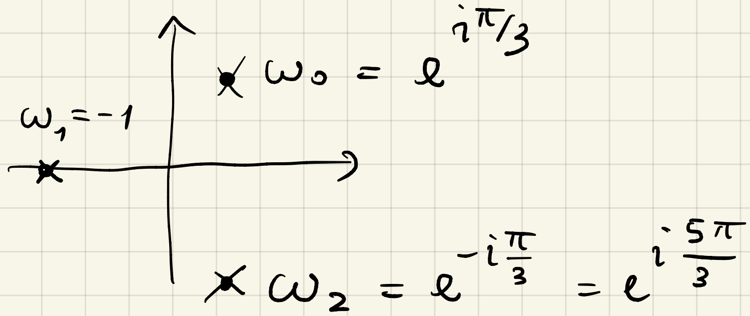
ESERCIZIO 1

I) $$f(z) = \frac{\sqrt{z}}{1+z^3}$$

Il denominatore si annulla per: $z^3 = -1$

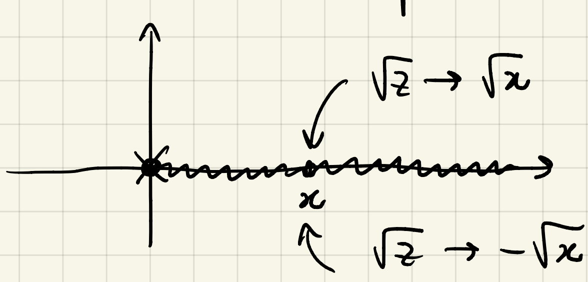
$\Rightarrow$ tre radici cubiche di -1 sono poli di ordine 1, sono date da:

$$\omega_k = e^{\frac{i\pi}{3} + \frac{2i\pi k}{3}}, \quad k = 0, 1, 2$$

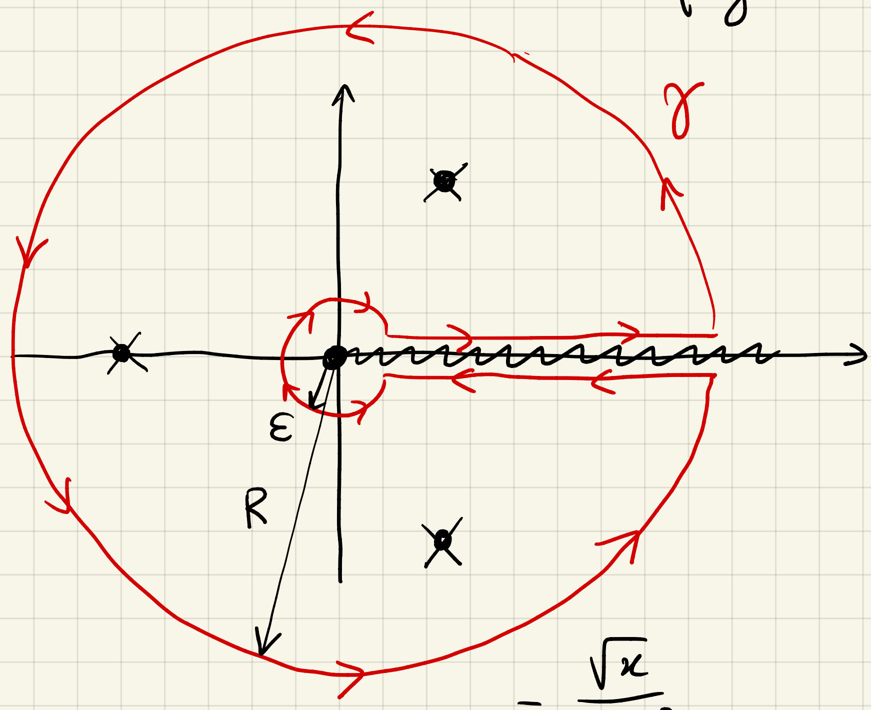


Inoltre al numeratore abbiamo $\sqrt{z}$ che causa un punto di diramazione in $z=0$ e in $z=\infty$.

