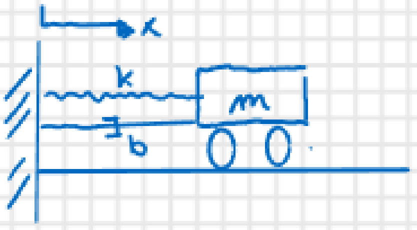
Titolo:

Oscillatore smorzato



- $m\ddot{x}(t) + b\dot{x}(t) + kx(t) = 0$
- $\ddot{x}(t) + \frac{b}{m}\dot{x}(t) + \frac{k}{m}x(t) = 0$

- $\ddot{x}(t) + 2\nu\omega\dot{x}(t) + \omega^2 x(t) = 0$ $\omega = \sqrt{\frac{k}{m}}$ pulsazione propria

$D = \frac{1}{2} b \dot{x}(t)^2$ $\nu = \frac{b}{2\sqrt{km}}$ fattore di smorzamento

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} + \frac{\partial U}{\partial x} + \frac{\partial T}{\partial q} + \frac{\partial D}{\partial \dot{q}} = 0$$

- $m\ddot{x}(t) + kx(t) + 0 + b\dot{x}(t) = 0$

Soluzione per l'equazione del moto

- $x(t) = \lambda_1 e^{\alpha_1 t} + \lambda_2 e^{\alpha_2 t}$ α_1 e α_2

$x(t) = e^{\alpha t}$ $\dot{x}(t) = \alpha e^{\alpha t}$ $\ddot{x}(t) = \alpha^2 e^{\alpha t}$

Sostituiamo nell'equazione del moto

$$\alpha^2 \cancel{e^{\alpha t}} + 2\nu\omega\alpha \cancel{e^{\alpha t}} + \omega^2 \cancel{e^{\alpha t}} = 0$$

$\alpha^2 + 2\nu\omega\alpha + \omega^2 = 0$ eq 2°grado in α

$$\alpha_{1,2} = -\nu\omega \pm \omega\sqrt{-1+\nu^2}$$

$\nu = 1$ Smorzamento critico

- $\nu < 1$ (Civile $\nu \simeq 3\% - 7\%$)

$$\alpha_{1/2} = -\nu\omega \pm \omega i \sqrt{1-\nu^2} = -\nu\omega \pm i\,\Omega$$

$$\Omega = \omega\sqrt{1-\nu^2} \qquad \text{libero } \Omega = \omega$$

$$x(t) = \lambda_1 e^{(-\nu\omega + \Omega i)t} + \lambda_2 e^{(-\nu\omega - \Omega i)t} = e^{-\nu\omega t}\left(\lambda_1 e^{\Omega i t} + \lambda_2 e^{-\Omega i t}\right)$$

$t \to \infty \quad x(t) \to 0$

$$x(t) = e^{-\nu\omega t} \cdot \left[\lambda_1(\cos\Omega t + i\,\text{sen}\,\Omega t) + \lambda_2(\cos\Omega t - i\,\text{sen}\,\Omega t)\right] =$$

$$= e^{-\nu\omega t}\left[\underbrace{(\lambda_1 + \lambda_2)}_{B}\cos\Omega t + \underbrace{(\lambda_1 i - \lambda_2 i)}_{A}\,\text{sen}\,\Omega t\right]$$

