Educational Studies in Mathematics (2020) 103:7–25
https://doi.org/10.1007/s10649-019-09921-7

Understanding and promoting students' mathematical thinking: a review of research published in *ESM*

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Published online: 12 November 2019
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Abstract
In this paper, we offer a comparative review of research on understanding and promoting students' mathematical thinking. The sources for the review are papers that were published in Educational Studies in Mathematics (ESM) during two windows of time: 1994–1998 and 2014–2018. Selection of these two time periods enables us to comment on the "state of the art" in research as well as identify changes over the past 25 years. The review is guided by an analysis of conceptualizations of "mathematical thinking" proposed in the research literature, selected curriculum documents, and international assessment programs such as the OECD's Programme for International Student Assessment (PISA). The review not only documents salient features of research studies, such as the country of origin of the authors, educational level of the participants, research aims, theoretical perspectives, and methodological approaches, but also identifies the contribution to knowledge made by this body of work as well as future research directions and opportunities.

Keywords Mathematical thinking · Problem solving · Reasoning · Proof · Problem posing · Comparative review

1 Introduction

This paper reports on a review of research on understanding and promoting mathematical thinking published in *Educational Studies in Mathematics* (*ESM*) during two windows of time: from 1994 to 1998 and 2014 to 2018[1]. The focus on mathematical thinking arises from contemporary views about the importance of mathematics for critical citizenship, with implications for what it means to learn mathematics successfully. In many countries, for example,

[1]This paper extends the review of research on mathematical thinking presented by the first author at the 2018 Regional Conference of the International Group for the Psychology of Mathematics Education (PME). The conference theme was *Understanding and promoting mathematical thinking*.

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curriculum reforms have initiated reconsideration of the nature of school mathematics, leading to changes in the selection and organization of mathematical content and increasing emphasis on mathematical thinking processes, practices, and ways of working (International Commission on Mathematical Instruction, 2017).

A comprehensive review of literature in the area of mathematical thinking would draw on a wide range of sources, such as research journals, books, and conference proceedings. However, this review is restricted to papers published in *ESM*. While such an approach could be criticized for limiting the review's scope and reducing the value of any conclusions that can be drawn from it, it does permit an analysis of how a particular research area has been represented in one of the most cited and respected journals in our field (Williams & Leatham, 2017). The paper was also planned with a view to marking the journal's fiftieth year in 2018 and publication of its one hundredth volume in January 2019. Together these events provide an occasion for reflection on the journal's past and future joining several commemorative articles of this type that have already been published (Bakker, 2019; Beckers, 2019; Mesa & Wagner, 2019).

Two separate 5-year windows of time have been chosen for this review. Examining papers published during the most recent time period, 2014–2018, allowed us to comment on the current "state of the art" in research on mathematical thinking. We wanted to select an earlier time frame for the purpose of comparison, and identified two reasons for ultimately deciding on 1994–1998. First, we anticipated that the 1994–1998 window would capture the maturation of research on mathematical problem solving as a key form of mathematical thinking. Second, the full review period, from 1994 to 2018, corresponds to the second half of *ESM*'s 50-year lifespan. Comparing these two "slices" of published research spanning 25 years enabled us to identify changes in emphasis over time, and to interpret the findings in terms of trends and research themes discussed in mathematics education research handbooks and seminal works in the area of mathematical thinking. Our aims were to produce a concise comparative analysis and an up-to-date snapshot of this area and to propose how research in mathematical thinking might develop.

This review is organized around the following sections. First, we consider the meaning of "mathematical thinking" and delineate the parameters and particular aspects that guided the literature search. Next, we outline the methodology used for the review—how the sources were selected, organized, and analyzed. The findings are then presented in response to the broad research questions that structured the analysis:

1. What similarities and differences can be observed in the contexts and features of studies published in *ESM* that investigate how to understand and promote students' mathematical thinking in the periods 1994–1998 and 2014–2018?
2. How have the theories and methodologies that framed the studies changed during these time periods?
3. What is the contribution to knowledge made by this body of work, and what future research directions are indicated?

2 Meaning of mathematical thinking

Mathematical thinking is considered to be an important goal of schooling across the world, but it is difficult to define in just a few words. Mathematical thinking gives attention to process rather than content, although both are clearly important for learning mathematics and both are typically represented in school mathematics curricula. Insights into the nature of mathematical thinking can be gained from examining research frameworks and curriculum frameworks that attempt to delineate its salient features.

In the 1980s, research focused attention on the processes involved in mathematical thinking, particularly in relation to mathematical problem solving. For example, Schoenfeld (1992) developed a framework for mathematical thinking that included mathematical knowledge and heuristics, metacognitive knowledge and control to guide problem solving activity, beliefs and affects and how these are influenced by the instructional environment. Ways of engaging in, and promoting, mathematical thinking while solving problems were addressed in a practical way by Mason, Burton, and Stacey (1985), whose book titled *Thinking Mathematically* identified two pairs of fundamental processes: (1) specialising and generalising, and (2) conjecturing and convincing.

While research on mathematical problem solving flourished in the 1980s and 1990s, more recent reviews have lamented "the lack of impact and cumulativeness" (Lesh & Zawojewski, 2007, p. 763) of research in this area, noting in particular that the literature on mathematical problem solving had not produced clear guidelines for school practice (Anderson & White, 2004; English & Gainsburg, 2016). Many possible reasons were suggested for these disappointments and shortcomings, such as:

- Uncertainty over what might be the most fruitful theoretical perspectives for understanding and promoting problem solving
- The breadth of the domain and the consequent difficulties in defining what is meant by "problem solving"
- Cyclic trends in education policy that lead to shifts in emphasis between problem solving and basic skills, resulting in corresponding shifts in research emphases
- Lack of agreement about the overarching goal of including problem solving in the mathematics curriculum

Nevertheless, research interest in mathematical problem solving had been sustained over the decades since the 1980s. For example, a double issue of *ZDM Mathematics Education* on the theme of "Problem solving around the world: Summing up the state of the art" was published in 2007, with papers from fifteen different countries (Törner, Schoenfeld, & Reiss, 2007). Research handbooks also regularly included chapters on problem solving research (e.g., English & Gainsburg, 2016), sometimes expanding the field to include perspectives on mathematical modeling (e.g., Lesh & Zawojewski, 2007) or problem posing (e.g., Weber & Leikin, 2016).

