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Making sense of the minus sign or becoming flexible in 'negativity'

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Abstract

This article focuses on the kinds of conceptual changes that occur when students have to deal with negative numbers in elementary algebraic operations. Interviews were carried out with twelve 8th-grade level students who were selected on the basis of their results in a test where they were required to reduce polynomials. The questions applied to their strategies and to the meaning they gave to the minus sign.

The analysis of the students' oral and written discourse attests to the presence of two major kinds of conceptual change: the first one results from students' attempts to reconcile their arithmetical presuppositions about natural numbers and the algebraic rules required to operate with negatives. The nature of the second kind of conceptual change relates to the minus sign and develops through an enlarged understanding and a flexible use of what we called 'negativity'. We argue that these two kinds of conceptual change cannot fully occur without the students developing a meta-conceptual awareness of their symbolizing activities.

1. Introduction

A few years ago, we experimented with situations in which 8th-grade level students (13–14 year olds) learned to solve equations (Vlassis, 2002b). In the French-speaking Community of Belgium, solving linear equations with one unknown (such as $2(x-3)=7$; $6x-3=-15-2x$) is part of the curriculum for 8th-grade students who have already covered basic algebraic operations in the 7th-grade (operations in N and Z like grouping like terms, and simple distributivity) (Ministère de la Communauté française de Belgique, 2000). This organization of the mathematical

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curriculum is justified by its designers in that, in order to be able to solve an equation, students have first to master the required algebraic techniques. Equations are never presented as the starting point to introduce algebra.

This research showed an obvious increase of errors as soon as negative numbers are introduced in operations such as the examples given above. Based on the literature (Gallardo, 2002; Gallardo & Romero, 1999; Glaeser, 1981; Vlassis, 2001), we could expect that negative whole numbers would create some difficulties. Nevertheless, we were surprised by the extent of the problem, the more so as the students participating in the research had been learning algebra for 18 months.

In that project, we identified several types of difficulties in solution procedures using negative numbers (Vlassis, 2001, 2002a). The main difficulty came from polynomial reduction operations, in simplifying equation members or in applying the solution procedures. However, the required reduction operations did not involve any particular difficulties such as brackets or two signs following each other. The terms were either a single number or a number with a one letter coefficient, with a maximum of four polynomial terms and never more than two different kinds, such as $2 + 7x - 3x + 8$; $6y - 20 + 3y - 12$; $4n - 3n$; $4 - 6n - 4n$; ...

In this paper, we propose an analysis of students' difficulties related to negative numbers in reducing polynomials such as those presented above. This analysis is crucial since these operations can be considered as basic elementary algebraic operations that are used not only to solve equations, but also in other kinds of algebraic activities such as the application of the operational properties, factorisation, and the study of functions.

2. What about conceptual change and negative numbers in polynomials?

When we consider the conceptual change perspective, the following question needs to be asked: what changes? And how?

According to Vosniadou (1994) and Vosniadou and Ioannides (1998), children construct, on the basis of their everyday experiences, a framework theory of physics, which is not available to conscious awareness and hypothesis testing. This theory constrains their understanding of scientific information which is often counter-intuitive and comes in conflict with everyday experience. In this perspective, Vosniadou (1999: p. 9) argues that conceptual change 'appears to be a gradual process during which information that comes through instruction is synthesized with information in the initial conceptual structure producing synthetic model or misconceptions'.

In the case of negative numbers, we can find evidence of errors in the literature due to students' attempts to assimilate negative numbers with their presuppositions about natural numbers (Gallardo, 2002; Gallardo & Rojano, 1994; Gallardo & Romero, 1999; Glaeser, 1981; Peled, Mukhopadhyay, & Resnick, 1989; Thompson & Dreyfus, 1988; Vergnaud, 1989). This kind of conceptual change is related to the fact that what students already know about natural numbers is inconsistent with the new numbers. These analyses, based on the cognitive research tradition, give an important insight into our comprehension of individual cognitive development in the context of negatives. The amount of data collected provides us with a solid basis to anticipate students' difficulties in teaching and learning operations with integers.

