbb{R}$ and (u_n) a sequence in $C^2(\mathbb{R})$ such that

(a) u_n is convex on $\mathbb{R}$ for every $n \in \mathbb{N}$;
(b) There exists $K \geq 0$ such that $|u_n(0)| + |u_n'(t)| \leq K$ for every $t \in I$ and for every $n \in \mathbb{N}$.

(1) Show that the sequence (u_n') is relatively compact in $L^1(I)$.
(2) Show that there exists a subsequence (u_{n_k}) and a map $u \in C^0(I)$ such that (u_{n_k}) converges uniformly to u on I.

63. Let $I = [0, 1] \subseteq \mathbb{R}$ and $\{e_n, n \in \mathbb{N}\}$ a Hilber basis $L^2(I)$. Set

$$(e_m \otimes e_n)(x, y) \doteq e_m(x) e_n(y); \quad m, n \in \mathbb{N}, \ (x, y) \in I \times I.$$

Show that the family $\{e_m \otimes e_n; \ m, n \in \mathbb{N}\}$ is a Hilbert basis in $L^2(I \times I)$.

Answer. It is an orthonormal family. Now, for $u, v \in L^2(I)$, we have

$$\sum_{n,m \in \mathbb{N}} |(u \otimes v, e_m \otimes e_n)|^2 = \sum_{n,m \in \mathbb{N}} |(u, e_m)|^2 |(v, e_n)|^2 = \sum_{m \in \mathbb{N}} |(u, e_m)|^2 \sum_{n \in \mathbb{N}} |(v, e_n)|^2$$
$$= \|u\|^2_{L^2(I)} \|v\|^2_{L^2(I)} = \|u \otimes v\|^2_{L^2(I \times I)}$$

This means that each $u \otimes v$ is in the closed space spanned by the above orthonormal family, hence also the linear conmbinations of these elements. Since the latter are dense in $L^2(I \times I)$, it follows that the space generated by the orthonormal family is $L^2(I \times I)$, and thus the orthonormal family is a Hilbert basis.

64. Let $I \doteq [-1, 1] \subseteq \mathbb{R}$ and consider the sequence of functions given by:

$$u_n(t) = e^{-n} \cdot e^{nt^2}; \quad t \in I, \quad n \in \mathbb{N}.$$

Study the convergence of the sequence (u_n) in the following spaces:

(i) $C^0(I)$ with uniform topology;

(*ii*) $L^1(I)$ with strong topology;
(*iii*) $L^1(I)$ with weak topology;
(*iv*) $L^\infty(I)$ with strong topology;
(*v*) $L^\infty(I)$ with weak* topology.

65. For every $n \in \mathbb{N}$ set

$$f_n(x) = \sin\left(\frac{x}{n}\right); \quad g_n(x) = \sin\left(n^2 x\right); \quad h_n(x) = \sin\left(\frac{nx}{n+1}\right); \quad x \in [0, 2\pi].$$

Study the equicontinuity of the sequences $\{f_n, n \in \mathbb{N}\}$, $\{g_n, n \in \mathbb{N}\}$ e $\{h_n, n \in \mathbb{N}\}$ on $[0, 2\pi]$.

66. Let $I = [0,1] \subset \mathbb{R}$ and, for every $n \in \mathbb{N}$, consider the subintervals of the form

$$I_n^m \doteq \left[\frac{m}{n}, \frac{m+1}{n}\right[, \quad m = 0, 1, \ldots, n-1.$$

Then set

$$u_n(t) \doteq (-1)^m \text{ for } t \in I_n^m.$$

Study the strong and weak convergence of the sequence (u_n) in $L^2(I)$.

67. Let D be te unit disk in $\mathbb{C}$. Study the equicontinuity of the following families of functions in $C(D)$:

(i) $\{f_a(z) = e^{iaz}, a \in \mathbb{R}\}$;
(ii) $\{f_a(z) = e^{i\frac{z}{a}}, a \in \mathbb{R}\ a \neq 0\}$;
(iii) $\{f_a(z) = e^{iaz}, a \in \mathbb{R}, |a| > 1\}$;
(iv) $\{f_a(z) = e^{iaz}, a \in \mathbb{R}, |a| < 1\}$.

68. Let $X = C([0,1], \mathbb{R})$ and (a_n) a sequence in $[0,1]$. Set

$$\langle f_n, u\rangle \doteq u(a_n), \ \forall n \in \mathbb{N}, \ \forall u \in X.$$

Show that $f_n \in X'$ for every $n \in \mathbb{N}$ and that there exists a subsequence (f_{n_k}) which converges in the topology $\sigma(X', X)$.

