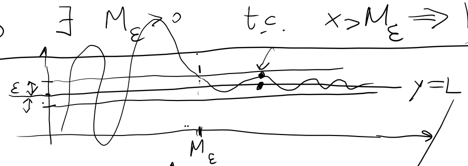
2 Ottobre

$\lim_{x\to+\infty} f(x) = +\infty$? $f: X \to \mathbb{R}$

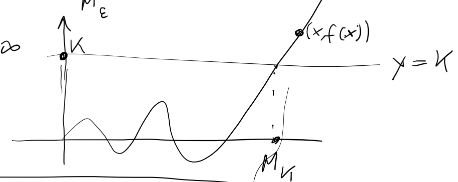
$\lim_{x\to+\infty} f(x) = -\infty$? $\sup X = +\infty$

Finora ho definito $\lim_{x\to+\infty} f(x) = L$ dove $L \in \mathbb{R}$

$$\boxed{\forall \varepsilon > 0 \quad \exists M_\varepsilon > 0 \quad t.c. \quad x > M_\varepsilon \Rightarrow |f(x) - L| < \varepsilon}$$



$\lim_{x\to+\infty} f(x) = +\infty$



$$\boxed{\begin{array}{l}\forall K > 0 \quad \exists M_K > 0 \quad t.c. \\ \quad x > M_K \Rightarrow f(x) > K\end{array}}$$

$\lim_{x\to+\infty} x = +\infty$?

Dobbiamo dimostrare che

$\forall K > 0 \quad \exists M_K > 0 \quad t.c. \quad x > M_K \Rightarrow x > K$

Basta $M_K = K$ ovviamente $x > K \Rightarrow x > K$

$\lim_{x\to+\infty} 2^x = +\infty$

$\forall K > 0 \quad \exists M_K > 0 \quad t.c. \quad x > M_K \Rightarrow 2^x > K$

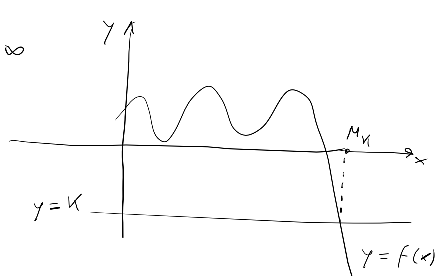
$2^x > K$ applico $\log_2$

$\Updownarrow$

$x = \log_2 2^x > \log_2 K$

$2^x > K \Leftrightarrow x > \log_2 K = M_K$

$\lim_{x\to+\infty} f(x) = -\infty$



$\forall K < 0 \quad \exists M_K > 0 \quad t.c.$

$x > M_K \Rightarrow f(x) < K$

Esercizio Dare la definizione di

$\lim_{x\to-\infty} f(x) = L \in \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$

Teor (Regola della somma) $f, g : X \longrightarrow \mathbb{R}$, $X \subseteq \mathbb{R}$ sup $X = +\infty$. Sia $\lim_{x \to +\infty} f(x) = a \in \overline{\mathbb{R}}$ e $\lim_{x \to +\infty} g(x) = b \in \overline{\mathbb{R}}$.

Allora
$$\lim_{x \to +\infty} \big(f(x) + g(x) \big) = a + b = \lim_{x \to +\infty} f(x) + \lim_{x \to +\infty} g(x)$$
salvo che nei casi in cui $a+b$ sia indefinito

cioè $(a,b) = \begin{cases} (+\infty, -\infty) \\ (-\infty, +\infty) \end{cases}$

Osservazione Nei casi indefiniti il limite ~~può~~ è in certi esempi un elemento qualsiasi di $\overline{\mathbb{R}}$ preassegnato oppure il limite potrebbe non esistere.