II) Scegliamo per $\sqrt{z}$ il ramo con un taglio sull'asse reale positivo e con $\mathrm{Arg}(\sqrt{z}) \in [0, \pi)$.



Scegliamo il cammino in figura:



$$\oint_\gamma f(z)\,dz = \int_\varepsilon^R dx\, \underbrace{f^{\text{sopra}}(x)}_{=\frac{\sqrt{x}}{1+x^3}} + \int_R^\varepsilon dx\, \underbrace{f^{\text{sotto}}(x)}_{=\frac{-\sqrt{x}}{1+x^3}}$$

$$+ \int_{\gamma_\varepsilon} dz\, f(z) + \int_{\gamma_R} dz\, f(z)$$

$$= \int_\varepsilon^R dx\, \frac{2\sqrt{x}}{1+x^3} + \int_{\gamma_\varepsilon} dz\, f(z) + \int_{\gamma_R} dz\, f(z)$$

$$\left| \int_{\gamma_\varepsilon} dz\, f(z) \right| \leq 2\pi\varepsilon \operatorname*{Max}_{\theta\in[0,2\pi)} \left|f(z)\right| \Big|_{z=\varepsilon e^{i\theta}}$$

$$= 2\pi\varepsilon \operatorname*{Max}_{\theta\in[0,2\pi)} \frac{\varepsilon^{1/2}}{|1+\varepsilon^3 e^{3i\theta}|} \underset{\varepsilon\to 0^+}{=} 2\pi\,\varepsilon^{3/2}\left(1+O(\varepsilon^3)\right) \xrightarrow[\varepsilon\to 0^+]{} 0$$

$$\Rightarrow \lim_{\varepsilon \to 0^+} \int_{\gamma_\varepsilon} dz\, f(z) = 0.$$

$$\left| \int_{\gamma_R} dz\, f(z) \right| \leq 2\pi R \max_{\theta \in [0, 2\pi)} |f(z)|\Big|_{z = Re^{i\theta}}$$

$$= 2\pi R \max_{\theta \in [0,2\pi)} \frac{R^{1/2}}{|1 + R^3 e^{3i\theta}|}$$

$$\underset{R\to+\infty}{=} 2\pi R\, R^{1/2} \frac{1}{R^3} \left(1 + \mathcal{O}\left(R^{-3}\right)\right) \xrightarrow[R\to+\infty]{} 0.$$

$$\Rightarrow \lim_{R\to+\infty} \int_{\gamma_R} dz\, f(z) = 0$$

$$\Rightarrow \lim_{\varepsilon\to 0^+} \lim_{R\to+\infty} \oint_\gamma f(z)\, dz = 2 \int_0^{+\infty} dx \frac{\sqrt{x}}{1+x^3}$$

Calcoliamo quindi $\oint_\gamma f(z)dz$ usando il teorema interno dei residui:

$$\oint_\gamma f(z) dz \underset{\substack{\downarrow \\ \text{assumendo } R > 1 > \varepsilon}}{=} 2\pi i \left[\mathrm{Res}_f(\omega_0) + \mathrm{Res}_f(\omega_1) + \mathrm{Res}_f(\omega_2) \right]$$

Due modi di calcolare i residui:

<u>MODO 1</u> :
$$f(z) = \frac{\sqrt{z}}{(z-\omega_0)(z-\omega_1)(z-\omega_2)}$$

$$\Rightarrow \quad Res_f(\omega_0) = \frac{\sqrt{\omega_0}}{(\omega_0-\omega_1)(\omega_0-\omega_2)} = \frac{e^{i\pi/6}}{(e^{i\pi/3}+1)(e^{i\frac{\pi}{3}}-e^{-i\pi/3})}$$

$$= \frac{1}{(e^{i\pi/6}+e^{-i\pi/6})(e^{i\pi/3}-e^{-i\pi/3})} = \frac{1}{(2\cos(\frac{\pi}{6}))(2i\sin(\frac{\pi}{3}))}$$

moltiplico numeratore e denominatore per $e^{-i\pi/6}$

$$= \frac{1}{4i}\,\frac{1}{\frac{\sqrt{3}}{2}\times\frac{\sqrt{3}}{2}} = \frac{1}{3i} = -\frac{i}{3}$$