69. Study the equicontinuity of the following families in $C(I)$ $(I \subseteq \mathbb{R})$).

(i) $\{f_a(x) = e^{ax}, a \in \mathbb{R}\}$, $I = \mathbb{R}$;
(ii) $\{f_a(x) = a(1-x)^2, a \in \mathbb{R}^+\}, I = [-1, 1]$;
(iii) $\{f_a(x) = x^{-a}, a \in \mathbb{R}^+, \}, I =]1, +\infty[$;
(iv) $\{f_a(x) = x^{-a}, a \in \mathbb{R}^+, \}, I =]0, +\infty[$.

70. Let $p \in [1, \infty[$. Consider the space $X = L^p([0,1])$ and set

$$(Tu)(x) = \int_0^x u(t)\, dt.$$

(i) Show that $T \in \mathcal{L}(X)$ and that $\|T\|_{\mathcal{L}(X)} \leq \left(p^{\frac{1}{p}}\right)^{-1}$.
(ii) Given a sequence (u_n) in X weakly converging to u in X, show that the sequence (Tu_n) converges strongly to Tu in X.

Answer. The bound follows from

$$|Tu(x)| \le \int_0^x |u(t)|dt \le x^{\frac{1}{p'}} \|u\|_{L^p([0,1])}$$

and from

$$\|x^{\frac{1}{p'}}\|_{L^p([0,1])} = \left(\int_0^1 x^{\frac{p}{p'}} dx\right)^{\frac{1}{p}} = \left(\frac{1}{\frac{p}{p'}+1} dx\right)^{\frac{1}{p}} = \left(p^{\frac{1}{p}}\right)^{-1} \left(\frac{1}{\frac{1}{p'}+\frac{1}{p}} dx\right)^{\frac{1}{p}} = \left(p^{\frac{1}{p}}\right)^{-1}.$$

For part (ii), the result follows from the fact that $T : L^p([0,1]) \to L^p([0,1])$ is compact. The case $p = 1$ is discussed in Cuccagna's notes. The case $p > 1$ is easier because we have for any $x_1 < x_2$

$$|Tu(x_1) - Tu(x_2)| \le \int_{x_1}^{x_2} |u(x)|dx \le \sqrt[p']{|x_1 - x_2|}\|u\|_{L^p([0,1])}.$$

From this and Ascoli Arzela we conclude that $T : L^p([0,1]) \to C^0([0,1])$ is compact for $p > 1$ and so, at fortiori, also $T : L^p([0,1]) \to L^p([0,1])$ is compact.

71. Let $C > 0$, $p \in [1, \infty[$, $\alpha \in]0, 1[$ and $B \doteq \{x \in \mathbb{R}^d : \|x\| \le 1\}$. Consider the set

$$U \doteq \{u \in C(B) : u(0) = 0, |u(x) - u(y)| \le C|x - y|^\alpha \ \forall x, y \in B\}.$$

Show that U is relatively compact in $L^p(B)$.

Answer. It is immediate that U is bounded and equicontinuous and so relatively compact in $C^0(B)$, and hence also in $L^p(B)$.

72. Let $I \doteq [0, 1]$ and (u_n) a sequence in $C^1([0, 1])$ such that

$$|u_n(0)| + \int_I |u_n'(t)|\, dt \le 1 \quad \forall n \in \mathbb{N}.$$

Show that there exist a subsequence (u_{n_k}) and a map $u \in L^1(I)$ such that $u_{n_k} \to u$ strongly in $L^1(I)$.

73. Let $E \subseteq \mathbb{R}^d$ be ameasurable set such that $0 < m(E) < +\infty$. For every $p \in [1, +\infty[$ and for every $f \in L^p(E)$ set

$$N_p[f] \doteq \left(\frac{1}{m(E)} \int_E |f(x)|^p\right)^{\frac{1}{p}}.$$

Show that $N_p[\cdot]$ is a norm on $L^p(E)$ and that, if $1 \le p \le q < +\infty$, we have

$$N_p[f] \le N_q[f] \qquad \forall f \in L^q(E).$$

74. Let X be a Banach space and set $\mathcal{K}(X) \doteq \{T \in \mathcal{L}(X) : T \text{ is compact}\}$. Show that $\mathcal{K}(X)$ is closed in $\mathcal{L}(X)$.