When we are studying conceptual change about negatives, we must bear in mind that these numbers do not only constitute a difficulty in themselves since they represent 'fictive' numbers (Glaeser, 1981), but, used in operations (as in the case of polynomials), they modify the role of the minus sign which becomes not only an 'operating' sign but also a 'predicative' sign (Glaeser, 1981). Based on students' difficulties that are reported in the literature above, we suggest that the minus sign plays a major role in the development of understanding and using negative numbers.

In order to examine in more detail the problems related to the minus sign, we think that taking into consideration the socio-cultural approaches coming from Vygotsky's works can be useful. These consider signs, such as language but also algebraic notations, graphs, and geometrical representations, as psychological tools mediating any communication activities. It is through social interactions that higher-level human mental functions develop by a transformation of interpersonal processes into intrapersonal processes. In this point of view, conceptual change is considered as a change in the use of signs in order to communicate more efficiently. In mathematics, the problem arises in a specific way since the topic concerns objects of an 'immaterial' nature (Saenz-Ludlow, 2001) that are not directly accessible through perception (Sfard, 2000). Therefore, signs are the only way to grasp these objects. Consequently, several authors such as Gravemeijer, Cobb, Bowers, and Whitenack (2000: p. 232) think that 'symbolizing is integral to the creation of new mathematical entities and thus to the knowledge construction'.

We must stress here the fact that, in our view, the cognitive and socio-cultural approaches are complementary rather than mutually exclusive. This point of view is also shared by researchers such as Mayer (2002) and Sfard (1998). Indeed, even considering, from an interpersonal point of view, that conceptual change has to develop through a more efficient use of signs in order to communicate more efficiently, we also have to keep in mind that, from an intrapersonal point of view, this will not happen in students unless they overcome obstacles such as those mentioned in the cognitive tradition.

3. A question about changes in 'negativity'

Against this wider theoretical background, the object of our analysis about conceptual change, rather than being restricted to the concept of negative numbers, will focus on the use of the minus sign in communication activities. We have to remember that in algebraic operations such as polynomials, the minus sign performs several functions. This fact makes it necessary to find a construct that encompasses these distinct roles; we will call this wider construct 'negativity'. Negativity maps the distinct uses of the minus sign in the context of elementary algebra (see Table 1). It can be considered as a tool that is required not only to understand students' oral and written discourse but also to keep in mind when planning teaching sequences. On the basis of Gallardo and Rojano's categorization

Table 1
Negativity: a map of the different uses of the minus sign in elementary algebra

Triple nature of the minus sign		
Unary	Binary	Symmetrical
Structural signifier	*Operational signifier*	*Operational signifier*
Subtrahend	Completing	Taking the opposite of or inverting the operation
Relative number	Taking away	
Isolated number	Difference between two numbers	
Formal concept of negative number	Movements on the number line	

(1994), negativity refers, first of all, to the minus sign's three major functions: unary, binary and symmetric.

- The *unary function* makes a number negative and corresponds to the sign as 'predicate' (Glaeser, 1981). The minus sign in this context is to be considered as a 'structural signifier', which refers to 'those mathematical symbols that appear in propositions dealing with objects' (Sfard, 2000: p. 49). This category is also divided into several levels of interpretation identified by Gallardo (2002: p. 179): '*Subtrahend* where the notion of a number is subordinated to the magnitude (for example, in $a - b$, a is always greater than b where a and b are natural numbers); *relative or directed number*, where the idea of opposite quantities in relation to a quality arises in the discrete domain and the idea of symmetry appears in the continuous domain; *isolated number*, that of the result of an operation or the solution to a problem or equation; and *formal concept of negative number*, a mathematical notion of negative number, within an enlarged concept of a number embracing both positive and negative numbers (today's integers)'.
- The *binary function* considers the minus sign as an 'operating' sign (Glaeser, 1981), which, in this case, corresponds to an 'operational signifier' (Sfard, 2000: p. 49) used in 'propositions that focus on, say, operations'. From the literature, we identified four uses of the minus sign as an operational signifier (see Table 1). The first three uses were defined from the works of Gallardo and Rojano (1994) who identified the triple nature of subtraction: *taking away* (in situations such as '6 marbles taken away from 12 marbles'), *completing* (in situations such as 'what number is missing from 24 to make 38'), and *the difference between two numbers* (in situations such as 'what number represents the difference between 16 and 22'). The minus sign as a *movement on the number line* refers to the work of Thompson and Dreyfus (1988).
- In the third role, the *symmetric one*, the minus sign is considered as an operational signifier and has to be distinguished from the 'relative number'. The symmetric includes the use of the minus sign as 'taking the opposite of' or 'indication of inversion' (Nunes, 1993) such as the role of the first minus sign in $2a - (4a - 5b + 2c)$. For relative numbers, the minus sign is a structural signifier and is used in contexts such as $(-x) + x = 0$.

