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Topological vector spaces

Def. A vector space X on a field $K = \mathbb{R}, \mathbb{C}$ is called a T.V.S when it is endowed witha a topology τ s.t.

1) $X \times X \longrightarrow X$
$(x, y) \longrightarrow x+y$ is continuous

2) $K \times X$
$(\lambda, x) \longrightarrow \lambda x$ is continuous.

We will also assume that (X, τ) is Haus.

<u>Def</u> Given $X \ni 0$, A subset $\Omega \subseteq X$ is

1) balanced if $x \in \Omega$ and $\lambda \in K$ with $|\lambda| \leq 1 \Rightarrow \lambda x \in \Omega$

2) absorbing if $\forall\ x \in X\ \exists\ \lambda \in K$ s.t. $\lambda \Omega = \{\lambda y : y \in \Omega\} \ni x$

Lemma Given a TVS X on K on $U \ni 0$ a neighb. of 0 there exists a balanced neigh of 0 V with $V \subseteq U$.

Pf $K \times X \longrightarrow X$ is continuous in $(0,0)$ $\Rightarrow$ $\exists$ $\delta > 0$ a $\tilde{V}$ neigh. of 0 in X s.t. if

$|\lambda| \leq \delta$ and $x \in \tilde{V}$

we have $\lambda x \in U$.

Set $\hat{V} = \{ \lambda x : |\lambda| \leq \delta$ and $x \in \tilde{V} \}$
$\subseteq U$

$\hat{V} \supset \delta \tilde{V}$ is a neigh. of 0 in X

It is easy to see that $\hat{V}$ is balanced

$\lambda x \in \hat{V}$ $x \in \tilde{V}$

$c\lambda, x \qquad |c| \leq 1$

$|c\lambda| = |\lambda| \leq \delta$

$\square$

<u>Def</u> Given two TVS X and Y we denote $\mathcal{L}(X, Y)$ the set of ~~all~~ linear maps $\mathcal{L}(X'$ $X \longrightarrow Y$ which are $Y = X$ continuous.

We denote by $X' = X^* = \mathcal{L}(X, K)$ the space of functionals.

<u>Remark</u> Suppose that X is a T.V.S on $\mathbb{C}$ and $v : X \longrightarrow \mathbb{R}$

Then we can define

$$f : X \longrightarrow \mathbb{C}$$

$$f(x) = v(x) - i\,v(ix)$$

$$X \otimes_{\mathbb{R}} \mathbb{C} \qquad\qquad z \otimes x$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr] & \mathbb{R} \\ & \mathbb{C} \end{tikzcd}$$

$$f(z \otimes x) = z\, f(x)$$

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "T"'] & \hat{X} \arrow[d, "\hat{T}"] \\ Y \arrow[r, hook] & \hat{Y} \end{tikzcd}$$

<u>Def</u> Given X TVS

a subset $B \subseteq X$ is bounded

if for any nbh. V of 0 in X

$\exists$ ~~$\lambda \in K$~~ $\lambda > 0$ s.t. $\lambda V \supseteq B$.

TVS

Def $T : X \to Y$ linear is bounded if for any bounded set in X, $B \subseteq X$ we also $TB \subseteq Y$ is bounded in Y.

Exercise T continuous $\Rightarrow$ T bounded.

$$\boxed{C^\infty(\mathbb{R})}$$

$\uparrow$

Lemma X TVS $T : X \to K$ linear $Tx \neq 0$ for some $x \in X$.

The following are equivalent

a) $T \in X'$

b) $\ker T$ ~~is~~ closed

c) $\ker T$ is not dense

d) $\exists$ U neigh. of ~~0~~ $0 \in X$ s.t.

$TU \subseteq K$ is bounded

Pf $a \Rightarrow b \Rightarrow c)$

$c) \Rightarrow d)$

$\ker T$ not dense $\Rightarrow$ x and a neigh. of X disjoint from $\ker T$

$$(x + \boxed{U}) \cap \ker T = \emptyset$$

U neigh of 0

U balanced $\lambda U \subseteq U$ $|\lambda| \leq 1$

$TU \subseteq K$ is balanced in K

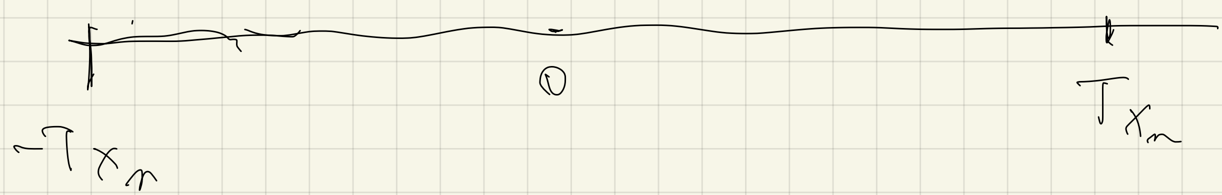
$$\lambda T U = T \lambda U \subseteq T U \quad \text{if } |\lambda| \leq 1$$

Then we claim that $T(U)$ is a bounded set in K. Otherwise we claim that $TU = K$

If TU is not bounded $\exists \{x_n\}$ in U st $|Tx_n| \xrightarrow{n \to +\infty} +\infty$

$$D_K(0, |Tx_n|) \subseteq TU \qquad \forall n$$

$$\overline{D_K(0, |Tx_n|)} \subseteq T \underbrace{\overline{D_K(0,1)}\, x_n}_{\subseteq U}$$



λTx_n

$$\Longrightarrow TU = K$$

$x \notin \ker T$. $\exists\, y \in U$ st $Ty = -Tx \Longrightarrow T(x+y) = 0$

$x + y \in \ker T$

$\boxed{(x + U) \cap \ker T = \emptyset}$

$\Rightarrow \quad x + y \in x + U$

$x + y \in \ker T$ contradiction

$d \Rightarrow a$

$\exists \; U$ balanced neigh. of 0 and $M > 0$

s.t. $\quad |Tx| < M \quad \forall \quad x \in U$

$\forall \varepsilon > 0 \quad \exists \; U_\varepsilon$ neigh of $0 \in X$

s.t. $\quad \boxed{T U_\varepsilon \subseteq D_K(0, \varepsilon)}$

$$U_\varepsilon = \frac{\varepsilon}{M} U$$

$x \in U_\varepsilon \qquad x = \frac{\varepsilon}{M} y \qquad y \in U$

$$|Tx| = \left| T \frac{\varepsilon}{M} y \right| = \left| \frac{\varepsilon}{M} T y \right| = \frac{\varepsilon}{M} \underbrace{|Ty|}_{< M} <$$

$$\frac{\varepsilon}{M} M = \varepsilon$$

$\forall\ x \in U_\varepsilon \qquad |Tx| < \varepsilon$

$\square$

<u>Def</u> A TVS X is metrizable if there is a metric on X which induces the topology of X.

A metric d on X is invariant by translation if

$$d(x+z, y+z) = d(x,y) \quad \forall\ x, y, z \in X$$

Rudin real complex anal
Funct anal.

<u>Theorem</u> A topological VS X is metrizable and admits a translation invariant metric if and only if every point of X admits a countable basis of neigh. s