75. Let $X = C_0(\mathbb{R}^2)$ and (a_n) a sequence in $\mathbb{R}^+$. For every $n \in \mathbb{N}$ and for every $u \in X$ set

$$T_n(u) = \int_{-a_n}^{+a_n} u(x, nx)\, dx.$$

Show that $T_n \in X'$ for every $n \in \mathbb{N}$ a find its norm and support. Study the convergence of the sequence (T_n) in the strong and weak* topology of X' in the cases $a_n = 1 + n^2$ and $a_n = e^{-\frac{1}{n}}$.

76. Let $I = [0,1]$ and H an equicontinuous subset of $C^0(I)$. Show that $\overline{H}$ is equicontinuous.

77. Let $I = [0,1]$, $B_r = B(0,r)$ the ball in $\mathbb{R}^d$ of center zero and radius r, $p \in [1,\infty]$, $X_p \doteq L^p(B_1)$ and $Y \doteq C^0(I,\mathbb{R})$. Given $u \in X_p$ and $t \in I$, set

$$(Tu)(t) \doteq \int_{B_t} u(y)\, dy.$$

Show that $T \in \mathcal{L}(X_p, Y)$ for every p and establish for which p it is compact.

78. For $(x,y) \in I \doteq [-1,1] \times [-1,1]$, consider the sequence of functions given by

$$u_n(x,y) = \left(\cos\left(\frac{nx^2}{n+1}\right)\sin(nx)\right)(1+e^{-ny^2}), \quad n \in \mathbb{N}.$$

Study the convergence of (u_n) in the strong and weak topology (weak* if $p=\infty$) of $L^p(I)$.

79. Let (a_n) and (b_n) sequence in $\mathbb{R}^+$ and set $R_n \doteq [-a_n, a_n] \times [-b_n, b_n] \subseteq \mathbb{R}^2$ and

$$u_n(x,y) \doteq \chi_{R_n}(x,y), \quad (x,y) \in \mathbb{R}^2.$$

Study the convergence of (u_n) in the strong and weak topology of $L^1(\mathbb{R}^2)$ and in the strong and weak* topology of $L^\infty(\mathbb{R}^2)$ in the following cases:

1. $a_n = n$, $b_n = n^{-1}$;
2. $a_n = n$, $b_n = n^{-\frac{1}{2}}$;
3. $a_n = \frac{n}{n+1}$, $b_n = n^{-1}$;
4. $a_n = \frac{n}{n+1}$, $b_n = \frac{n}{n+1}$.

80. Let $X = C_0(\mathbb{R}^2, \mathbb{R})$, endowed with the uniform norm, and $(a_n), (b_n)$ sequences in $\mathbb{R}^+$. Define

$$\langle f_n, u\rangle \doteq \int_0^{2\pi} u(a_n\cos\theta, b_n \sin\theta)\, d\theta, \quad n \in \mathbb{N},\ u \in X.$$

Show that $f_n \in X'$ for every $n \in \mathbb{N}$ and find its norm and support.

Suppose $a_n \to 1, b_n \to 0$ and study the convergence of the sequence (f_n) in the strong and weak* topology of X'.

81. Let $I = [0,1]$, $M > 0$ and (u_n) a sequence in $C^1(I)$ such that

1. $\int_I |u_n(t)|^2 \le M\ \forall n \in \mathbb{N}$;
2. $u_n'(t) + t \ge 0\ \forall t \in I, \forall n \in \mathbb{N}$.

Show that the sequence (u_n) is relatively compact in $L^1(I)$. *Answer. Here too, like in exercise 86 below, it is possible to restrict to case* $u_n(0) \equiv 0$. *Next, let* $u_n^{'+} = \max\{u_n', 0\}$ *and* $u_n^{'-} = \max\{-u_n', 0\}$. *Then*

$$u_n = u_n^{'+} - u_n^{'-} \text{ and } u_n = v_n - w_n \text{ with}$$
$$v_n(x) := \int_0^x u_n^{'+}(t)dt \text{ and}$$
$$w_n(x) := \int_0^x u_n^{'-}(t)dt.$$

Form $u_n^{'-}(t) \le t \le 1$ we have for any $x_1 < x_2$

$$|w_n(x_1) - w_n(x_2)| \le \int_{x_1}^{x_2} dt = |x_2 - x_1|.$$

So (w_n) is relatively compact in $C^0(I)$ by Ascoli–Arzelá, and hence also in $L^2(I)$ (where, hence, it is bounded) and $L^1(I)$. Also (v_n) is relatively compact in $L^1(I)$, by Exercise 86 below.