With these considerations in mind, we can see how discussing negative numbers only would be a narrow viewpoint in a discussion about conceptual change in the context of polynomials, since it only concerns the unary function of the minus sign. Indeed, the presence of negative numbers in algebraic operations requires that a larger set of modifications be taken into account than those concerned by the unary function.

4. Research aims and questions

Up to now, studies about negative numbers have often occurred in numerical contexts only: they analyzed students' conceptions of negative numbers or their difficulties in operating with them. Rarely did studies focus on literal contexts, i.e. equations or algebraic operations with letters. Gallardo and Rojano (1993) worked on negative numbers in the context of equations but they mainly focused on the negative solution. In this article, we propose concentrating our analysis of the minus sign on operations in a literal context, the context of polynomial reductions.

More particularly, our presentation will try to analyze students' written and oral discourse in relation to the following three questions:

- Q1 How do students explain their procedures in polynomial reductions? To which models or rules do they refer?
- Q2 Which meaning do they confer to the minus sign?
- Q3 What is the resistance level of their initial models? This questions aims to analyse resistance to change in the reasoning of both students who fail (by reminding them of the correct procedures) and those who succeed (by asking their opinion on the erroneous procedures often suggested by their peers).

5. Methods

5.1. Subjects

Twelve 8th-grade students (13–14 years) were individually interviewed during the last two months of the school year. These students came from six classes in three schools which differed in terms of the student's socio-cultural background: two classes (consisting respectively of 23 and 25 students) came from a school with students from an advantaged socio-cultural background, two classes (consisting respectively of 20 and 22 students) from a school with students from a disadvantaged socio-cultural background, and two classes (consisting respectively of 19 and 24 students) from a school with students from an average socio-cultural background.

The students were selected according to the results they achieved in a test that required performing 28 polynomial reductions. The polynomials were constructed on the same structure as those described in the Introduction (four terms maximum and no particular difficulties). Every item was scored with either a 1 for a correct answer or 0 if it was incorrect. In order to select the students, we calculated the mean of the score in order to have percentages. We selected four lower-level students (mean of 50% or less); four middle-level students (mean between 60 and 70%); and four higher-level students (mean of 80% or more).

These students were introduced to algebra at the beginning of the 7th grade, through the generalisation of the properties of operations. During this school year, they learned operations in N and Z (grouping like terms, simple distributivity). Negative numbers were introduced in numerical contexts with the help of models such as the number line or debts and gains. Operations with negatives were justified by algebraic rules such as the signs rule for multiplication and rules based on the opposite and the absolute value for addition and subtraction. In the 8th grade, the focus of the mathematical course was on operations in R, more complex operations in Z (double distributivity and factorisation) and solving linear equations with one unknown.

5.2. Procedures and test materials

The interviews took place during the students' lunchtime two to five weeks after the test. Each interview lasted about half an hour. The author was responsible for taking the interviews. They were semi-structured: the major steps were planned with the main questions. Due to the heterogeneity of the students' levels, the interviews had to be planned flexibly. Students were presented with 16 items from the 28 in the test. These were selected on the basis of the number of terms (two, three or four), the place of like terms (grouped or not) and the distribution of negative terms. Examples of polynomials given to the students during the interview are shown in Table 2.