Analogamente: $$Res_f(\omega_2) = \frac{\sqrt{\omega_2}}{(\omega_2-\omega_1)(\omega_2-\omega_0)}$$

$$= \frac{e^{i5\pi/6}}{(e^{-i\frac{\pi}{3}}+1)(e^{-i\frac{\pi}{3}}-e^{i\frac{\pi}{3}})} = \frac{e^{i\pi}}{(e^{-i\frac{\pi}{6}}+e^{i\frac{\pi}{6}})(e^{-i\frac{\pi}{3}}-e^{i\pi/3})}$$

moltiplico numeratore e denominatore per $e^{i\frac{\pi}{6}}$

$$= \frac{-1}{2\cos(\frac{\pi}{6})(-2i)\sin(\frac{\pi}{3})} = \frac{1}{4i}\,\frac{1}{\frac{\sqrt{3}}{2}\times\frac{\sqrt{3}}{2}} = \frac{1}{3i} = -\frac{i}{3}$$

$$Res_f(\omega_1) = \frac{\sqrt{\omega_1}}{(\omega_1-\omega_0)(\omega_1-\omega_2)} = \frac{i}{(-1-e^{i\frac{\pi}{3}})(-1-e^{-i\frac{\pi}{3}})}$$

$$= \frac{i}{\left(e^{i\frac{\pi}{6}} + e^{-i\frac{\pi}{6}}\right)\left(e^{i\frac{\pi}{6}} + e^{-i\frac{\pi}{6}}\right)} = \frac{i}{2\cos\left(\frac{\pi}{6}\right) 2\cos\left(\frac{\pi}{6}\right)}$$

$$= \frac{i}{4} \frac{1}{\frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2}} = \frac{i}{3}$$

<u>MODO 2</u> : $\mathrm{Res}_f(\omega_k) = \lim_{z \to \omega_k} (z - \omega_k) \frac{\sqrt{z}}{z^3 + 1}$

$$= \sqrt{\omega_k} \frac{1}{\lim_{z \to \omega_k} \frac{z^3 - \omega_k^3}{z - \omega_k}} = \sqrt{\omega_k} \frac{1}{\frac{d}{dz} z^3 \Big|_{z=\omega_k}}$$

$$= \sqrt{\omega_k} \frac{1}{3\omega_k^2} = \frac{\omega_k \sqrt{\omega_k}}{3\omega_k^3} = -\frac{1}{3} \omega_k \sqrt{\omega_k}$$

$$\Rightarrow \mathrm{Res}_f(\omega_0) = -\frac{1}{3} e^{i\frac{\pi}{3}} e^{i\frac{\pi}{6}} = -\frac{1}{3} e^{i\frac{\pi}{2}} = -\frac{i}{3}$$

$$\mathrm{Res}_f(\omega_1) = -\frac{1}{3}(-1)\, i = \frac{i}{3}$$

$$\mathrm{Res}_f(\omega_2) = -\frac{1}{3} e^{-i\frac{\pi}{3}} e^{i\frac{5\pi}{6}} = -\frac{i}{3}$$

Usando il risultato per i residui otteniamo:

$$\oint_\gamma f(z)\,dz = 2\pi i \left[-\frac{i}{3} + \frac{i}{3} - \frac{i}{3}\right] = \frac{2\pi}{3}$$

da cui : $\int_0^{+\infty} dx \frac{\sqrt{x}}{1+x^3} = \frac{\pi}{3}$. ✓

ESERCIZIO 2

I) $$f^P(x) = \begin{cases} f(x) & , x > 0 \\ f(-x) & , x \leq 0 \end{cases} \quad , f^P(x) = +f^P(-x)$$

$$f^D(x) = \begin{cases} f(x) & , x > 0 \\ -f(-x) & , x \leq 0 \end{cases} \quad , f^D(x) = -f^D(-x)$$