82. Let (x_n) be a sequence in a Hilbert space H endowed with the inner product $\langle \cdot, \cdot \rangle$. Show that, if the sequence $(\langle x_n, y \rangle)$ converges for every $y \in H$, then the sequence (x_n) converges weakly.

83. Let $I = [0,1]$ and call X the Banach space $C(I)$, endowed with the uniform norm. Introduce the space

$$Y \doteq \{u \in X, u \text{ differentiable on } I \text{ with } u' \in X\}$$

and set

$$\|u\|_Y \doteq \|u\|_\infty + \|u'\|_\infty, \; u \in Y.$$

Prove that $(Y, \|\cdot\|_Y)$ is a Banach space.

Let α be a nonzero element of X and set

$$(Tu)(x) \doteq \alpha(x) u'(x) \quad u \in Y, \; x \in I.$$

(i) Prove that $T \in \mathcal{L}(Y, X)$ and find its norm.
(ii) Establish if T is compact and justify the answer.

Answer. $(Y, \|\cdot\|_Y)$ is a Banach space. Indeed, let $(u_n, u_n') \xrightarrow{n \to +\infty} (u, v)$ in $C(I) \times C(I)$. Then, since for any $x \in I$ we have

$$u_n(x) = \int_0^x u_n'(t) dt,$$

it follows, taking the limit in n, that

$$u(x) = \int_0^x v(t) dt,$$

and so by the Fundamental Theorem of calculus, $v = u'$. We have $T \in \mathcal{L}(Y, X)$ with a bound

$$\|Tu\| \le \|\alpha\|_{L^\infty(I)} \|u'\|_{L^\infty(I)} \Longrightarrow \|T\| \le \|\alpha\|_{L^\infty(I)}.$$

Consider now that map $X \hookrightarrow Y$ given by $v \to (u, v)$, with $u' = v$ with $u(0) = 0$. Then, if T is compact, also the multiplier map $X \ni v \to Sv := \alpha v \in X$, is compact. Recall that $\sigma(S) = \alpha(I)$. It is clear, from the Spectral Theorem of compact operators, that $\alpha(I)$ must contain 0, be at most countable, and have 0 as unique accumulation point. Since $\alpha(I)$ is connected, it follows that we must have $\alpha(I) = \{0\}$, that is $\alpha \equiv 0$, hence T, if compact, is the 0 operator.

84. Let H be a Hilbert space. For $T \in \mathcal{L}(H)$ denote by $R(T)$ and $N(T)$, respectively, the range and the kernel of T. Calling T^* the adjoint of T, prove that $N(T) = (R(T^*))^\perp$ and $\overline{(R(T))} = (N(T^*))^\perp$.

85. Let $B_r = B(0,r)$ be the ball in $\mathbb{R}^d$ of center zero and radius r and $X \doteq C_0(\mathbb{R})$. Let m be a map in $C(\mathbb{R})$, with $m(x) \geq 0$ for every $x \in \mathbb{R}$, and, for every $t > 0$, set

$$T_t(u) \doteq t^{-d} \int_{B_t} m(y)u(y)\,dy.$$

Prove that $T_t \in X'$ for every $t > 0$ and find its norm and support. Study the convergence of T_t as $t \to 0+$ in the strong and weak* topology of X'. *Answer. It*

is obvious that

$$|T_t(u)| \leq t^{-d} \int_{B_t} m(y)\,dy \|u\|_{L^\infty(\mathbb{R}^d)}$$

so that this yields an element in $(L^\infty(\mathbb{R}^d))'$, in fact $t^{-d}1_{B_1}\left(\frac{x}{t}\right) m(x) \in L^1(\mathbb{R}^d)$. As an element in X', we have $t^{-d}1_{B_1}\left(\frac{x}{t}\right) m(x) \rightharpoonup m(0)\delta(x)$ in the weak topology of X'. If we have strong convergence in X', this implies strong convergence in $L^1(\mathbb{R}^d)$ to an element $u \in L^1(\mathbb{R}^d)$. It is easy to see that $u = 0$ a.e. in $\mathbb{R}^d$, which implies $u = 0$. So we conclude that strong convergence in X' is necessarily to 0 and implies $m(0) = 0$. Viceversa, if $m(0) = 0$, we know that for any $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that $|x| < \delta_\epsilon$ implies $|m(x)| < \epsilon$. Then, for $0 < t < \delta_\epsilon$ we have*