Particular emphases in relation to mathematical thinking can also be discerned in international assessment programs and curriculum frameworks. For example, the Programme for International Student Assessment (PISA) uses the term "mathematical literacy" in its assessment of the ability of 15-year-old students to apply mathematics in real world contexts. For the PISA 2021 assessment framework, the domain of mathematical literacy is organized around three inter-related aspects: (1) mathematical reasoning and problem solving processes, (2) mathematical content, and (3) contexts for the assessment items and selected 21st century skills (see OECD, 2017, for a summary of key publications and reports conceptualising 21st century skills). In this framework, mathematical literacy is defined as "an individual's capacity to *reason mathematically* and to formulate, employ, and interpret mathematics to *solve problems* in a variety of real-world contexts" (OECD, 2018, p. 6, emphasis added).

Problem solving and reasoning processes are also given prominence in curriculum frameworks, such as the *Principles and Standards for School Mathematics* developed in the USA by the National Council of Teachers of Mathematics (2000). In this document, conjecturing, justification, and argument were identified as fundamental reasoning processes that all students can learn, and that can ultimately be expressed in a formal way as mathematical proof. A contrasting approach to curriculum design for promoting mathematical thinking was illustrated by the Singapore mathematics framework, which since 1990 has been centrally focused on problem solving. In its current version the Singapore framework additionally identifies reasoning, communication and connections, as well as applications and modeling, as important processes for learning (Ministry of Education, 2012).

The *Adding it up* report prepared by the US National Research Council (2001) introduced the notion of "mathematical proficiency" to propose a comprehensive view of what is necessary for all students to learn mathematics successfully. Mathematical proficiency was considered to have five interwoven and interdependent strands:

- Conceptual understanding—comprehension of mathematical concepts, operations, and relations
- Procedural fluency—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- Strategic competence—ability to formulate, represent, and solve mathematical problems
- Adaptive reasoning—capacity for logical thought, reflection, explanation, and justification
- Productive disposition—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy. (p. 116)

The notion of mathematical proficiency had been taken up in mathematics curriculum frameworks in various countries as a way of specifying the process domain for mathematics learning (e.g., in Australia, see ACARA, n.d.; in Ireland, see NCCA, 2015). Although all five strands arguably have a bearing on mathematical thinking, the strands of *strategic competence* and *adaptive reasoning* seemed to us to be most closely related to mathematical thinking as conceptualized by previous research. It was noteworthy that strategic competence included the ability to pose and formulate problems as well as to solve them, and that adaptive reasoning referred to intuitive and inductive reasoning as well as informal explanation and justification, formal proof and other forms of deductive reasoning. Research interest in mathematical reasoning reflected this breadth of emphasis, from development of frameworks for understanding students' conceptions of proof (e.g., Harel & Sowder, 2007) to investigations of instruction that supports students' informal reasoning through explanation and justification (e.g., Yackel & Hanna, 2003).

This overview of research pointed to two broad aspects of mathematical thinking that became the focus for our comparative review of *ESM* papers: mathematical *problem solving* and (2) mathematical *reasoning*. These dimensions guided the literature search, further details of which are presented in the next section.

3 Methodology of the review

This comparative review drew on the research questions, methodology, and approach to reporting findings that were developed for a literature survey undertaken by the ICME 13 Survey Team on "Teachers Working and Learning Through Collaboration" (Jaworski et al., 2017; Robutti et al., 2016). For our own review of research on mathematical thinking, we identified sources from a manual online search of *ESM* issues, including special issues, published from (a) 1994 to 1998, that is, Volumes 26(1) to 37(3) and (b) 2014 to 2018, that is, Volumes 85(1) to 99(3). Excluded from the search and subsequent analysis were editorials, book reviews, letters to the Editor, and brief communications comprising announcements, errata, obituaries and calls for papers. Research papers published in French were also removed from the analysis (ten papers published in 1994–1998 and one in 2014–2018). This process left us with a total of 164 English-language research papers published in *ESM* in 1994–1998 and 284 published in 2014–2018. In order to identify an initial set of papers in the area of mathematical thinking, the second author searched the titles, keywords and abstracts of these papers for the following broad range of terms related to problem solving and reasoning: problem solving, problem posing, reasoning, proof, argumentation, explanation, justification, generalization, and abstraction. Following the process described by Robutti et al. (2016), she then read the papers and recorded information about each in a spreadsheet under a set of column headings (e.g., country, research aims, theoretical perspective, number and type of participants, methodological approach, duration), that allowed us to address the first and second research questions, as described below. Citation details, including hyperlinks to the online versions of the papers, were also recorded in the spreadsheet.

In analyzing contexts and features of the studies, we recorded the geographical region in which the study was conducted, and for empirical studies the target educational level (primary school, secondary school, vocational education, tertiary study of mathematics, teacher education) and scale (duration and number of participants). We also wanted to document the substantive research features of these studies. To do so, we developed three categories that we labeled research aim, research focus, and research orientation, and added these as columns in the spreadsheet.

To classify the research aim of the papers, we drew on the survey of published research in mathematics teacher education undertaken by Adler, Ball, Krainer, Lin, and Novotna (2005). This survey had distinguished between two types of studies: those aiming to understand how teachers learn, and those seeking to improve teachers' opportunities to learn. We adapted this idea for our review to record two kinds of research aim in the papers we identified: to *understand* or to *promote* mathematical thinking. We assigned papers to the former category if their aim was only to describe and analyze students' reasoning or problem solving strategies, and we used the latter category for papers that studied the effect of some kind of teaching intervention on students' thinking.

To classify the research focus of the papers, we recorded the broad type of mathematical thinking that was the subject of investigation: *problem solving* or *reasoning*. We also found it useful to identify papers that extended problem solving research into the associated area of problem posing (cf Weber & Leikin, 2016). In addition, we identified papers with a research focus on proof as a kind of formalized mathematical reasoning.

