27 Sept

Norms

<u>Def</u> X a vector space on $K = \mathbb{R}, \mathbb{C}$

$\|\cdot\| : X \longrightarrow [0, +\infty)$

1) $\|x\| = 0 \iff x = 0$

2) $\|x+y\| \leq \|x\| + \|y\| \quad \forall x, y \in X$

3) $\|\lambda x\| = |\lambda| \, \|x\| \quad \forall \lambda \in K, \ x \in X$

$d(x,y) = \|x - y\|$

The metric induces a on X a structure of TVS.

$(X, \|\cdot\|)$ normed space

If it is complete it is a Banach (B) space.

Example (Ω, μ)

$X = L^p(\Omega, d\mu)$ $\quad p \geq 1$

$\mu(\Omega) > 0$

Then $f \in L^p(\Omega, d\mu)$ $\quad 1 \leq p < +\infty$

$$\|f\|_{L^p(\Omega, d\mu)} = \left(\int_\Omega |f(x)|^p \, d\mu \right)^{\frac{1}{p}}$$

$$\|f\|_{L^\infty(\Omega, d\mu)} := \sup \{ c \geq 0 : \mu\{ x : |f(x)| \geq c \} > 0 \}$$

Ex $\quad$ if $f = 0$ for $c > 0$

$$\mu\{ \underbrace{x : |f(x)| \geq c}_{\emptyset} \} = 0$$

If $c = 0$

$$\{ x : |f(x)| \geq 0 \} = \Omega$$

$$\{ c \geq 0 : \mu\{ x : |f(x)| \geq 0 \} \} = \{0\}$$

$\sup\{0\} = 0$

$\Omega \subseteq \mathbb{R}^d$ open

$C^0(\Omega)$

$C_c^0(\Omega) = \{f \in C^0(\Omega) : \operatorname{supp} f \subset\subset \Omega\}$

$BC^0(\Omega) = C^0(\Omega) \cap L^\infty(\Omega)$

$BC^0(\Omega) \subseteq L^\infty(\Omega)$

$BC^0(\Omega)$ with L^∞ norm is a Banach Space.

$$|f|_{L^\infty} = \sup_{x \in \Omega} |f(x)|$$

Is a Banach space

$\{f_n\}$ a Cauchy sequence in $BC^0(\Omega)$. Since I am assuming that $L^\infty(\Omega)$ is complete

$\exists \ f \in L^\infty(\Omega)$ s.t. $f_n \longrightarrow f$

in $L^\infty(\Omega)$. We need to show

$f \in C^0(\Omega)$. $\quad x \in \Omega$

$\{f_n(x)\}$

By $|f|_{BC^0(\Omega)} = \sup_{x\in\Omega} |f(x)|$

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_{L^\infty}$$

$$\Rightarrow \quad \left.\begin{array}{l} \{f_n(x)\}_n \text{ is a Cauchy} \\ f_n \longrightarrow f \quad \text{in } L^\infty \end{array}\right\}$$

$$\Longrightarrow \quad f(x) = \lim_{n\to+\infty} f_n(x)$$

$f \in C^0(\Omega)$

$\cancel{x}_0 \in \Omega$,

$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad$ s.t.

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

$$\lim_{n \to \infty} \| f - f_n \|_{L^\infty(\Omega)} = 0$$

$\exists \cancel{\forall}\ n \quad$ s.t. $\quad \| f - f_n \|_{L^\infty} < \frac{\varepsilon}{3}$

$$|f(x) - f_n(x)| \le \| f - f_n \|_{L^\infty} < \frac{\varepsilon}{3} \quad \forall x \in \Omega$$

Choose $\delta > 0$ s.t.