$$\|t^{-d}1_{B_1}\left(\frac{x}{t}\right) m\|_{L^1(\mathbb{R}^d)} = \|t^{-d}1_{B_1}\left(\frac{x}{t}\right) m\|_{L^1(B_t)} \leq \|t^{-d}1_{B_1}\left(\frac{x}{t}\right) m\|_{L^1(B_{\delta_\epsilon})} \leq \epsilon$$

and so, indeed, we conclude $t^{-d}1_{B_1}\left(\frac{\cdot}{t}\right) m \xrightarrow{t\to 0^+} 0$ in $L^1(\mathbb{R}^d)$.

86. Let $I = [0,1]$ and (u_n), (v_n) be two bounded sequences in $L^2(I)$. Assume in addition that the maps $I \ni x \mapsto u_n(x)$ and $I \ni x \mapsto v_n(x)$ are continuous and monotone non decreasing for every $n \in \mathbb{N}$; then define

$$f_n(x,y) \doteq u_n(x)v_n(y), \quad (x,y) \in Q \doteq I \times I.$$

Prove that f_n lies in $L^2(Q)$ for every $n \in \mathbb{N}$ and that the sequence (f_n) is relatively compact in $L^1(Q)$. *Answer. First of all, it is sufficient to show that*

(u_n) is relatively compact in $L^1(I)$. The argument which follows is rather complicated. Notice, incidentally, that here it is crucial that relative compactness is in $L^1(I)$, since it is easy to obtain a non relatively compact set $L^2(I)$ using for example $n^{\frac{1}{2}}\chi_{[0,1]}(n(1-x))$ (notice that $n^{\frac{1}{2}}\chi_{[0,1]}(n(1-x)) \xrightarrow{n\to+\infty} 0$ in $L^1(I)$). I will also assume that $u_n(0) \equiv 0$, since it is easy to see that we can reduce to this case and I will also assume $\|u_n\|_{L^2(I)} \leq 1$ for all n. Next, for any $M > 0$ let $x_M^{(n)} = \inf\{x : u_n(x) \geq M\}$. Then $1 - x_M^{(n)} \leq M^{-1/2}\|u_n\|_{L^2(I)} \leq M^{-1/2}$ follows from the Chebyshev inequality. So $x_M^{(n)} \geq 1 - M^{-1/2}$. Next, split

$$\begin{aligned} u_n &= v_n + w_n \text{ where} \\ v_n &:= \chi_{[0,1-M^{-1/2}]}u_n \\ w_n &= \chi_{[1-M^{-1/2},1]}u_n \end{aligned}$$

Notice that

$$\|w_n\|_{L^1(I)} \leq \|\chi_{[1-M^{-1/2},1]}\|_{L^1(I)}\|u_n\|_{L^2(I)} \leq M^{-1/4}.$$

Now, we have $v_n(0) \equiv 0$ *and* $v_n(1-M^{-1/2}) \leq M$*. Then, see*

https://math.stackexchange.com/questions/1003580/a-bounded-monotonic-function-on-an-closed-interval-has-fourier-coefficient-decay

it is easy to see that there exists a fixed $C > 0$ *such that the Fourier series*

$$v_n(x) \sim \sum_{j\in\mathbb{Z}} \widehat{v}_n(j) e^{i\frac{2\pi}{1-M^{-1/2}}jx} \text{ satisfies}$$
$$|\widehat{v}_n(j)| \leq \frac{C}{\langle j \rangle} \text{ for all } n \in \mathbb{N}.$$

Notice that this implies that (v_n) *defines a bounded sequence in* $H^s(\mathbb{T}_M)$ *where* $\mathbb{T}_M = \frac{\mathbb{R}}{(1-M^{-1/2})\mathbb{Z}}$ *for any* $s \in (0,1/2)$*. Since the immersion* $H^s(\mathbb{T}_M) \hookrightarrow L^2(0,1-M^{-1/2})$ *is compact, we conclude that* (v_n) *is relatively compact in* $L^2(0,1-M^{-1/2})$*, and so also in* $L^1(0,1-M^{-1/2})$*. So for any* $\epsilon > 0$ *we conclude that there is a finite covering of* (v_n) *in* $L^1(0,1-M^{-1/2})$ *with balls of radius* $\epsilon/2$*. Now, choosing* $M^{-1/4} < \epsilon/2$ *we conclude that there exists a finite covering of* (u_n) *in* $L^1(0,1)$ *with balls of radius* ϵ*. This yields the desired result.*