The category we labeled research orientation picks up a theme that emerged from Weber and Leikin's (2016) review of research on problem solving and problem posing. They distinguished between research in which problem solving is the *object of study* and that which uses problem solving as a *research tool* to investigate other aspects of mathematics learning. They explained that studies of the former type reported on how students solved problems and how these skills could be taught, while studies of the latter type used problem solving tasks to investigate something else, such as students' understanding of mathematical concepts or their beliefs about mathematics. Although Weber and Leikin's classification referred only to research on problem solving, we found it useful for recording the research orientation of *ESM* papers about mathematical reasoning as well.

As the search progressed, we met frequently to screen the papers that had been entered into the spreadsheet, removing any that did not seem to fit the search criteria and flagging those about which we were uncertain so that we could return to reconsider them later. This was an iterative process lasting several weeks, during which time we proposed and revised insights from the emerging patterns of similarity and difference between papers. The process of screening and discussion also enabled us to clarify the dimensions of the review and articulate what seemed to be important distinctions, such as the difference in emphasis between investigating mathematical thinking processes as the object of study and using problem solving or reasoning as a tool for teaching mathematical content. When the initial search was completed, the first author skimmed the full version of every paper recorded in the spreadsheet and removed any that did not fall within the scope of the review. As a final check, she also inspected the titles of all research papers published in *ESM* during the two time periods of interest and skimmed the full version of any paper that had not been selected but seemed relevant to the search criteria. She then added to the spreadsheet the small number of extra papers that were deemed to meet the criteria for inclusion, that is, papers that were concerned with *understanding* or *promoting* students' mathematical *problem solving* and mathematical *reasoning*. This process resulted in further clarification of the dimensions for review. For example, it led to sharpening the focus on students' mathematical thinking rather than on teachers' pedagogical reasoning, and to removal of papers about concept formation via reflective abstraction as these were considered less relevant to problem solving and reasoning.

The search and screening process yielded two sets of papers. For the period from 1994 to 1998, there were 24 papers (21 empirical and 3 theoretical), representing 15% of the 164 English-language research papers published during that time. For the period from 2014 to 2018, we identified 57 papers (51 empirical and 6 theoretical), representing 20% of the 284 English-language research papers published in *ESM* during that time. This initial comparison suggested that research on mathematical thinking disseminated via *ESM* had continued to flourish over the last 25 years. Further analysis of both sets of papers revealed interesting characteristics of this research and some changes over time. We used descriptive statistics to summarize findings in relation to our first two research questions, concerning contexts and features of studies and the theories and methodologies that framed them. For the third research question, which asked about contributions to knowledge and future research directions, we identified qualitative features of the sets of papers and selected illustrative examples for discussion.

4 Themes from the analysis

Findings are presented as sets of themes corresponding to the three research questions.

4.1 Theme 1: contexts and features

The *geographical distribution* of papers is represented in Fig. 1. The definition of geographical regions was adapted from the methodology of the United Nations Statistics Division (UNSD, 2019), which identifies the following regions: Africa, Americas, Asia, Europe, and Oceania. To distinguish between papers from North and South America, we used the corresponding UNSD sub-regions of Northern America and Latin America and the Caribbean. In 2014–2018, three regions contributed 43 papers, representing three quarters of the total: Northern America,

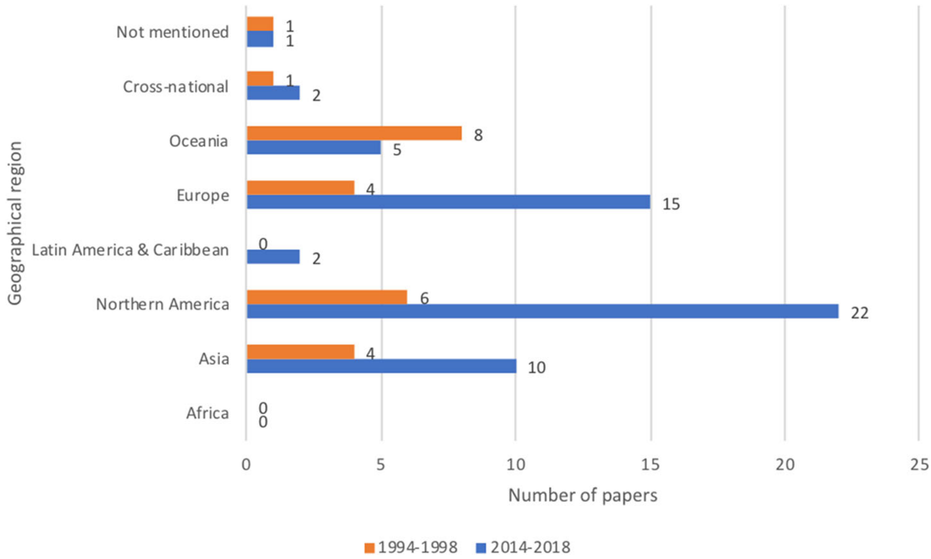

Fig. 1 Comparison of the geographical distribution of papers on mathematical thinking published in *ESM* from 1994 to 1998 and 2014 to 2018

Europe, and Asia, while in 1994–1998 Oceania, Northern America, and Europe together had contributed the same proportion (18 papers). It is difficult to draw any conclusions from this comparison, especially in light of the relatively small sample of papers from 1994–1998, but the dominance of research papers from Australia and New Zealand (the only Oceania countries that were the source of published papers) during this time is noteworthy in light of the small population of mathematics education researchers in the region.

The *research aims* of both sets of papers were classified as being either to understand or to promote mathematical thinking, with the latter type of studies typically involving an educational intervention of some kind. Results of this analysis are summarized in Fig. 2. In the period from 2014 to 2018, there was a somewhat larger number of studies seeking to understand mathematical thinking than those aiming to promote mathematical thinking. As an example of the first kind of aim, the study by Heino (2015) sought to understand how Japanese secondary school students attended to and compared multiple solutions proposed by their classmates in structured problem-solving lessons. The study reported by Mata-Pereira and da Ponte (2017) exemplifies the second kind of aim, describing principles for design research where whole class mathematical discussions were conducted in order to enhance primary school students' mathematical reasoning processes. The profile of *research aims* in the earlier period from 1994 to 1998 was quite different from that in 2014 to 2018, with the vast majority of studies seeking to understand mathematical thinking. Typical of this research emphasis is the study by Chinnappan (1998), which used schema theory to examine the organizational quality of secondary school students' geometric knowledge and use of that knowledge during problem solving. It may be that in earlier studies researchers were more concerned with developing an understanding of mathematical thinking before designing studies aiming to promote problem solving or proving, for example, as specific forms of mathematical thinking.