$$x(t) = e^{-\nu\omega t}\left[A\,\text{sen}\,\Omega t + B\cos\Omega t\right]$$

$$x(t) = e^{-\nu\omega t} X\cos(\Omega t - \varphi)$$

- Condizioni al iniziali

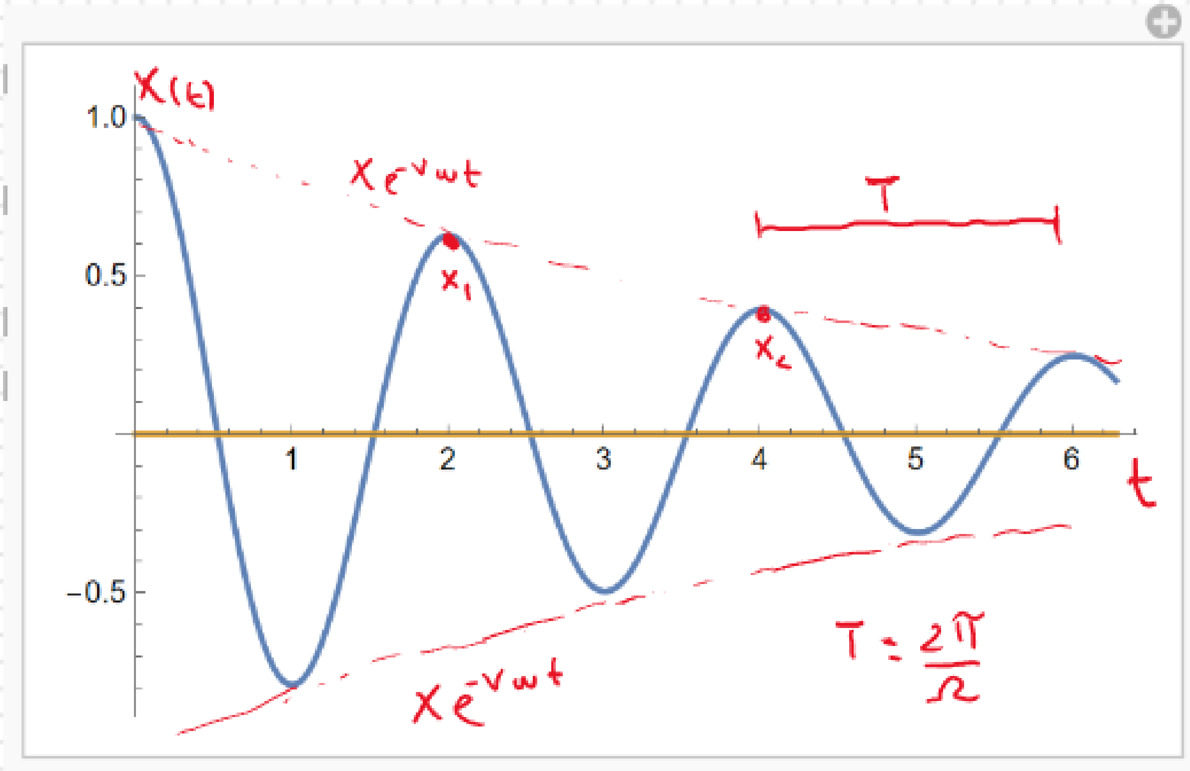
1) $X(0) = x_0 \qquad X(0) = B = x_0 \qquad B = x_0$

2) $\dot{x}(t) = -\nu\omega e^{-\nu\omega t}(A\,\text{sen}\,\Omega t + B\cos\Omega t) + e^{-\nu\omega t}(\Omega A\cos\Omega t - B\Omega\,\text{sen}\,\Omega t)$

$$\dot{x}(0) = -\nu\omega B + \Omega A = v_0 \longrightarrow A = \frac{v_0 + \nu\omega x_0}{\Omega}$$

Titolo:

$$x(t) = e^{-\nu\omega t}\left(x_0 \cos\Omega t + \frac{v_0 + \nu\omega x_0}{\Omega} \operatorname{sen}\Omega t\right)$$ soluzio- $\nu < 1$



$x_1 = x(t_1)$ coincide con un max

$$x_2 = x(t_1 + T) = e^{-\nu\omega(t_1+T)}\left(A \sin(\Omega(t_1+T)) + B\cos(\Omega(t_1+T))\right)$$

$$x_2 = e^{-\nu\omega T} \underbrace{e^{-\nu\omega t_1}(\ldots\ldots\ldots)}_{x_1} = \underline{e^{-\nu\omega T}} x_1$$

$\vdots$

$$x_n = e^{-\nu\omega T} x_{n-1} \qquad \frac{x_{n-1}}{x_n} = e^{\nu\omega T} = e^{\delta} \longrightarrow \delta = \log\frac{x_{n-1}}{x_n}$$

$$\delta = \frac{\nu \cancel{\omega} 2\pi}{\omega\sqrt{1-\nu^2}} \longrightarrow \nu \qquad \delta \simeq 2\pi\nu \quad (\nu \text{ sono piccoli})$$