87. Let $I = [0,1]$, $Q \doteq I \times I$ and (a_n), (b_n) sequences in $]0,1]$. Define the family of sets $R_n \doteq [0,a_n] \times [0,b_n] \subseteq Q$ and set

$$u_n(x,y) \doteq (1+\sin(nx))(1+e^{-ny})\chi_{R_n}(x,y), \quad (x,y) \in Q.$$

Study the convergence of (u_n) in the strong and weak topology of $L^1(Q)$ and in the strong and weak* topology of $L^\infty(Q)$ in the following cases:

1. $a_n = n^{-2}$, $b_n = 1 - n^{-1}$;
2. $a_n = 1 - n^{-2}$, $b_n = 1 - n^{-1}$.

Answer. In the first case, we have

$$|u_n(x,y)| \leq 4\chi_{[0,n^{-2}]}(x)\chi_{[I]}(y) \xrightarrow{n\to+\infty} 0 \text{ in } L^1(Q) \text{ by Dominated Convergence.}$$

We have $\|u_n\|_{L^\infty(Q)} = 2$ *and this and the above imply that* u_n *is not strongly convergent in* $L^\infty(Q)$*, however* $u_n \rightharpoonup 0$ *in the weak* topology of* $L^\infty(Q)$*. In the second case, set*

$$v_n(x,y) := (1+\sin(nx))(1+e^{-ny})\chi_{[0,b_n]}(y).$$

Then

$$v_n(x,y) - u_n(x,y) = (1+\sin(nx))(1+e^{-ny})\chi_{[1-n^{-2},1]}(x)\chi_{[0,b_n]}(y) \xrightarrow{n\to+\infty} 0$$
$$\text{in } L^1(Q) \text{ by Dominated Convergence.}$$

Next,

$$v_n(x,y) = (1+\sin(nx)) + w_n(x,y) \text{ for}$$
$$w_n(x,y) := -(1+\sin(nx))\chi_{[b_n,1]}(y) + (1+\sin(nx))e^{-ny}\chi_{[0,b_n]}(y)$$
$$\text{where } w_n \xrightarrow{n\to+\infty} 0 \text{ in } L^1(Q) \text{ by Dominated Convergence.}$$

We have $1+\sin(nx) \rightharpoonup 1$ *in* $L^1(Q)$ *in the weak* topology of* $L^\infty(Q)$ *by the Riemann–Lebesgue Lemma. On the other hand*

$$\|1+\sin(nx)-1\|_{L^1(Q)} = \int_0^1 |\sin(nx)|dx \xrightarrow{n\to+\infty} \frac{2}{\pi}$$

which implies that $1+\sin(nx)$ *does not converge to* 1 *in* $L^1(Q)$. *We also have*

$$v_n - u_n \rightharpoonup 0 \text{ in the weak* topology of } L^\infty(Q) \text{ and}$$
$$w_n \rightharpoonup 0 \text{ in the weak* topology of } L^\infty(Q).$$

There is no strong convergence of u_n *in* $L^\infty(Q)$, *since this would imply strong convergence to* 1, *in particular also in* $L^1(Q)$, *which has just been excluded.*

88. Let H be a complex Hilbert space with inner product $(\cdot,\cdot)$. Prove that we have

$$4(x,y) = (\|x+y\|^2 - \|x-y\|^2) - i(\|x+iy\|^2 - \|x-iy\|^2) \quad \forall x,y\in H.$$

89. Let $I=[0,1]$ and call X the Banach space $C(I)$, endowed with the uniform norm. Let $g\in C(I\times I)$ and set

$$(Tu)(x) \doteq \int_I g(x,y)u(y)\,dy \quad u\in X,\ x\in I.$$

(i) Prove that $T\in\mathcal{L}(X)$ and estimate its norm.
(ii) Establish if T is compact and justify the answer.
(iii) Compute the norm of T in the case $g(x,y)=e^{x+y}$.