Dim (con $a, b \in \mathbb{R}$)

$\lim_{x \to +\infty} f(x) = a$ significa che

$$\boxed{\forall \varepsilon > 0 \ \exists\, M_\varepsilon^{(1)} > 0 \text{ t.c. } x > M_\varepsilon^{(1)} \Rightarrow |f(x) - a| < \varepsilon}$$

$\lim_{x \to +\infty} g(x) = b$

$$\boxed{\forall \varepsilon > 0 \ \exists\, M_\varepsilon^{(2)} > 0 \text{ t.c. } x > M_\varepsilon^{(2)} \Rightarrow |g(x) - b| < \varepsilon}$$

Fisso $\varepsilon > 0$

Sia $N_\varepsilon = \max\{M_\varepsilon^{(1)}, M_\varepsilon^{(2)}\}$ e sia $x \in X$ t.c. $x > N_\varepsilon \geq M_\varepsilon^{(j)}$ per $j = 1, 2$

$$|(f(x) + g(x)) - (a+b)| = |(f(x) - a) + (g(x) - b)| \leq$$
$$\leq \underbrace{|f(x) - a|}_{< \varepsilon} + \underbrace{|g(x) - b|}_{< \varepsilon} < 2\varepsilon$$

da $x > M_\varepsilon^{(1)}$ da $x > M_\varepsilon^{(2)}$

Abbiamo dimostrato che $\forall \varepsilon > 0 \ \exists\, N_\varepsilon > 0$ t.c. $x > N_\varepsilon \Rightarrow |(f(x) + g(x)) - (a+b)| < 2\varepsilon$ (1_ε)

E' vero allora che
$\forall \varepsilon > 0 \ \exists\, M_\varepsilon > 0$ t.c. $x > M_\varepsilon$
$\Rightarrow |(f(x) + g(x)) - (a+b)| < \varepsilon$?

Sì per $\boxed{M_\varepsilon = N_{\frac{\varepsilon}{2}}}$!

Infatti $\varepsilon > 0 \Rightarrow \frac{\varepsilon}{2} > 0$. Se scelgo $N_{\frac{\varepsilon}{2}}$ dalla (1_ε) ricavo

$$x > N_{\frac{\varepsilon}{2}} \Rightarrow |(f(x) + g(x)) - (a+b)| < \cancel{2} \frac{\varepsilon}{\cancel{2}} = \varepsilon$$

<u>Teor</u> (Regola del prodotto) $f, g : X \to \mathbb{R}$, $X \subseteq \mathbb{R}$

$\sup X = +\infty$, $\lim_{x \to +\infty} f(x) = a \in \overline{\mathbb{R}}$, $\lim_{x \to +\infty} g(x) = b \in \overline{\mathbb{R}}$

Allora

$$\lim_{x \to +\infty} f(x)\, g(x) = a\, b = \left(\lim_{x \to +\infty} f(x)\right)\left(\lim_{x \to +\infty} g(x)\right)$$

salvo nei casi in cui il prodotto non è definito.

$$(a,b) = \begin{cases} (\pm\infty, 0) \\ (0, \pm\infty) \end{cases}.$$

<u>Osservazione</u> Nel caso $a = +\infty$, $b = 0$ per ogni prefissato $L \in \overline{\mathbb{R}}$ esistono esempi t.c. $\lim_{x \to +\infty} f(x) g(x) = L$

e vi sono anche esempi in cui $\lim_{x \to +\infty} f(x)\, g(x)$ non esiste.

<u>Es</u> $\lim_{x \to +\infty} (x^3 - x^2)$ $\qquad +\infty - (+\infty)$ non è definito

$\lim_{x \to +\infty} x^3 = (+\infty)^3 = +\infty$ $\qquad \lim_{x \to +\infty} \frac{1}{x} = \frac{1}{+\infty} = 0$

$\lim_{x \to +\infty} x^2 = (+\infty)^2 = +\infty$

$$x^3 - x^2 = x^3 \left(\frac{x^3}{x^3} - \frac{x^2}{x^3}\right) = \underset{\downarrow\, +\infty}{x^3} \underbrace{\left(1 - \frac{1}{x}\right)}_{\xrightarrow{x \to +\infty} 1} \xrightarrow{x \to +\infty} +\infty \qquad \frac{1}{x} \xrightarrow{x \to +\infty} 0$$

$\lim_{x \to +\infty} (x^3 - x^2) = \lim_{x \to +\infty} x^3 = +\infty$

$\lim_{x \to +\infty} (x^3 + x^2) = +\infty + (+\infty) = +\infty$

$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

$a_n \neq 0$ $\qquad \deg p = n$

degree

$p(x) =$

$= a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 =$

$\lim_{x \to +\infty} (x^3 + x^2 - x) = +\infty + \cancel{+\infty - \infty}$

$$= a_n x^n \left(\frac{a_n x^n}{a_n x^n} + \frac{a_{n-1} x^{n-1}}{a_n x^n} + \dots + \frac{a_1 x}{a_n x^n} + \frac{a_0}{a_n x^n} \right)$$