The *research focus* of the papers, classified as various combinations of problem solving, problem posing, reasoning, and proof, is shown in Fig. 3.

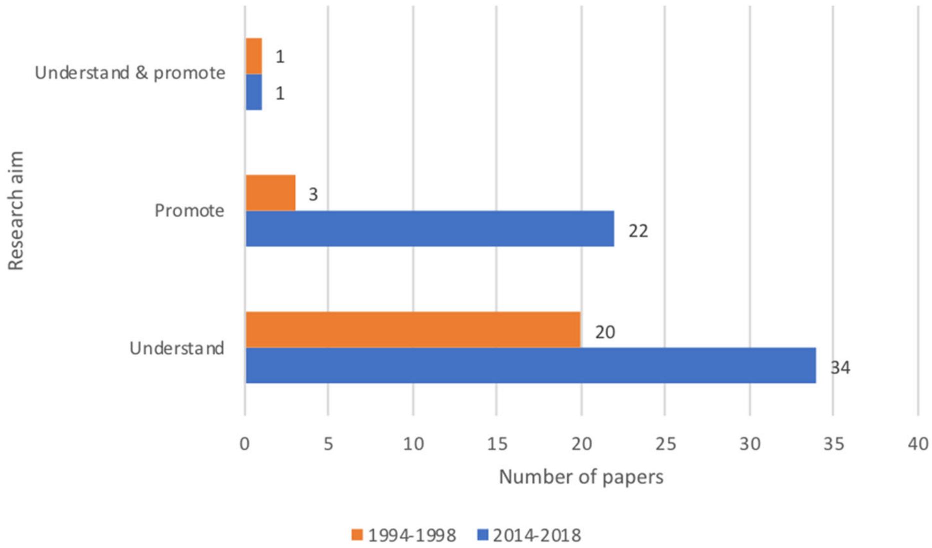

Fig. 2 Distribution of the research aims of papers on mathematical thinking published in *ESM* from 1994 to 1998 and 2014 to 2018

Research on reasoning and proof comprised the focus for more than three-quarters of the mathematical thinking papers in the 5-year period from 2014 to 2018, while during the period from 1994 to 1998 more than half the papers were concerned with problem solving/posing. In both sets of papers, research on mathematical problem solving (and sometimes problem posing) was spread across studies in which primary school students, secondary school students, university undergraduate students and pre-service teachers were the participants. An example of a study involving pre-service teachers is that of Xie and Masingila (2017), who examined mutual effects and supports between problem solving and problem posing and how these interactions supported pre-service primary teachers' conceptual understanding of fractions.

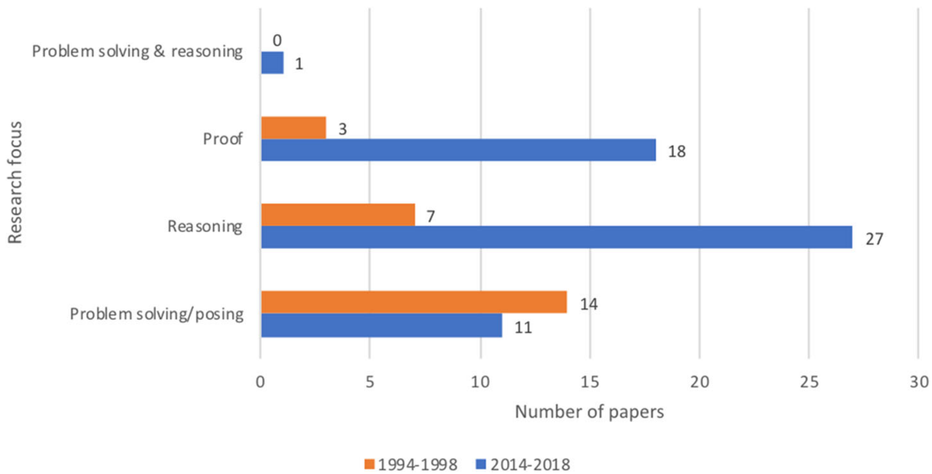

Fig. 3 Distribution of the research focus of papers on mathematical thinking published in *ESM* from 1994 to 1998 and 2014 to 2018

A different pattern with respect to research participation was observed in the studies that had reasoning and proof as their focus. For papers published from 2014 to 2018, those classified as focusing on proof mainly involved secondary school students, university undergraduate students and lecturers, while primary and secondary school students were the most frequent participants in studies of mathematical reasoning. This finding is consistent with research themes identified in recent handbooks and curriculum frameworks, which give equal attention to informal and formal reasoning processes, and teaching approaches that develop children's reasoning capabilities. For example, at the primary school level, Downton and Sullivan (2017) conducted a study with 8- and 9-year-old children working on tasks that prompted multiplicative thinking. In contrast to this study, formal proofs were the subject of the research reported by Ramos and Weber (2014), who investigated how and why mathematicians read proofs. For papers published from 1994 to 1998, the small number that investigated mathematical proof processes focused only on university students and lecturers, while the much more numerous problem-solving/posing studies displayed a similar pattern of participants to that found in the more recent time period. Thus, over the past 25 years there has been an expansion of interest in studies of mathematical reasoning among younger students and of mathematical proof at the secondary and tertiary educational levels.