$$\nu = \frac{\delta}{2\pi}$$

$t_1 \longrightarrow t_2$ · Considerazioni energetiche

$$\Delta U_1 = \frac{1}{2} k x_1^2 - \frac{1}{2} k x_2^2 = \frac{1}{2} k x_1^2 - \frac{1}{2} k \left(x_1 e^{-\nu\omega T}\right)^2 =$$

$$= \frac{1}{2} k x_1^2 \left(1 - e^{-2\nu\omega T}\right)$$

$t_2 \to t_3$ " $\longrightarrow \Delta U_1 > \Delta U_2$ perché $x_1^2 > x_2^2$

$$\Delta U_2 = \frac{1}{2} k x_2^2 \left(1 - e^{-2\nu\omega T}\right)$$

$\nu = 1 \longrightarrow b = 2\sqrt{km}$

$\alpha_{1,2} = -\nu\omega \pm \omega\sqrt{\nu^2 - 1} \longrightarrow \alpha_{1/2} = -\omega$ Reali e coincidenti

$$x(t) = \lambda_2 e^{\alpha t} + \lambda_1 t e^{\alpha t} = e^{-\omega t}\left(\lambda_1 t + \lambda_2\right)$$

$$\dot{x}(t) = -\omega e^{-\omega t}\left(\lambda_1 t + \lambda_2\right) + e^{-\omega t}\lambda_1$$

Condizioni iniziali

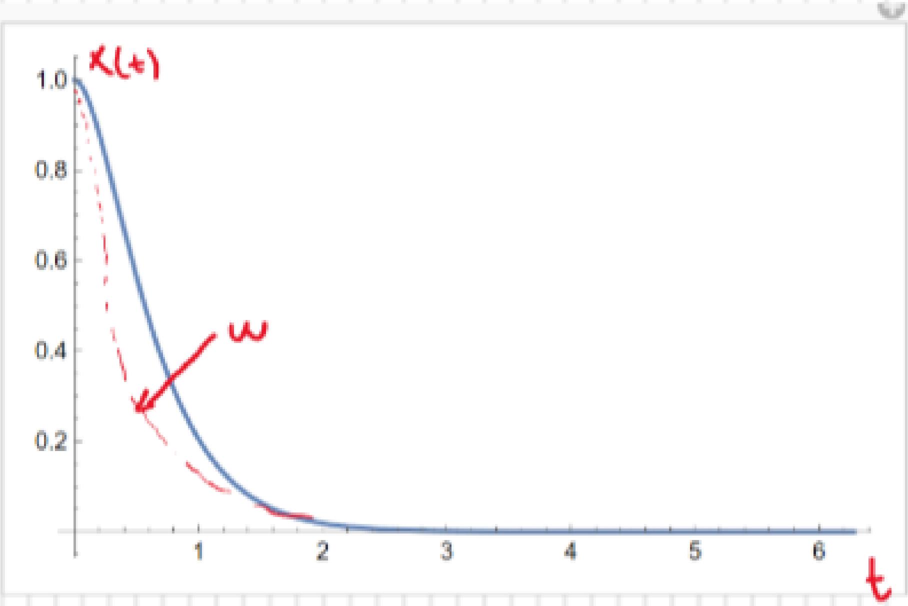
$x(0) = x_0 \qquad x_0 = \lambda_2$

$\dot{x}(0) = -\omega\lambda_2 + \lambda_1 = v_0 \qquad \lambda_1 = v_0 + \omega x_0$

$$x(t) = e^{-\omega t}\left(\left(v_0 + \omega x_0\right)t + x_0\right)$$

· $\nu = 4$

$$x(t) = e^{-\omega t}\left((v_0 + \omega x_0)t + x_0\right) \quad \bullet \; \nu = 1$$



$\nu > 1$

$$\alpha_{1/2} = -\nu\omega \pm \omega\sqrt{\nu^2 - 1} = -\nu\omega \pm \Omega$$

$$x(t) = \lambda_1 e^{(-\nu\omega + \Omega)t} + \lambda_2 e^{(-\nu\omega - \Omega)t} = \underline{e^{-\nu\omega t}}\left(\lambda_1 e^{\Omega t} + \lambda_2 e^{-\Omega t}\right)$$