$x^n = x\, x^{n-1}$

$$= a_n \underset{\downarrow\, +\infty}{x^n} \left(1 + \underset{\downarrow\, 0}{\frac{a_{n-1}}{a_n x}} + \dots + \underset{\downarrow\, 0}{\frac{a_1}{a_n x^{n-1}}} + \underset{\downarrow\, 0}{\frac{a_0}{a_n x^n}} \right)$$

$$\xrightarrow{x \to +\infty} a_n (+\infty) = \begin{cases} +\infty & \text{se } a_n > 0 \\ -\infty & \text{se } a_n < 0 \end{cases}$$

Teor (regola del Quoziente) $f, g : X \to \mathbb{R}$ sup $X = +\infty$

$\lim_{x \to +\infty} f(x) = a \in \overline{\mathbb{R}}$, $\lim_{x \to +\infty} g(x) = b \in \overline{\mathbb{R}}$.

Sia $\boxed{g(x) \neq 0}$ $\forall x \in X$, Allora

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \frac{a}{b}$$

salvo quando $\boxed{b = 0 \text{ e } (a,b) = (\pm\infty, \pm\infty)}$

Es $\lim_{x \to +\infty} \frac{1}{x^3} = 0 = \frac{1}{+\infty}$

$$p(x) = a_n x^n + \boxed{a_{n-1} x^{n-1} + \dots + a_1 x + a_0}$$

$$\lim_{x \to +\infty} p(x) = \lim_{x \to +\infty} a_n x^n$$

$$\lim_{x \to +\infty} \frac{-x^3 + x^2 + x}{2x^7 + x + 1} \qquad \frac{-\infty}{+\infty}$$

$$\frac{-x^3 + x^2 + x}{2x^7 + x + 1} = \frac{-x^3 \left(1 - \frac{1}{x} - \frac{1}{x^2}\right)}{2x^7 \left(1 + \frac{1}{2x^6} + \frac{1}{2x^7}\right)}$$

$$= \frac{-x^3}{2x^7} \cdot \frac{1 - \frac{1}{x} - \frac{1}{x^2}}{1 + \frac{1}{2x^6} + \frac{1}{2x^7}} \xrightarrow[x \to +\infty]{} 1$$

$$\frac{1}{x} \to 0, \quad \frac{1}{x^2} \to 0, \quad \frac{1}{2x^6} \to 0, \quad \frac{1}{2x^7} \to 0$$

$$-\frac{1}{2x^4} \xrightarrow{x \to +\infty} 0$$

$$\lim_{x \to +\infty} \frac{a_n x^n + \dots + a_0}{b_m x^m + \dots + b_0} = \lim_{x \to +\infty} \frac{a_n x^n}{b_m x^m}$$

$X \subseteq \mathbb{R}$ card $X < +\infty$

Allora esiste $\max X$

Lo dimostriamo per induzione sulla cardinalità

1) ~~Sia~~ card $X = 1$.

Se card $X = 1$ allora esiste $x_1 \in \mathbb{R}$

t.c. $X = \{x_1\}$ singleton

$\sup\{x_1\} = x_1 = \inf\{x_1\}$

$\cap$

$X \Rightarrow \max X = x_1$

2) Supponiamo di sapere che se card $X = n$ allora $\max X$ esiste.

Consideriamo un X con card $X = n+1$

Scegliamo un elemento di $X = \{x_1, x_2, \dots x_n, x_{n+1}\}$

$x_{n+1} \in X$

~~ed~~ e poniamo $Y = \{x_1, \dots, x_n\}$

$X = \{x_{n+1}\} \cup Y$

Siccome card $Y = n$ $\exists$ $\max Y$

$\exists$ $x_{j_0} \in Y$ $1 \leq j_0 \leq n$ t.c.

$x_{j_0} \geq x_j$ $\forall$ $1 \leq j \leq n$

$\max\{x_{n+1}, x_{j_0}\}$