We defined the *research orientation* of the papers in terms of whether mathematical thinking was the object of study or used as a research tool to investigate the learning or teaching of mathematical concepts (English & Gainsburg, 2016; Weber & Leikin, 2016). Figure 4 summarizes the distribution of papers in both samples. Papers published from 2014 to 2018 were fairly evenly divided between these orientations. Among the former category, there was more of a research focus on proof (17 out of 31 papers), with somewhat less attention given to problem solving (8 papers) and reasoning (6 papers) as objects of study. There were several theoretical papers in this group: for example, Simpson (2015) analyzed a model solution to a proof question using Toulmin's scheme of argumentation in order to provide insight into what examiners might be expecting of students. In the latter category of papers,

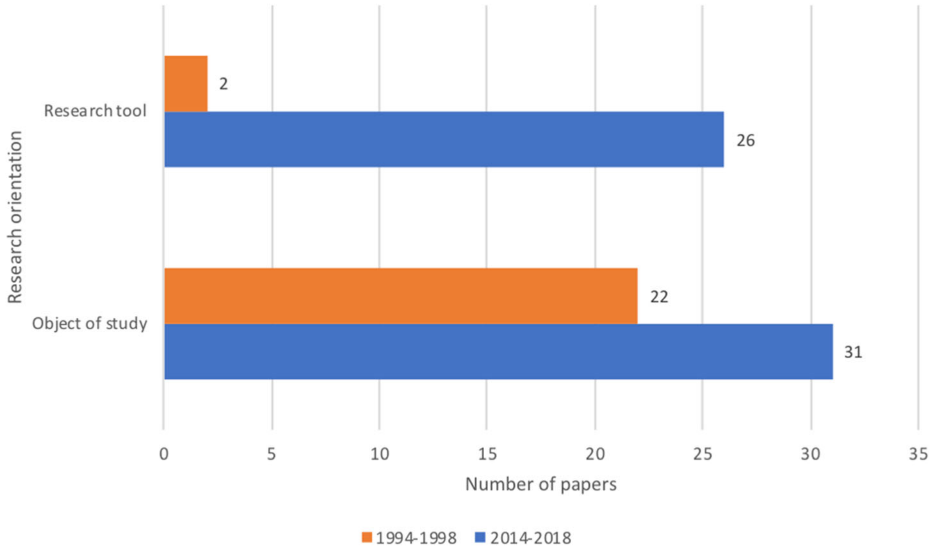

Fig. 4 Distribution of the research orientation of papers on mathematical thinking published in *ESM* from 1994 to 1998 and 2014 to 2018

reasoning was overwhelmingly the focus (21 out of 26 papers): that is, reasoning tasks were used to investigate students' learning of mathematical concepts in different areas of the curriculum, such as statistics, algebra, number, and geometry. The large-scale professional development project conducted by Hilton, Hilton, Dole, and Goos (2016) exemplifies such studies. These researchers worked with middle school teachers to devise strategies for improving students' proportional reasoning abilities and demonstrated the effectiveness of their approach through analysis of student pre- and posttest data.

The *research orientation* of papers published from 1994 to 1998 displayed a distinctly different pattern from that observed in the more recent time period, with almost all papers treating mathematical thinking as the object of study, and largely with a focus on problem solving (13 out of 22 papers). The study reported by Campbell, Collis, and Watson (1995) is illustrative of this approach, investigating secondary school students' use of visual schemata and logical operational ability in problem solving.

Analysis of the *educational level* targeted by the sample of empirical research papers revealed a slight change over time in the proportions of research participants who were studying either at school or at university. Figure 5 illustrated this comparison. In the earlier time period, from 1994 to 1998, two thirds (16 out of 21 empirical papers) of the identified studies involved participants who were primary or secondary school students. By comparison, only a little over half (33 out of 51 empirical papers) of the papers published from 2014 to 2018 reported on studies conducted with school children; there was a corresponding increase in the proportion of studies targeting undergraduate university students during this later time period. The increasing focus on research into proof over the past 20 years may explain this shift in educational levels (see Fig. 3).

We analyzed the *scale* of the research sample by recording the duration of the empirical studies and the number of participants in each study. In the 2014–2018 set of papers, it was surprising to find that only two studies lasted longer than a year when a large proportion of the studies were aiming to promote mathematical thinking (see Fig. 2). This finding suggested that

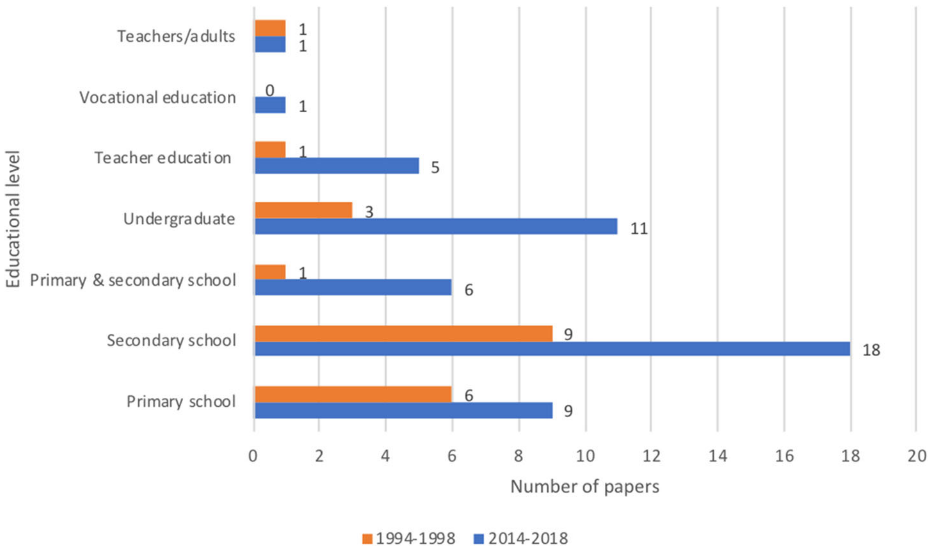

Fig. 5 Comparison of the educational level in empirical studies of mathematical thinking published in *ESM* from 1994 to 1998 and 2014 to 2018

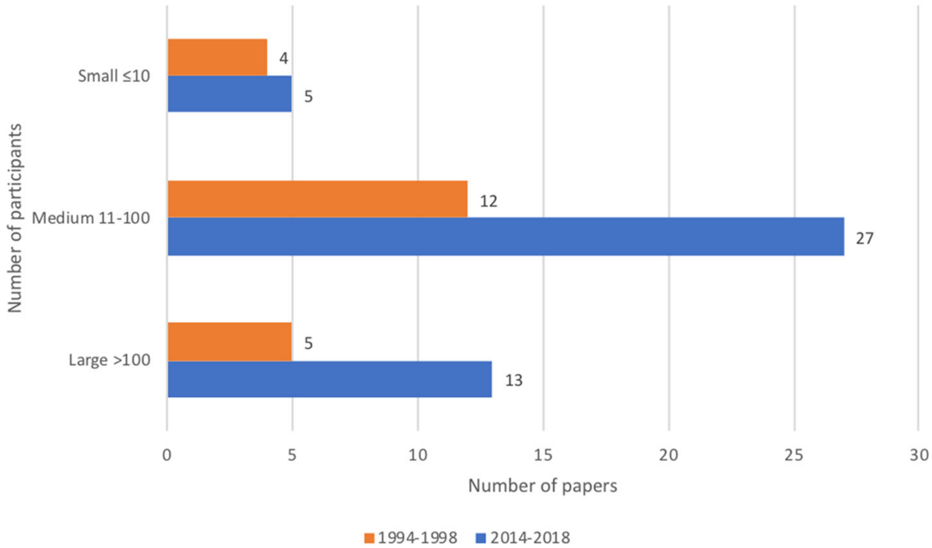

Fig. 6 Comparison of the number of participants in empirical studies of mathematical thinking published in *ESM* from 1994 to 1998 and 2014 to 2018