In order that all the students reduce a set of representative polynomials, the first six items of the interview presented in Table 2 had major features of the set of polynomials: three polynomials were composed of two, three and four terms and had already grouped the like terms (see items 2, 4, and 6 in Table 2), one had two pairs of like terms but one pair was separated (see item 3 in Table 2), and two polynomials had two separated pairs of like terms (see items 1 and 5 in Table 2).

The 10 other interview items, constructed according to the same principles, were intended to explore students' discourse in more detail with the help of other cases.

The interview questions were organized according to the three research questions:

(a) the procedures: the students were invited to reduce polynomials on a new and blank sheet of paper and to explain their reasoning (related to Q1);
(b) the meaning given to the minus sign (related to Q2); and
(c) the resistance to change (related to Q3).

During the interview, the students were initially asked to reduce the first six polynomials without any intervention from the interviewer. After that, the interviewer asked some general questions about the rules or the models they used. When students failed to reply to one or the other, they were reminded of the context of the rule that they applied erroneously and were shown the number line in

Table 2
Main procedures used by students in order to reduce polynomials

	Items	% Test	Observed incorrect approaches			Observed correct approaches		
			Brackets reasoning	Signs rules	Confusions in selecting the sign	Not expressed	Algebraic rule	Number line
1	$6 - 5a - 3 - 4a$	66	L2	L1∗–L3–L4				
			M1–*M2*	*M2*		M4		M3
				G1			G4	G2–G3
2	$4 - 6n - 4n$	73	*L1∗*–L2–L3–*L4*	*L1∗*	*L1∗–L4*			
			M1∗–*M2*	*M2*		M4		M3
			G1				G4	G2–G3
3	$-15 - 9y - 4y + 10$	73	*L1∗*–L2–L3	*L1∗*–L4				
			M1–M2			M4		M3
							G4	G1–G2–G3
4	$20 + 8 - 7n - 5n$	75	L1∗–L2–L3–L4					
			M2			M1–M4		M3
						G4		G1–G2–G3
5	$6y - 20 + 3y - 12$	77	*L1∗–L4*	*L4*–L3	*L1∗*–L2–*L4*			
			M1–*M2*	*M2*		M4		M3
							G4	G1–G2–G3
6	$-4n - 3n$	89	L2	L1∗–L3		L4		
				M2		M1–M4		M3
							G4	G1–G2–G3

Notes: (1) The letters G, M and L refer to the student's score: G: the student's score is equal to or above 80%; M: the student's score is between 60% and 70%; L: the student's score is equal to or below 50%. (2) Italics apply to students who suggest different approaches for a single item. (3) The asterisk indicates the students who erroneously grouped unlike terms.

order to help them reduce the polynomial and test their resistance to change. When students succeeded in reducing all the polynomials, the interviewer showed them erroneous reasoning from other students and asked them what they thought about it. The students then continued, thus reducing the other polynomials.

After being questioned about the procedures they used and being tested on their resistance to change, all students were questioned about the meaning of the minus sign by using questions like 'what do you think the minus sign is used for?'. Three situations were investigated: when the minus sign was between two like terms, when it was between two unlike terms, when it was at the beginning of the polynomial before the first term. For each situation, one or two illustrative polynomials was/were given to the students.

6. Results

This section presents the results and an analysis of the first six interview items. Our analysis is presented according to the three research questions.

6.1. How do students explain their procedures in polynomial reductions?

Table 2 shows the procedures spontaneously suggested by the students before any intervention.

6.1.1. Brackets reasoning

This is a widespread type of reasoning which puts imaginary brackets around like terms when preceded by the minus sign. For instance, to reduce $20 + 8 - 7n - 5n$, some students change the expression to $20 + 8 - (7n - 5n) = 28 - 2n$. This procedure is very frequent in polynomials where like terms are presented in pairs (items no. 2, 3, and 4). This is often implemented by lower-level students, but is also observed with students M1, M2, and G1.