Non oscillatorio

$$e^{i\alpha} = \cos\alpha + i \operatorname{sen}\alpha$$

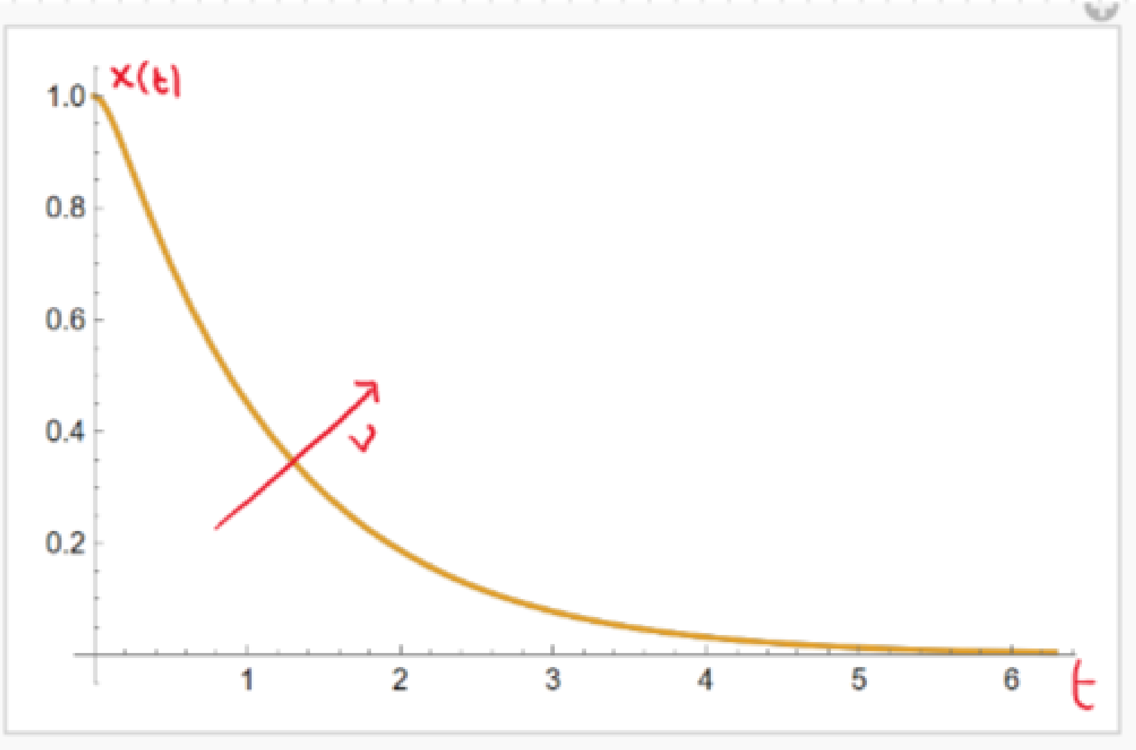
Condizioni iniziali

- $x(0) = \lambda_1 + \lambda_2 = x_0$

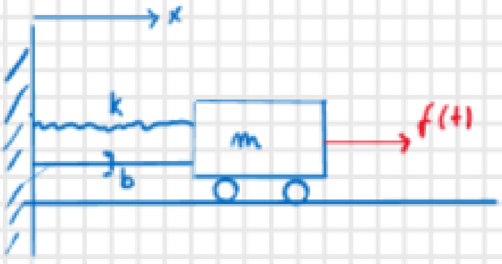
$$\dot{x}(t) = -\nu\omega e^{-\nu\omega t}\left(\lambda_1 e^{\Omega t} + \lambda_2 e^{-\Omega t}\right) + e^{-\nu\omega t}\left(\Omega\lambda_1 e^{\Omega t} + \Omega\lambda_2 e^{-\Omega t}\right)$$

- $\dot{x}(0) = -\nu\omega(\lambda_1 + \lambda_2) + \Omega\lambda_1 + \Omega\lambda_2 = 0$

$$\lambda_1 = \frac{v_0 + \nu\omega x_0 + \Omega x_0}{2\Omega} \qquad \lambda_2 = \frac{\Omega x_0 - v_0 - \nu\omega x_0}{2\Omega}$$