Answer. We have

$$|(Tu)(x)| \le \int_I |g(x,y)u(y)|dy \le \int_I |g(x,y)|dy\|u\|_{L^\infty(I)}.$$

So we have the bound

$$\|T\| \le \sup_{x\in I}\int_I |g(x,y)u(y)|dy.$$

Using the fact that $g: I\times I\to\mathbb{R}$ *is uniformly continuous, it is easy to show that* $TD_{C(I)}(0,1)$ *is bounded and equicontinuous, and so relatively compact by Ascoli Arzela.*

90. Let $X=C_0(\mathbb{R}^2)$ and, for every $n\in\mathbb{N}$, consider the set

$$R_n \doteq]-n,n[\times]-n^{-1},n^{-1}[\subseteq\mathbb{R}^2.$$

Given $u\in X$ and $n\in\mathbb{N}$ set

$$(T_nu)(x) = \frac{1}{n}\int_{R_n} e^{-(x^2+y^2)}u(x,y)\,dxdy.$$

Prove that $T_n\in X'$ for every $n\in\mathbb{N}$ and find its norm and support. Study the convergence of the sequence (T_n) in the strong and weak* topology of X'.

Answer. It is pretty straightforward that

$$\|T_n\| = \frac{1}{n}\int_{R_n} e^{-(x^2+y^2)}\,dxdy \le \frac{1}{n}\int_{R_n} dxdy = \frac{1}{n}Area(R_n) = \frac{4}{n} \xrightarrow{n\to+\infty} 0.$$

91. Let $Q = [0,1]^d \subseteq \mathbb{R}$ and consider (u_n), (v_n), two relatively compact sequences in $L^2(Q)$. Define

$$f_n(x) \doteq u_n(x)v_n(x), \quad x \in Q,\ n \in \mathbb{N}.$$

Prove that f_n lies in $L^1(Q)$ for every $n \in \mathbb{N}$ and that the sequence (f_n) is relatively compact in $L^1(Q)$.

92. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be map of class C^1 such that $\varphi(0) = 0$ and $1 \leq \varphi'(t) \leq 2$ for every $t > 0$. Let $I = [0,1]$ and (u_n) a sequence in $L^1(\mathbb{R})$.

(i) Prove that the sequence (v_n) defined by $v_n(t) \doteq u_n(\varphi(t))$ for $t \in I$ and $n \in \mathbb{N}$ lies in $L^1(I)$.
(ii) Assuming that $u_n \to u$ strongly in $L^1(\mathbb{R})$, study the convergence of (v_n) in the strong and weak convergence of $L^1(I)$.
(iii) Assuming that $u_n \rightharpoonup u$ weakly in $L^1(\mathbb{R})$, study the convergence of (v_n) in the strong and weak convergence of $L^1(I)$.

93. Let $Q = [0,1] \times [0,1]$ and X the Banach space $C^0(Q)$, endowed with the uniform norm. Set

$$(T_n u) \doteq \int_0^1 n\, e^{-nx}\, u(x, x^2)\, dx, \quad u \in X.$$

Prove that $T_n \in X'$ for every $n \in \mathbb{N}$ and find its norm and support. Study the convergence of (T_n) in the strong and weak* topology of X'.

Answer. We have the map $I : C^0(Q) \to C^0([0,1])$ and $n\, e^{-nx}$ a sequence in $L^1([0,1])$. If this sequence was convergent to a $f \in L^1([0,1])$, then there would be a subsequence converging almost everywhere point wise to f. But then $f = 0$, impossible in view of the fact that $\|n\, e^{-nx}\|_{L^1([0,1])} \xrightarrow{n\to+\infty} 1$. This is equivalent to say that (T_n) is not strongly convergent. However we have $T_n u \xrightarrow{n\to+\infty} u(0,0)$, so there is weak convergence in X'.*

94. Let H be a Hilbert space, $T \in \mathcal{L}(H)$ and (T_n) a sequence in $\mathcal{L}(H)$.

(i) Prove that $T_n \to T$ if and only if $T_n^* \to T^*$.
(ii) Prove that the sequence $(T_n x)$ converges weakly to Tx for every $x \in H$ if and only if the sequence $(T_n^* x)$ converges weakly to $T^* x$ for every $x \in H$.

95. Let $I = [0,1] \subseteq \mathbb{R}$ and $X = C^0(I)$. Given a map $m \in L^2(I)$, set

$$Tu(x) \doteq \int_0^{x^2} m(y)u(y)\, dy.$$

Prove that $T \in \mathcal{L}(X)$ and establish if T is compact in $\mathcal{L}(X)$, justifying the answer.