Here is an example of an oral explanation about this reasoning:

Student L1 when reducing $20 + 8 - 7n - 5n$:

> *You put the 'n' with the 'n' and the numbers alone with the numbers alone, so I make* $20 + 8$*, that gives* <u>28 *minus*</u> … *and* $7n - 5n$*,* $2n$*; so,* $28 - 2n$*.*

We also observed that reasoning, but less frequently, in the items where like terms are not grouped by pairs. This happens after the student has grouped like terms. The minus sign before these fictive brackets (which are never represented in written discourse) is considered as a boundary, which divides the polynomial in two operations made on natural numbers.

6.1.2. The signs rule

The signs rule is regularly proposed by students. The version 'minus by minus gives plus' seems to strike the students more than the version 'minus by plus'. The use of this rule is more common for the items in which several minus signs are present (like in item no. 1).

Here is an example:
Student L4 when reducing $6 - 5a - 3 - 4a$:

5a, it will be +4a because there are only minus signs. Minus by minus gives plus. I think it will give 9a minus ... So now, we make 6, it will give 9 because minus by minus gives plus. (She writes 9a–9).

6.1.3. Confusions in selecting the minus sign

This error can have two forms:

(1) Operating from the right to the left side:

Students L1 and L4 write simplified expression of $4 - 6n - 4n$ (item 2) from right to left. Student L1 explains that $6n - 4n$ equals $2n$ and so the answer is $2n - 4$. In another example which is not given in Table 2 ($7 - 6n + 13$), L1 performs the operations from right to left and explains the reasoning as follows: *'I made 7 with 13, so 13–7 gives 6. So it is 6–6n'*. This kind of response, for this item, is given by three other students. It also appears in the collective paper-and-pencil test in polynomials such as $\mathbf{2x} - 7 - \mathbf{6x} - 4$ where the answer $\mathbf{4x} - 11$ is given by 9 of 131 students. In this particular case (a large number subtracted from a smaller one), it seems that some students are tempted to invert the operation in order to have a more 'comfortable' form like $13 - 7$ instead of $7 - 13$ or $6x - 2x$ instead of $2x - 6x$. This kind of behaviour concerns a well-known buggy procedure in subtraction with numbers. It has also been pointed out by De Corte and Verschaffel (1981) in the context of younger children's solutions of canonical and non-canonical arithmetic sums.

(2) Taking the following sign:

In item 5 ($\mathbf{6y} - 20 + \mathbf{3y} - 12$), three students (L1, L2, L4) reduce the y terms to obtain $6y - 3y$ by using the $6y$ and the minus immediately after it, followed by $3y$. This predominance of the 'sign after' behaviour is also found in the discourse of student L1 (words underlined) in Section 6.1.1 and in that of student M1 in Section 6.2 (expression underlined at point 1). In their oral discourse, they put the sign that follows the first pair of like terms immediately after the result of the reduction, while they have not yet simplified the second pair and do not know the sign of the reduced expression. This tendency to select the sign after can be related to the first arithmetical manipulations when, for example, children who had to make $6 - 4$ were told that they had six apples from which four apples had to be taken away. This discourse around these manipulations where the minus sign appears to be a signal for an action to be performed on the first term (six), from which the second one (four) is going to be taken away, could give some students the idea that the minus sign is associated with the term placed before it.

6.1.4. Not expressed

Some students cannot explain the rule they used. In the present study, this only concerns students who have no difficulties in reducing the polynomials. Student M4 is quite typical of this type of behaviour. When the interviewer asked him whether he used a particular rule or the number line in his mind, he answered: *no, it comes to me spontaneously*. The difficulties good-level students can have in explaining their strategies can come from the fact that they perform some tasks so quickly and automatically that they find it difficult to express the underlying reasoning in words.

6.1.5. Algebraic rule

A good-level student (G4) refers to an algebraic rule,[1] which concerns the sum of two whole numbers. This student has difficulties explaining this procedure.