Da cui: $$|f^P(x)|^2 = |f^P(-x)|^2$$
$$|f^D(x)|^2 = |f^D(-x)|^2$$

$$\Rightarrow \begin{cases} \int_{-\infty}^{+\infty} dx \, |f^P(x)|^2 = 2 \int_0^{+\infty} dx \, |f(x)|^2 < \infty \\ \int_{-\infty}^{+\infty} dx \, |f^D(x)|^2 = 2 \int_0^{+\infty} dx \, |f(x)|^2 < \infty \end{cases}$$

perché $f \in L^2((0,+\infty))$

Quindi sia f^P che $f^D \in L^2(\mathbb{R})$ ed è definita la loro trasformata di Fourier, anch'essa in $L^2(\mathbb{R})$. Usando la parità/disparità otteniamo:

$$\widehat{f^P}(k) \equiv \int_{-\infty}^{+\infty} dx\, f^P(x)\, e^{ikx}$$

$$= \int_0^{+\infty} dx\, f^P(x)\, e^{ikx} + \int_{-\infty}^{0} dx\, f^P(x)\, e^{ikx}$$

$$= \int_0^{+\infty} dx\, f^P(x)\, e^{ikx} + \int_0^{+\infty} dx\, f^P(-x)\, e^{-ikx}$$

$$= \int_0^{+\infty} dx\, f^P(x)\, 2\cos(kx)$$

$$= \int_0^{+\infty} dx\, f(x)\, 2\cos(kx) \underset{k>0}{=} \hat{f}^C(k)$$

$$\widehat{f^D}(k) = \int_{-\infty}^{+\infty} dx\, f^D(x)\, e^{ikx}$$

$$= \int_0^{+\infty} dx\, f^D(x)\, e^{ikx} + \int_{-\infty}^{0} dx\, f^D(x)\, e^{ikx}$$

$$= \int_0^{+\infty} dx\, f^D(x)\, e^{ikx} + \int_0^{+\infty} dx\, f^D(-x)\, e^{ikx}$$

$$= \int_0^{+\infty} dx\, f^D(x)\, 2i\sin(kx)$$

$$= i\int_0^{+\infty} dx\, f(x)\, 2\sin(kx) \underset{k>0}{=} i\,\hat{f}^S(k)$$

$$\Rightarrow \quad \hat{f}^C(k) = \widehat{f^P}(k), \quad \hat{f}^S(k) = -i\,\widehat{f^D}(k). \quad \checkmark$$

$\hat{f}^P$ e $\hat{f}^D$ sono in $L^2(\mathbb{R})$. Inoltre:

$$\widehat{f^P}(k) = \widehat{f^P}(-k) \quad e \quad \widehat{f^D}(k) = -\widehat{f^D}(-K).$$

Quindi:
$$\int_0^{+\infty} dk \, |\hat{f}^c(k)|^2 = \frac{1}{2} \int_{-\infty}^{+\infty} dk \, |\widehat{f^P}(k)|^2 < +\infty$$

$$\int_0^{+\infty} dk \, |\hat{f}^s(k)|^2 = \frac{1}{2} \int_{-\infty}^{+\infty} dk \, |\widehat{f^D}(k)|^2 < +\infty$$

$\Rightarrow$ $\hat{f}^c$ e $\hat{f}^s$ sono in $L^2((0,+\infty))$.