short-term interventions were a common approach and raised questions about the enduring impact of such research. Similarly, only four of the papers published from 1994 to 1998 reported on studies that lasted more than 1 year; however, the majority of these studies aimed to understand rather than promote mathematical thinking, often using one-off data collection methods such as questionnaires or clinical interviews. The number of participants per study varied widely: we classified these as being small-scale studies if there were fewer than 10 participants, medium-scale studies if there were 11–100 participants, and large-scale studies if there were more than 100 participants. However, in both time periods, the distribution of participant numbers was similar, with just under half the papers reporting on medium-scale studies (12 out of 21 papers in 1994–1998 and 27 out of 45 papers in 2014–2018; see Fig. 6).

4.2 Theme 2: theories and methodologies

One of the criticisms of research in this area is the lack of a strong theoretical base (Lesh & Zawojewski, 2007). In recent years, the proliferation of theories in mathematics education in general has also been identified as both a challenge and a rich resource for our research community. For example, Prediger, Bikner-Ahsbahs, and Arzarello (2008) argued that a variety of theoretical approaches is needed in order to do justice to the complexity of mathematics learning and teaching. However, they also claimed that there is not yet any agreement among mathematics education researchers about how to define what a theory is. Instead, they suggested that researchers drew on an eclectic mix of broad research paradigms and comprehensive theories as well as local conceptual tools in order to investigate phenomema of interest. This rather pragmatic eclecticism was evident in the 57 papers selected for analysis in the most recent time period of our review, from 2014–2018. We adapted Lerman's (2006) categorization of theories in mathematics education to identify the following broad families of theories used in this sample of papers (with example papers given in parentheses):

- Individual cognitive and constructivist perspectives: for example, learning trajectories (Blanton, Brizuela, Gardiner, Sawrey, & Newman-Owens, 2017), expert-novice thinking (Garfield, Le, Zieffler, & Ben-Zvi, 2015), Piagetian and neo-Piagetian studies (Jurdak & El Mouhayar, 2014)
- Cultural psychology perspective: for example, cognitive apprenticeship (Schukajlow, Krug, & Rakoczy, 2015), activity theory (Hitt, Saboya, Zavala, & GRUTEAM, 2017)
- Discourse perspective: for example, argumentation (Gabel & Dreyfus, 2017), semiotics (Samper, Perry, Camargo, Sáenz-Ludlow, & Molina, 2016), commognitive theory (Jeannotte & Kieran, 2017)

There were also papers that used mathematics-specific conceptual tools, such as geometric transformations (e.g., Fan, Qi, Liu, Wang, & Lin, 2017) or multiplicative reasoning (e.g., Downton & Sullivan, 2017) in order to illuminate a particular mathematical phenomenon. Some papers combined broad theories and local conceptual tools: for example, Alberracin and Gorgorio (2014) drew on several bodies of literature about problem solving, estimation, representational models, and Fermi problems to ground their study of the plans made by secondary school students for solving Fermi problems involving large numbers. Two papers, by Fiallo and Gutierrez (2017) and Johnson and McClintock (2018), showed evidence of an explicit attempt to network theories in the manner proposed by Prediger et al. (2008) in order to combine different theoretical ideas in a principled way. This networking approach shows promise of bringing some coherence to a very diverse theoretical landscape.

It is perhaps not surprising to see less theoretical diversity in the set of 24 papers on mathematical thinking published from 1994 to 1998.These papers were largely concerned with understanding problem solving as the object of study. Research conducted during this time was influenced by schema theory (e.g., Chinnappan, 1998), constructivism (e.g., English, 1997) and by the cognitive/metacognitive models of problem-solving proposed by Schoenfeld (1992) and Mason et al. (1985) (e.g., Goos & Galbraith, 1996; Stillman & Galbraith, 1998; Yusof & Tall, 1999). Studies of mathematical reasoning frequently drew on Piagetian and neo-Piagetian theories of cognitive development (e.g., Jones, Langrall, Thornton, & Mogill, 1997; Markovits & Hershkowitz, 1997; Watson & Moritz, 1998). Within this period, a number of significant theory-generating studies were published. For example, Jones et al. (1997) developed a framework for assessing probabilistic thinking, based on their synthesis of current literature and observations of young children over 2 years. Although they subsequently validated the framework through careful case studies of children completing problem tasks in clinical interview settings, they noticed "static" in children's thinking that was created by instruction and led to decreasing internal consistency in the students' assessment profiles over time. Nevertheless, the framework developed by Jones and colleagues served to illuminate children's thinking and provided information that was useful to teachers and curriculum developers.