INT: Explain to me how you simplify $4 - 6n - 4n$.
G4: 4, is on its own, I left it alone. Then $-6n - 4n$*, that makes* $-10n$.
INT: And how do you know that $-6n - 4n$*, makes* $-10n$*?*
G4: Because there are two minus signs, that makes plus.
INT: But you did put minus…
G4: Yes, because you must leave the minus sign, I don't know how to explain it.
INT: And if I ask you $7 - 9$*, how do you do it?*
G4: It would make -2 *because the highest is 9.*
INT: Does the number line help you to make your calculations?
G4: No.

6.1.6. The number line

Three students use this model to justify their polynomial reductions. They never drew a number line, but they only refer to it or to an image of it, like student G2 who speaks about the number line in terms of a 'scale'. She explains the reduction of $6 - 5a - 3 - 4a$ as follows:

G2: That gives $3 - 9a$
INT: How do you know whether it is $-9a$ *or* $+9a$*?*
G2: In my mind, I group both numbers with a and then, mm … if I take the 'scale' then I am in the negatives, at 5a, so I go down by 4, for me it is obvious when I see it like that.

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[1] The whole rule is presented as follows in a very common textbook in the French Community of Belgium: (*Adam, A., Close, P., Janssens, R., Lousberg, F., 2001, Espace Math 1—De Boek Wesmael, p. 84.*). In order to calculate the sum of two whole numbers:

- first we determine the sum sign
  - if both terms are positive, the sum is positive;
  - if both terms are negative, the sum is negative;
  - if both terms are of different signs, the sum takes the sign of the term which has the highest absolute value.
- secondly, we determine the absolute value of the sum
  - if the signs of the terms are similar, we calculate the sum of the absolute values;
  - if the signs of the terms are different, we calculate the absolute value of the difference of the absolute values.

6.2. What is the meaning given to the minus sign?

In this stage of the interview, students were asked to explain what the minus sign is used for. These analyses highlighted three main features:

(1) For most students, the minus sign has its meaning only in relation to the procedure applied in polynomial reductions.

- All the students agree about the fact that, generally, the minus sign is used to subtract. Furthermore, this is the only meaning attributed by two low-level students (L1 and L3). Most of the time, the students refer to that subtraction function, only in cases in which the minus is placed between two like terms.
- Three students (L2, M1, M3) think that the minus sign between two unlike terms is used to split. For example, according to M1, the first minus in $4 - 6n - 4n$, *is only to split. I first make that operation, then the other one. I made 4, there was no other without a letter. I wrote* $\underline{4-}$, *I kept the* $-$, *then I made* $6n - 4n$, *I found 2n, then I have* $4 - 2n$.
- According to student L4, who applied the signs rule, the minus is used for making minus by minus gives plus.

(2) No student considers explicitly that the minus sign could have a double status, neither when the minus sign is between two unlike terms nor when it is between two like terms.

- When the minus sign is between two unlike terms, some students think it is used for operating (G1, G2, G4), for splitting (L2, M1, M3), or for making the following term negative (M1, M3, G3). Only students M1 and M3 present two different discourses about the meaning of the minus sign between two unlike terms, depending on the items. Five students cannot say anything about the meaning of the minus sign in that context.
- Between two like terms, 10 students explain that the minus sign is used for subtracting. Two good-level students (G1 and G2) think that the minus sign cannot be attached to the number because there would not be any operation sign anymore. When questioned about this subject, these good-level students were puzzled since, furthermore, when they have to move the term, they correctly take the sign before it. Here is an extract of the interview with G2 who is a very-good level student.

INT: What's the minus sign used for in $-7x - 8x$. (the second minus sign)
G2: The minus sign is used for showing you go down on the 'scale', you have to subtract another number and, mmm,... this number is positive, you have to go down again.
INT: Where is there a positive number?