Usando l'antitrasformata in $L^2(\mathbb{R})$ abbiamo:

$$f^P(x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \, \widehat{f^P}(k) \, e^{-ikx}$$

$$= \int_0^{+\infty} \frac{dk}{2\pi} \, \widehat{f^P}(k) \, e^{-ikx} + \int_{-\infty}^{0} \frac{dk}{2\pi} \, \widehat{f^P}(k) \, e^{-ikx}$$

$$= \int_0^{+\infty} \frac{dk}{2\pi} \, \widehat{f^P}(k) \, e^{-ikx} + \int_0^{+\infty} \frac{dk}{2\pi} \, \widehat{f^P}(-k) \, e^{+ikx}$$

$$= \int_0^{+\infty} \frac{dk}{2\pi} \, \hat{f}^P(k) \, 2\cos(kx)$$

$$= \int_0^{+\infty} \frac{dk}{2\pi} \, \hat{f}^c(k) \, 2\cos(kx)$$

Valutando per $x>0$ otteniamo:

$$f(x)=\int_0^{+\infty}\frac{dk}{2\pi}\,\hat{f}^c(k)\,2\cos(kx).$$

Analogamente:

$$f^D(x)=\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\,\hat{f}^D(k)\,e^{-ikx}$$

$$=\int_0^{+\infty}\frac{dk}{2\pi}\,\hat{f}^D(k)\,e^{-ikx}+\int_{-\infty}^{0}\frac{dk}{2\pi}\,\hat{f}^D(k)\,e^{-ikx}$$

$$=\int_0^{+\infty}\frac{dk}{2\pi}\,\hat{f}^D(k)\,e^{-ikx}+\int_0^{+\infty}\frac{dk}{2\pi}\,\hat{f}^D(-k)\,e^{-ikx}$$

$$=\int_0^{+\infty}\frac{dk}{2\pi}\,\hat{f}^D(k)\,(-2i)\sin(kx)$$

$$=\int_0^{+\infty}\frac{dk}{2\pi}\,\hat{f}^s(k)\,2\sin(kx),$$

e valutando per $x>0$ otteniamo:

$$f(x)=\int_0^{+\infty}\frac{dk}{2\pi}\,\hat{f}^s(k)\,2\sin(kx).$$

II)

$$\widehat{\frac{df}{dx}}^{s}(k) = \int_0^{+\infty} dx \, \frac{df}{dx}(x) \, 2\sin(kx)$$

$$= 2f(x)\sin(kx)\Big|_0^{+\infty} - \int_0^{+\infty} dx \, f(x) \, 2k\cos(kx)$$

$$= \underset{\substack{\swarrow \\ f(+\infty)=0}}{0} - \underset{\substack{\downarrow \\ \sin(0)=0}}{0} - k\int_0^{+\infty} dx \, f(x) \, 2\cos(kx)$$

$$= -k \, \hat{f}^{c}(k). \quad \checkmark$$

$$\widehat{\frac{df}{dx}}^{c}(k) = \int_0^{+\infty} dx \, \frac{df}{dx}(x) \, 2\cos(kx)$$

$$= 2f(x)\cos(kx)\Big|_0^{+\infty} - \int_0^{+\infty} dx \, f(x) \, 2k(-\sin(kx))$$

$$= \underset{\substack{\swarrow \\ f(+\infty)=0}}{0} - 2f(0) + k\int_0^{+\infty} dx \, f(x) \sin(kx)$$

$$= -2f(0) + k\hat{f}^{s}(k). \quad \checkmark$$

III)

$$f(x) = e^{-x} \quad , \quad \frac{df}{dx}(x) = -e^{-x} = -f(x)$$

$$\hat{f}^c(k) = \int_0^{+\infty} dx \, e^{-x} \, 2\cos(kx)$$

$$= \int_0^{+\infty} dx \left(e^{-(1-ik)x} + e^{-(1+ik)x} \right)$$

$$= \frac{1}{1-ik} + \frac{1}{1+ik} = \frac{2}{1+k^2}$$

$$\hat{f}^s(k) = \int_0^{+\infty} dx \, e^{-x} \, 2\sin(kx)$$

$$= -i \int_0^{+\infty} dx \left(e^{-(1-ik)x} - e^{-(1+ik)x} \right)$$

$$= -i \left(\frac{1}{1-ik} - \frac{1}{1+ik} \right) = \frac{2k}{1+k^2}$$

$$\widehat{\frac{df}{dx}}^s(k) = -\hat{f}^s(k) = -\frac{2k}{1+k^2} = -k\hat{f}^c(k) \quad ✓$$