Relative to classifying theories, it was an easier task to classify the methodologies used in the selected papers. Figure 7 provides a comparison of methodologies across the two time periods under analysis. In the 2014–2018 set of papers, the most common approach was a form of classroom intervention, design experiment, or teaching experiment that involved analysis of lesson video-recordings, interviews with teachers and students, or students' written work. A smaller number of classroom-based studies adopted an experimental or quasi-experimental design. Studies that aimed to promote mathematical thinking typically used one of the latter two methodological approaches. Other common methodological approaches, most often associated with the aim of understanding mathematical thinking, included analysis of student

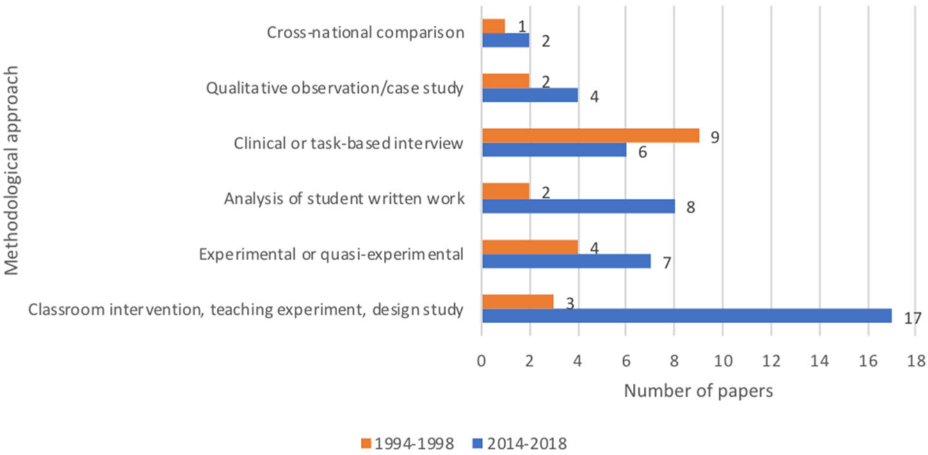

Fig. 7 Distribution of methodological approaches in papers on mathematical thinking published in *ESM* from 1994 to 1998 and 2014 to 2018

written work (without being part of a classroom intervention), clinical or task-based interviews, and qualitative case studies. Two studies were cross-national comparisons: one investigated problem-solving strategies used by Chinese and Singaporean students (Jiang, Hwang, & Cai, 2014) and the other compared the nature of proof taught in secondary school geometry classes in France and Japan (Miyakawa, 2017).

The methodological landscape in research on mathematical thinking looked different in the papers published in *ESM* from 1994 to 1998. During this time period, clinical and other task-based interviews were most commonly used to elicit students' thinking during problem solving. For example, Campbell et al. (1995) interviewed secondary school students while they worked individually on three problems that could be solved using either visual or nonvisual strategies. Goos and Galbraith (1996) instead adopted a less interventionist think-aloud strategy to elicit problem solving protocols from a pair of secondary school students as they worked on unfamiliar applied mathematics tasks.

4.3 Theme 3: contribution to knowledge

In comparing the contribution to knowledge made by the papers on mathematical thinking published in *ESM* from 1994 to 1998 and 2014 to 2018, three features stand out. First, the majority of earlier studies contributed to a better *understanding* of mathematical thinking and involved detailed analysis of individual students' *problem solving*. In these studies, mathematical thinking was typically elicited via clinical or task-based interviews, verbal problem-solving protocols, or carefully designed questionnaires, tasks, or tests that aimed to reveal or assess different levels of thinking. Second, few of the earlier studies attempted to *promote* mathematical thinking in natural classroom settings. English's (1997) research on primary school children's problem posing provides a rare example of this type of study. In this 1-year study, English implemented a problem posing program with fifth-grade children and tracked the progress of individual children who had previously displayed different patterns of achievement in number sense and solving novel problems. Interviews conducted before and after the program provided evidence that it was successful in developing the children's problem posing abilities. Third, in the earlier time period, there were significantly fewer published studies of *reasoning and proof* than we observed in the more recent window of time. An early and insightful study of this type was conducted by Selden and Selden (1995), who examined the difficulties experienced by their own first year university students in unpacking the logic of formal mathematical statements. Their grounded theory approach, drawing on their experience as university mathematics teachers, raised many interesting issues and questions about the perceived ritualistic nature of proof and how school mathematics could help students learn how to make arguments supporting their conjectures.

Turning now to the set of papers selected from the period 2014–2018, the foremost impression gained from surveying this work was of the diversity in research focus, scale, educational level, theoretical perspectives, and methodological approaches. One could view this diversity as productive, suggesting a broad interest in understanding and promoting mathematical thinking in all its guises, and for all students. Alternatively, too much diversity makes it difficult to synthesize findings, identify the most useful theories, and generate guidelines for classroom practice—adding fuel to existing criticisms concerning the lack of impact and cumulativeness of research in this field (English & Gainsburg, 2016; Lesh & Zawojewski, 2007).

Despite these concerns, it is possible to identify some theoretical, methodological, and practical contributions to knowledge emerging from the research published from 2014 to 2018 and reviewed for this paper. The first theoretical contribution comes from frameworks that organize ideas in the field in new ways in order to improve our understanding of mathematical thinking and how it can be promoted. One example of such a framework is provided in the study by Jeannotte and Kieran (2017), which developed a conceptual model of mathematical reasoning (MR) for school mathematics. These authors noted that although curricula around the world identify mathematical reasoning as an important goal of schooling, the way in which reasoning is described in these documents "tends to be vague, unsystematic, and even contradictory from one document to the other" (p. 2). Jeannotte and Kieran conceptualized mathematical reasoning as a discursive activity, using the commognitive framework of Sfard (2008) to construct:

> …a coherent theoretical model that synthesizes and builds upon the convergences to be found in the main types and characteristics of MR described in the mathematics education research literature and that can thereby serve as a conceptual tool for both teachers and researchers. (p. 4)

The resulting model highlights the dialectical relationship between structural and process aspects of mathematical reasoning, with the former aspect foregrounding deductive, inductive, and abductive modes of inference and the latter aspect identifying processes of searching for similarities and differences, validating, and exemplifying. Interestingly, Jeannotte and Kieran (2017) were clear to state that they did not create their model in order to provide practical advice on classroom tasks to encourage development of mathematical reasoning. Instead, the aim was to improve communication within the field by promoting a common discourse.

A second example of a new theoretical framework incorporates the dual aims of understanding and promoting mathematical thinking. Dawkins and Weber (2017) developed a framework for conceptualising proof in terms of mathematical values and norms. Motivated by the observation shared by many researchers that it is difficult to foster classroom proving practices, they drew on philosophical and sociocultural writings to identify epistemic values held by the community of mathematicians and discuss how norms with respect to proof and proving can work to uphold these values. The four values identified were expressed as follows:

(1) Mathematical knowledge is justified by a priori arguments.
(2) Mathematical knowledge and justifications should be a-contextual and specifically be independent of time and author.
(3) Mathematicians desire to increase their understanding of mathematics.
(4) Mathematicians desire a set of consistent proof standards. (p. 128)

Norms claimed to uphold each of these values were also identified. The authors' central argument was that "students are being asked to adopt mathematicians' proof norms, but students may not perceive the mathematicians' values that those norms are intended to uphold" (p. 133). They went on to discuss possible reasons for students' misunderstanding of, or resistance to, mathematical norms, and for mathematicians' views that students are incapable of producing proofs. They concluded that many of the challenges of proof instruction could be understood by acknowledging that teachers and students in these classrooms are engaging in cross-cultural interactions underpinned by distinct, and differing, sets of values.