G2: If you look at the numbers like that, if you don't consider the signs, you say that the 8x is positive, and you have already a negative (−7x), you subtract a positive one, you go down, if 8x had been −8x, you would have gone up to 1x.
INT: So, is the minus sign separated from the 8x?
G2: It depends on the way you solve the difference. For me, since it is a difference between like terms, I would put it separately. If you put it together then there is no more sign between the both of them and you will have to write a plus sign.

(3) The minus sign at the beginning of a polynomial is always considered as the sign of a negative number.

Ten students out of 12 maintain this without any hesitation. The two students who do not think like that are low-level students (L1 and L3) who can only justify the role of the minus sign when it is considered as a subtraction sign between two similar terms. The first negative term of the expression seems to be considered as the prototype of negative number.

6.3. How strong is the resistance to change?

The resistance to change has been measured after the students reduced the six first polynomials in the interviews. When they failed to perform a reduction correctly, they were reminded about the number line. When they succeeded in reducing all the polynomials, the interviewer confronted them with erroneous reasoning and asked them what they thought about it.

An analysis of the students' discourse leads us to categorize them in three profiles:

Profile 1 concerns students who have great difficulties in changing the incorrect use of the rules they used (students L1, L3, L4). Student L1 is a typical example of this profile. He cannot understand the irrelevance of the signs rules in the context of polynomials. His explanations about the reduction of $-15 - 9y - 4y + 10$, after being reminded twice about the number line and twice about the usage context of the signs rule, are the following:

L1: $-15 + 10$*, that gives 5 because it is the highest number, which is stronger, isn't it? (He refers to a correct algebraic rule but not to the number line about which he was reminded just before)*
$-9y - 4y$ *...mm... minus by minus (in a very low voice). That makes 13.*
INT: Plus or minus?
L1: Plus because minus by minus gives plus.

Profile 2 features students who can efficiently re-orientate their approaches when they are reminded about relevant rules and models (students L2, M1, M2, G1).

Profile 3 covers students M3, M4, G2, G3, G4 who show a great coherence in their procedures and who do not let themselves be disturbed by counter-arguments.

They have an efficient and flexible use of the minus sign in the polynomial reductions, while a rigid idea of that sign appears in their discourse (a sign – a single meaning).

Theoretically, it would be possible to find a *profile 4* (we did not meet any student of that type) for which written and oral discourse would be coherent. This double competence could belong to a kind of meta-conceptual knowledge which would feature students aware of the efficient use of their own knowledge, in contrast with profile 3 students who have an efficient use of the symbol, but who do not seem to be aware of it.

7. Discussion and conclusion

Our analysis of the students' procedures in reducing polynomials and their discourse about the minus sign show evidence of the conceptual change highlighted by Vosniadou (1994). This conceptual change consists of synthesizing new information with existing presuppositions, leading to the so-called 'synthetic models'. In our algebraic context, we find these synthetic models in some erroneous procedures that keep track of arithmetical practices with natural numbers. The brackets reasoning, where the minus sign is considered as a barrier that divides the polynomial in two operations made on natural numbers, is a typical example. Selecting the 'sign after' also attests to early manipulations with natural numbers. The tendency to operate from right to left is also a reminiscence of an approach followed in the early years of learning arithmetic. It is possible that the context of polynomials, where reduction operations do not necessarily follow a sequential path from the first term to the last one, reactivates this kind of error. Typical examples of synthetic models also emerge from the students' discourse about the minus sign. No students imagine that the minus sign could have a double status, namely unary and binary. Some low-level students consider the minus sign only as a subtraction sign whatever its place in a polynomial. Even some good-level students claim that the minus sign placed inside the expression could not be attached to the sign, *otherwise there would not be any operation sign anymore*.

We also observed another hybrid conception resulting from students' attempts to synthesize two kinds of information when they apply different rules or models to reduce polynomials, even in the same polynomial, depending on the item's features: e.g., the presentation of like terms by pairs preceded by a minus sign, which leads to a 'brackets' reasoning or, the sequence of several minus signs, which activates the signs rule. We can relate this tendency to the notion of mixed models outlined by Vosniadou (1994) and Vosniadou and Ioanides (1998) in the context of forces. Those mixed models appear when the students produce conflicting answers, because they use criteria coming from distinct models. This could partly justify why the students do not see inconsistency in their own reasoning. Each polynomial is considered as different.