Oscillatore forzato in presenza di smorzamento



$f(t) = F \operatorname{sen} \bar{\omega} t$

ω

Ω

$\bar{\omega}$

- $m\ddot{x} + b\dot{x} + kx = f(t) = F \operatorname{sen} \bar{\omega} t$ $\qquad x_{st} = \dfrac{F}{k}$
- $\underbrace{\ddot{x}(t) + 2\nu\omega\dot{x}(t) + \omega^2(t)}$ $= x_{st}\,\omega^2 \operatorname{sen} \bar{\omega} t$

Soluzione eq. moto

$$x_{tot}(t) = x_{om}(t) + x_p(t)$$

$\downarrow$

- soluzione del sistema senza forzante $f(t) = 0$

- $x_p(t) = C \operatorname{sen}(\bar{\omega} t - \varphi)$
- $\dot{x}_p(t) = \bar{\omega} C \cos(\bar{\omega} t - \varphi)$
- $\ddot{x}_p(t) = -\bar{\omega}^2 C \operatorname{sen}(\bar{\omega} t - \varphi)$

Sostituiamo nell'equazione del moto

$$-\bar{\omega}^2 C \operatorname{sen}(\bar{\omega} t - \varphi) + 2\nu\omega\bar{\omega} C \cos(\bar{\omega} t - \varphi) + \omega^2 C \operatorname{sen}(\bar{\omega} t - \varphi) = x_{st}\,\omega^2 \operatorname{sen} \bar{\omega} t$$

Ricavare C e φ

$t = 0$

$$+\bar{\omega}^2 C \operatorname{sen}\varphi + 2\nu\omega\bar{\omega} C \cos\varphi - \omega^2 C \operatorname{sen}\varphi = 0 \qquad 1)$$

$\bar{\omega} t - \varphi = 0 \longrightarrow t = \varphi/\bar{\omega}$

$$2\nu\omega\bar{\omega} C = x_{st}\,\omega^2 \operatorname{sen}\varphi \qquad 2)$$

Dalla 1 $\quad \frac{\sin\varphi}{\cos\varphi} = \frac{2\nu\omega\bar{\omega}}{\omega^2-\bar{\omega}^2} : tg\varphi = \frac{2\nu\beta}{1-\beta^2} \qquad \beta = \frac{\bar{\omega}}{\omega}$

Dalla 2 $\quad C = \frac{x_{st}\,\omega^2}{2\nu\omega\bar{\omega}} \operatorname{sen}\varphi \qquad \operatorname{sen}\varphi = \frac{tg\,\varphi}{\sqrt{1+tg\varphi^2}}$

$$C = x_{st} \cdot \underbrace{\frac{1}{\sqrt{(1-\beta^2)^2 + 4\nu^2\beta^2}}}_{N} = x_{st} \cdot N \qquad \underline{\varphi = \operatorname{arctg}}\left(\frac{2\nu\beta}{1-\beta^2}\right)$$

$$x_p(t) = x_{st} \cdot N \cdot \operatorname{sen}(\bar{\omega}t - \varphi)$$

$\nu < 1$ Ci concentriamo su questo caso

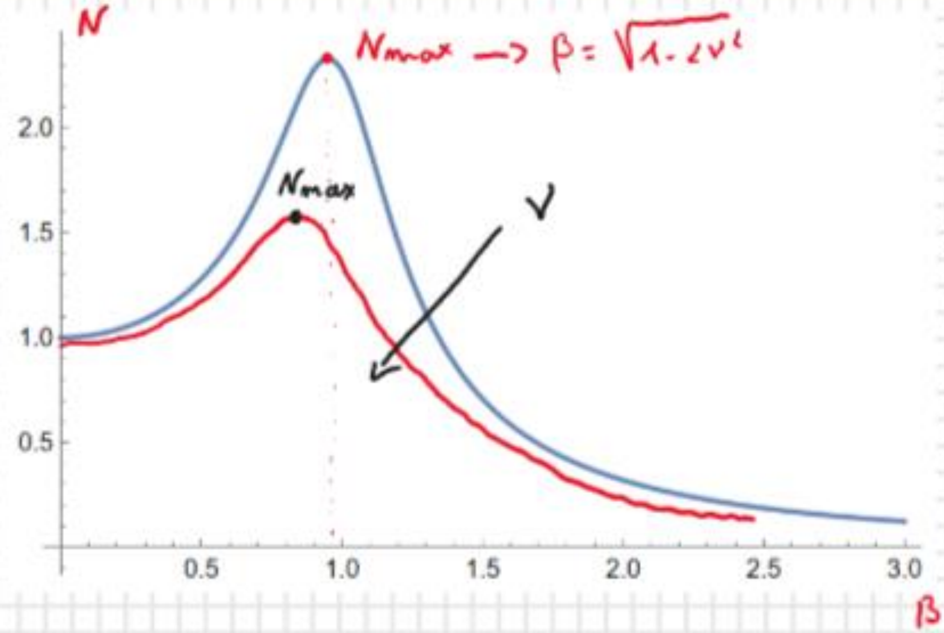
$$x(t) = e^{-\nu\omega t}\left(A \operatorname{sen} \Omega t + B \cos \Omega t\right) + x_{st} N \operatorname{sen}(\bar{\omega}t - \varphi)$$