$$\widehat{\frac{df}{dx}}^c(k) = -\hat{f}^c(k) = -\frac{2}{1+k^2} \overset{?}{=} \quad ✓$$

$$k\hat{f}^s(k) - 2f(0) = \frac{2k^2}{1+k^2} - 2 = -\frac{2}{1+k^2}$$

Esercizio 3

I) $\forall v, w$, $\forall \alpha \in \mathbb{C}$

$$(\alpha w, T[v]) = (v, T^\dagger[\alpha w])$$

D'altra parte: $(\alpha w, T[v]) = \alpha^* (w, T[v])$

$$= \alpha^* (v, T^\dagger[w])$$

Dunque: $(v, T^\dagger[\alpha w]) = \alpha^* (v, T^\dagger[w])$

$$= (v, \alpha^* T^\dagger[w])$$

Visto che questo è valido $\forall v \in H$, segue che:

$$T^\dagger[\alpha w] = \alpha^* T^\dagger[w], \ \forall w \in H, \forall \alpha \in \mathbb{C}$$

ovvero $T^\dagger$ è anti lineare. ✓

(Ad esempio, possiamo far variare v in un s.o.c. e quindi

$$T^\dagger[\alpha w] = \sum_{n=1}^{+\infty} (e^{(n)}, T^\dagger[\alpha w]) e^{(n)}$$

$$= \sum_{n=1}^{+\infty} (e^{(n)}, \alpha^* T^\dagger[w]) e^{(n)}$$

$$= \alpha^* T^\dagger[w] \ ).$$

T_1, T_2 antilineari, allora $\forall \alpha \in \mathbb{C}, \forall v \in H$

$$T_1 T_2 [\alpha v] = T_1 [T_2 [\alpha v]]$$
$$= T_1 [\alpha^* T_2 [v]] = \alpha^{**} T_1 [T_2 [v]]$$
$$= \alpha T_1 T_2 [v] \Rightarrow T_1 T_2 \text{ è lineare.}$$

$\forall w, v \in H:$
$$(w, T_1 T_2 [v])$$
$$= (T_2 [v], T_1^\dagger [w])$$
$$= (T_1^\dagger [w], T_2 [v])^*$$
$$= (v, T_2^\dagger T_1^\dagger [w])^*$$
$$= (T_2^\dagger T_1^\dagger [w], v)$$

$$\Rightarrow (T_1 T_2)^\dagger = T_2^\dagger T_1^\dagger.$$ ✓

Se U è antiunitario: $\forall v, w \in H$

$$(U[w], U[v]) = (v, w)$$

D'altra parte: $(U[w], U[v]) = (v, U^\dagger U[w])$

$\Rightarrow \forall v, w \in H \quad (v, w) = (v, U^\dagger U[w])$

$\Rightarrow \quad U^\dagger U = \mathbb{I} \quad \Rightarrow \quad U^\dagger = U^{-1}.$ ✓

Se U_1, U_2 sono anti unitari allora:

$$(U_1 U_2)^\dagger = U_2^\dagger U_1^\dagger = U_2^{-1} U_1^{-1} = (U_1 U_2)^{-1}.$$

Quindi $U_1 U_2$ è lineare (visto prima) e unitario. ✓

II)
$$\begin{aligned} C : L^2(\mathbb{R}) &\to L^2(\mathbb{R}) \\ f &\mapsto f^* \end{aligned}$$

$U : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ operatore unitario, allora

$\forall f, g \in L^2(\mathbb{R})$

$$(UC[f], UC[g]) = (C[f], C[g])$$

$$= \int_{-\infty}^{+\infty} dx \, (C[f](x))^* \, C[g](x)$$

$$= \int_{-\infty}^{+\infty} dx \, (f(x))^{**} (g(x))^*$$

$$= \int_{-\infty}^{+\infty} dx \, g(x)^* f(x) = (g, f)$$

$\Rightarrow UC$ è antiunitario. ✓