With a view to further examining the phenomena that we just discussed, we argue that the debate about conceptual change can benefit from involving socio-cultural approaches in focusing our analysis on the signs as mediating tools. In this perspective, recent works about mathematical symbolism give an insight into the development of the use of mathematical symbols. Following Sfard (2000), their use goes from a template-driven use to an object-mediated one. Template-driven refers to a kind of meaning which takes place in the first contexts where the signifier appears; students will use the symbols in 'ready-made sentences' coming from these contexts. Therefore, at that stage, the new signifier does not have an existence of its own. The object-mediated use is characterized by the ability of using the sign as a self-sustained entity, which is detached from the 'ready-made sentences'. Our results show that students have a template-driven use of the symbols used in the polynomials. Their erroneous use thereof is due to their interpretation of the minus sign, which prompts the students to use different templates; different situations around the minus (such as a minus sign followed by a pair of like terms; or a succession of minus signs) will redirect the discourse in one direction or another and lead to the emergence of mixed models as explained above. This can explain why some algebraic rules (like the signs rules) appear to be part of a resistance to change which is as strong as that coming from initial arithmetic conceptions.

This analysis emphasizes that conceptual change involved in reducing polynomials not only goes through synthesizing new knowledge with old presuppositions, as discussed in the beginning of this section. It also has to integrate a second kind of change related to the use of the minus sign. Indeed, we argue that, in algebraic operations in general, and in polynomials in particular, an object-mediated use of the symbols presupposes an adequate use of the minus, featured by a capacity for flexibility in what we called the 'negativity'. This capacity involves knowing and using in a flexible manner the different roles of the minus sign whether structural or operational, depending on the context.

We think that an important part of the difficulties with the minus sign can be related to the fact that students could not subordinate their use of mathematical symbolism to the need for efficient communication. The students in our interviews have been given traditional teaching as described by Gravemeijer et al. (2000) where they are introduced to the ready-made use of signs without experiencing a need for it themselves. This kind of teaching gives rise to certain kinds of 'meta-rules' (Sfard, 2000) or 'underlying cultural rules' (Verschaffel, Greer & De Corte, 2000), which are internalized by the students. As discussed in Section 5.1, a teacher gives models for negative numbers in a numerical context (e.g. number line) and other means for an algebraic context (rules and only rules). Therefore, low-level students try to select the 'good rule' without considering the number line as a possible model (see the interview extract of student L1 in Section 6.3). Even good-level students do not present a fully object-mediated use of the symbols: they use them appropriately in their procedures but their oral discourse about the minus remains inflexible.

Following Vosniadou (1994), an important determinant of conceptual change is the development of a meta-conceptual awareness. For Vygotsky (1997), a concept cannot be fully developed into conscious form without language. In our interviews, some students are able to solve polynomials but are totally unable to express their reasoning (category 'not expressed') and other students can explain their reasoning but when questioned about the minus sign, present a very poor and inflexible discourse about it.

Everything that has been said shows that conceptual change in algebraic operations will 'not be a sudden or radical shift but a gradual and slow one' (Vosniadou, 1999: p. 11). Two major conceptual changes were identified: the first one concerns the students' ability to reconcile their initial conceptions about operating with natural numbers, and the algebraic rules required to operate with negatives. The second one is related to an enlarged and flexible use of negativity. For these changes to really occur, we agree with Vosniadou (2003) that the development of metaconceptual awareness will be determinant. This can only be developed in discursive practices where students can experience their own symbolizing activities in order to understand the need for conventional discourse in a perspective of 'take-as-shared meaning' (Gravemeijer et al., 2000). This context, where ordinary language has its place in the process of mathematical learning, differs greatly from traditional teaching where mathematical symbolism and formal language are considered the only good way of talking about mathematics.

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