$$x_{tot}(t) = \qquad x_{om}(t) \qquad + \quad x_p(t)$$

$$N = \frac{1}{\sqrt{(1-\beta^2)^2 + 4\nu^2\beta^2}} \qquad \frac{\partial N(\beta)}{\partial \beta} = 0 \qquad \beta^2 = 1 - 2\nu^2$$

$$\beta = \sqrt{1-2\nu^2} \longrightarrow N_{max} \qquad N_{max} = \frac{1}{2\nu\sqrt{1-2\nu^2}}$$

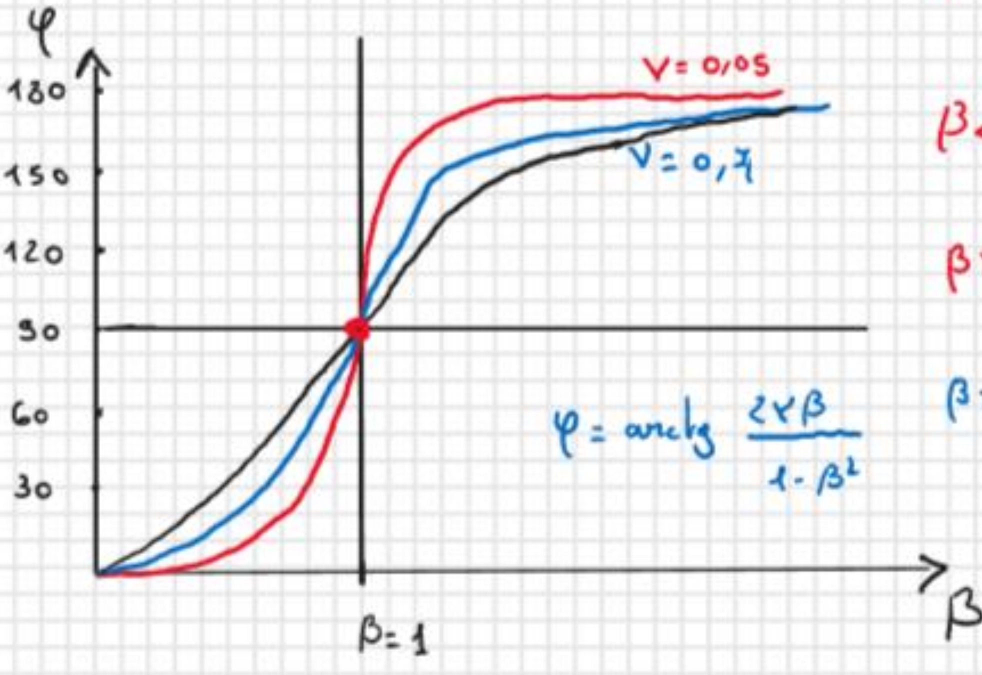
$$\nu \ll 1 \qquad N_{max} \simeq \frac{1}{2\nu}$$