An innovative methodological contribution to advancing research on mathematical thinking was found in Bruce et al. (2017). Their study used network analysis to understand current patterns of communication across the fields of Education, Mathematics, Psychology, and Neuroscience in research on spatial reasoning. The analysis identified connection gaps, that is, blockages or other limitations on communication between the disciplines, that the authors—a multidisciplinary team—suggest might be frustrating efforts to understand and promote spatial reasoning. The methodology used citation analysis of 7,200 articles to examine the extent of bibliographic coupling, where two works were considered to be coupled if they cited the same source. They argued that this coupling, or journal co-citation, was a measure of bidirectional information flow between disciplines. The results of this analysis were displayed in a visual representation of the distance-based citation network connecting the articles. The analysis pinpointed weak bidirectional information flow between Education and the other three disciplines; the authors presented case studies of research related to each discipline to illustrate some negative consequences of this lack of flow for the development of the field of mathematics education, and especially for research on spatial reasoning. They suggested that "the reasons for the connection gaps include differences in sources of research validity and outcome expectations across disciplines, unfamiliarity with bodies of research in similar or complementary constructs across disciplines, and limited awareness of research activity across relatively distant research domains" (p. 155). These authors argued that transdisciplinary research is needed to close connection gaps so that researchers can effectively address complex issues in the teaching and learning of mathematics.

We selected two papers from our sample that illustrate the potential for research on mathematical thinking to make practical contributions to knowledge and impact positively on classroom practice. Both feature innovative approaches to assessing students' reasoning or problem solving. The first study, by Hilton et al. (2016), has already been mentioned. This was a 2-year professional development study involving more than 130 Australian primary and secondary school teachers and their students, with the focus on improving students' proportional reasoning skills. An assessment approach involving development of a two-tier diagnostic instrument was used to collect baseline data on students' proportional reasoning skills at the start of the study and at the end of the first year, for students in participant (n = 1,026) and control (n = 277) classes in Years 5, 6, 7, and 8 (students 10–13 years of age). The first tier of each item required a true-false response, while the second tier required a selection from four possible reasons to justify the first tier response. The options in the second tier used research literature concerning students' proportional reasoning errors and the findings from previous studies undertaken by this research team. While there were some differences between results for the various Year levels, in general, the participant and control classes had similar pretest scores while the participant groups recorded statistically significant higher scores than the control groups for the posttest. In addition, mean posttest scores for the participant classes were consistently higher than the pretest scores of control group students at least 2 years older. This large-scale design-based study, with its multiple cycles of design, enactment and evaluation, successfully combined research with a practical purpose of promoting change in teachers' knowledge and classroom practices. As well as presenting evidence from the diagnostic assessment instrument, the paper provides a detailed account of the professional development approach. The study therefore contributes to the limited literature on the effect of teacher professional development on mathematics students' learning.

The second example of potential for practical impact from research on mathematical thinking comes from a UK study by Jones and Inglis (2015). These authors pointed out that traditional examination papers comprise mainly short, structured items that are not suited to assessing students' reasoning or problem solving. Such examinations have high reliability (via a specified marking scheme, or scoring rubric) but low validity when it comes to assessing the problem solving skills specified in the school curriculum. The authors worked with four experienced examination paper writers to produce a problem solving paper that aligned with mathematics curriculum expectations but deliberately did not have a marking scheme. The paper was administered to 750 secondary school students, whose work was assessed by 20 mathematics education professionals with teaching experience ranging from one to more than 10 years. The assessors had been trained to use comparative judgment, an alternative to traditional marking involving pairwise global judgments of the quality of students' work. The study found that the four examination paper writers, when freed from the constraints of using a marking scheme, were able to design more open-ended, less structured questions, and that the comparative judgment approach yielded assessments that were both reliable and valid. This study is significant because it demonstrated a practical and affordable approach to large-scale assessment of mathematical problem solving. In addition, Jones and Inglis speculated that comparative judgement has potential as a teaching tool if used to encourage discussion about what makes a good solution to an unstructured problem.

5 Concluding comments and future research directions

This review of research on mathematical thinking published in *Educational Studies in Mathematics* in two different time periods separated by 20 years is necessarily brief and limited by our methodological choices and our own understanding of the field. In addition to the tentative conclusions offered about the contributions to knowledge made by research in this field, we first offer some practical observations directed at authors of research papers. Our review of research literature required us to begin with a search of the titles, keywords, and abstracts of published papers to identify papers of interest. It was surprising to find that there was often little alignment, and sometimes even contradiction, between these three important elements that together communicate what a paper is about. It was also common to find little or no information in the abstract about the theoretical perspective informing the study or the methodological approach that was taken. This observation might serve to remind journal editors (and reviewers) of the importance of checking the content of abstracts of submitted manuscripts, and it should also encourage authors to ensure that they create informative titles, keywords, and abstracts so readers—and reviewers—can easily judge the relevance of papers to their own research interests.

We conclude with observations on what was *not* represented in this review of research on mathematical thinking published in *ESM*. Some gaps can be attributed to the decisions we made in designing the review methodology: in particular, studies of "applied" mathematical thinking were not captured by our review. By this, we mean studies of mathematical modeling, statistical literacy, and numeracy—all topics of papers recently published in *ESM*. Future research might usefully explore how mathematical thinking can be deployed to deal with the ill-defined problems of modern work and life (English & Gainsburg, 2016). Studying mathematical thinking in real world contexts might yield insights into the nature of critical mathematical thinking in workplace settings, the role of digital technologies in affording new problem solving and reasoning strategies, and approaches to working on interdisciplinary problems that require synthesis of knowledge across Science, Technology, Engineering, and Mathematics (STEM) domains.

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