Answer. For any $x_1 < x_2$ we have

$$|Tu(x_2) - Tu(x_1)| \leq \|u\|_{L^\infty(I)} \int_{x_1^2}^{x_2^2} |m(y)| dy \leq \|u\|_{L^\infty(I)} \|m\|_{L^2(I)} \sqrt{x_2^2 - x_1^2}$$
$$\leq \sqrt{2}\|u\|_{L^\infty(I)} \|m\|_{L^2(I)} \sqrt{x_2 - x_1}$$

So $T\{u \in X : \|u\|_{L^\infty(I)} \leq 1\}$ is a bounded equiconcontinuous family, and hence also a relatively compact one by Ascoli–Arzela.

96. Let $Q = [0,1]^d \subseteq \mathbb{R}$. Consider two relatively compact families U and V in $C^0(Q)$ and define

$$F \doteq \{f : f(x) = \sin(u(x) \cdot v(x)),\ x \in Q, u \in U, v \in V\}.$$

Prove that F is a relatively compact family in $C^0(Q)$.

97. Let $I = [0,1] \subseteq \mathbb{R}$, $p > 1$ and $X = L^\infty(I)$. Given a map $m \in L^p(I)$, set

$$Tu(x) \doteq \int_0^x m(y)u(y)\,dy.$$

Prove that $T \in \mathcal{L}(X)$ and establish if T is compact in $\mathcal{L}(X)$, justifying the answer.
Answer. We have that $T : L^\infty(I) \to C^{\frac{1}{p'}}(I)$ is bounded, by the formula below, and hence the map from X into itself is compact. The formula is, for $x_1 < x_2$,

$$|Tu(x_1) - Tu(x_2)| \leq \int_{x_1}^{x_2} |m(x)|dx \|u\|_{L^\infty(I)} \leq |x_1 - x_2|^{\frac{1}{p'}} \|m\|_{L^p(I)} \|u\|_{L^\infty(I)}.$$

98. Let X be the Banach space $C_0(\mathbb{R}^2)$, endowed with the uniform norm, and let (g_n) be a sequence in $C_b(\mathbb{R}^2)$ such that

$$0 \leq g_n(x,y) \leq (1 + x^2 + y^2)^{-1} \quad \forall (x,y) \in \mathbb{R}^2, \forall n \in \mathbb{N}$$

and

$$g_n \longrightarrow g \quad \text{in} \quad C_b(\mathbb{R}^2).$$

Set

$$(T_n u) \doteq \int_{\mathbb{R}} g_n(x,x)u(x,x)\,dx, \quad u \in X.$$

Prove that $T_n \in X'$ for every $n \in \mathbb{N}$ and find its norm and support. Study the convergence of (T_n) in the strong and weak* topology of X'.

Answer. We are considering the continuous map $I : C_0(\mathbb{R}^2) \ni u(x,y) \hookrightarrow u(x,x) \in C_0(\mathbb{R})$. Then, $T_n = S_n \circ I$ with S_n identifies with $L^1(\mathbb{R}) \ni v_n(x) = g_n(x,x)$. Set also $v(x) = g(x,x)$. It is easy to see that $g(x,y)$ satisfies

$$0 \leq g(x,y) \leq (1 + x^2 + y^2)^{-1} \quad \forall (x,y) \in \mathbb{R}^2.$$

So, by dominated convergence,

$$\lim_{n\to+\infty} \int_{\mathbb{R}} |v_n - v|dx = 0.$$

Hence $T_n \xrightarrow{n\to+\infty} T$ strongly in X', with

$$(Tu) \doteq \int_{\mathbb{R}} g(x,x)u(x,x)\,dx, \quad u \in X.$$

99. Let $f \in L^2(\mathbb{R})$ and set

$$(Tu)(x) \doteq \int_{\mathbb{R}} f(x-y)u(y)\,dy.$$

Establish for which indices $p, q \in [1, +\infty]$ we have $T \in \mathcal{L}(L^p(\mathbb{R}), L^q(\mathbb{R}))$.

100. Let $I = [0,1] \subseteq \mathbb{R}$, $X = C^0(I)$ and $Y = L^1(I)$. Set

$$Tu(x) \doteq \int_0^x xyu(y)\,dy.$$

a) Prove that $T \in \mathcal{L}(X)$ and $T \in \mathcal{L}(Y)$.
b) Establish if T is compact in $\mathcal{L}(X)$ and in $\mathcal{L}(Y)$, justifying the answer.