$\beta = \frac{\bar{\omega}}{\omega}$

$\beta \gg 1$

$N \to 0$

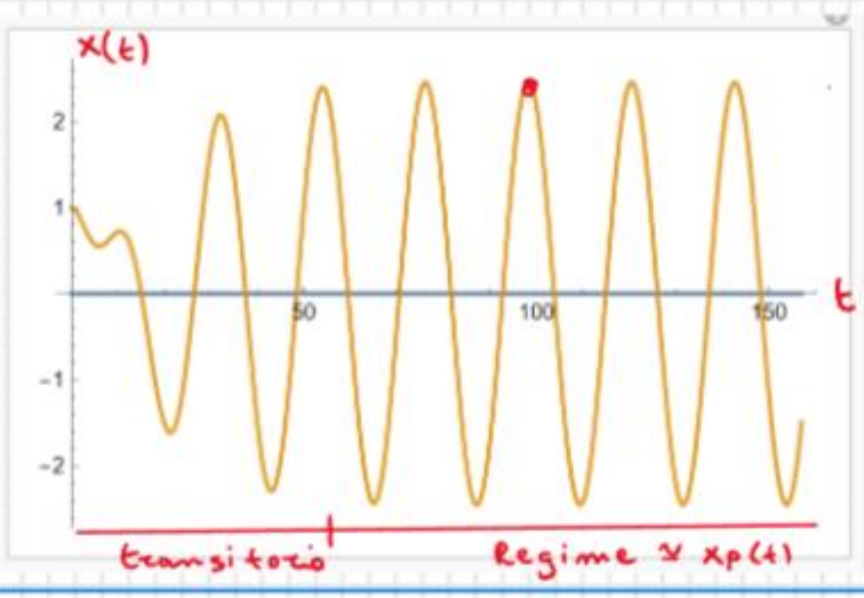


$\beta < 1 \quad \varphi < \frac{\pi}{2}$

$\beta > 1 \quad \varphi > \frac{\pi}{2}$

$\beta \gg 1 \quad \varphi = \pi$

$\beta = 1$
$\varphi = \frac{\pi}{2}$

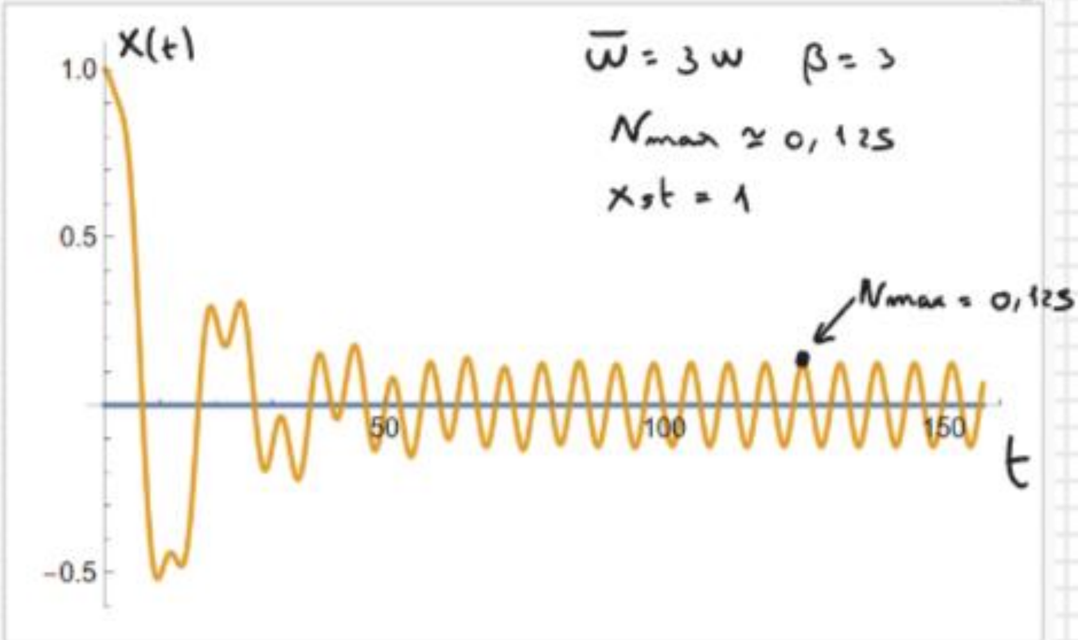


$\beta = 0{,}9 = \frac{\bar{\omega}}{\omega}$

$X_{st} = 1$

$\nu = 0{,}2$

$N_{max} \simeq 2{,}46$



X(t)
1.0
0.5
-0.5
50
100
150
t
$\bar{\omega} = 3\omega$ $\beta = 3$
$N_{max} \simeq 0,125$
$x_{st} = 1$
$N_{max} = 0,125$