$$|x - x_0| < \delta \Rightarrow |f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3}$$

$$|f(x) - f(x_0)| \le |(f(x) - f_n(x)) + (f_n(x) - f_n(x_0)) + (f_n(x_0) - f(x_0))|$$

$$\le |f(x) - f_n(x)| + |f_n(x_0) - f(x_0)| + |f_n(x) - f_n(x_0)|$$

$$\leq 2 \underbrace{\| f - f_n \|_{L^\infty}}_{\varepsilon/3} + \underbrace{| f_n(x) - f_n(x_0) |}_{< \varepsilon/3} \leq \varepsilon$$

$$|x - x_0| < \delta$$

$\vartheta \in (0,1)$ $\qquad$ $\Omega \subset \mathbb{R}^d$

$C^\vartheta(\Omega)$ $\qquad$ $\mathring{\Omega} = \Omega$

$\| f \|_{C^\vartheta}$

$f \in C^\vartheta(\Omega) \Longleftrightarrow$

1) $f \in L^\infty(\Omega)$.

2) $\exists\, C > 0$ dependent on f, s.t.

$$\frac{|f(x) - f(y)|}{|x - y|^\vartheta} \leq C \quad \forall\, x \neq y$$

$$\| f \|_{C^\vartheta} = \| f \|_{L^\infty} + [f]_{C^\vartheta}$$

$$[f]_{C^\vartheta} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\vartheta}$$

$[c]_{C^\infty} = 0$

The key is completeness.

$\{f_n\}$ a Cauchy sequence

$C^\vartheta \supseteq BC^0$

$\{f_n\}$ is also Cauchy in BC^0 $\|\cdot\|_{L^\infty}$

$\forall \varepsilon > 0 \quad \exists \; N_\varepsilon$ s.t. $n, m \geq N_\varepsilon$

$$\varepsilon > \|f_n - f_m\|_{C^\vartheta} = |f_n - f_m|_{L^\infty} + [f_n - f_m]_{C^\vartheta}$$

$$\Rightarrow \quad |f_n - f_m|_{L^\infty} < \varepsilon$$

$\Rightarrow$ $\{f_n\}$ Cauchy in BC^0

$\Rightarrow$ $\exists \; f \in BC^0$ st

$$f_n \longrightarrow f \text{ in } BC^0.$$

We need to show $f \in C^\vartheta$, $\exists \; C > 0$

$$\frac{|f(x)-f(y)|}{|x-y|^{\nu}} \leq C \quad \forall \; x \neq y$$

$$\frac{|f(x)-f(y)|}{|x-y|^{\nu}} =$$

$$= \frac{|(f(x)-f_n(x)) + (f_n(x)-f_n(y)) + (f_n(y)-f(y))|}{|x-y|^{\nu}}$$

$$\leq \frac{|f(x)-f_n(x)|}{|x-y|^{\nu}} + \frac{|f(y)-f_n(y)|}{|x-y|^{\nu}} + \boxed{\frac{|f_n(x)-f_n(y)|}{|x-y|^{\nu}}}$$

$\leq C_0$

Since f_n is Cauchy in C^{ν} then

$\forall \varepsilon > 0 \; \exists \, M_\varepsilon$ st $n, m \geq M_\varepsilon$

$$\underline{[f_n - f_m]_{C^{\nu}} \leq \varepsilon}$$

$\Rightarrow \; \exists \; C_0$ s.t.

$$[f_n]_{C^{\nu}} \leq C_0 \quad \forall n.$$

$$\frac{|f(x)-f(y)|}{|x-y|^{\nu}} \lessgtr$$

$$\leq \frac{|f(x)-f_n(x)|}{|x-y|^{\nu}} + \frac{|f(y)-f_n(y)|}{|x-y|^{\nu}} + C_0$$

$\searrow 0$

$n \to +\infty$

$$\sup_{x \neq y} \frac{|f(x)-f(y)|}{|x-y|^{\nu}} \leq C_0$$

$$[f - f_n]_{C^{\nu}} \xrightarrow{n \to +\infty} 0$$

$\forall \varepsilon > 0 \quad \exists \; N_\varepsilon \quad$ s.t.

$$n > N_\varepsilon \Rightarrow [f - f_n]_{C^{\nu}} < \varepsilon$$

$$\frac{|(f(x)-f_n(x))-(f(y)-f_n(y))|}{|x-y|^{\vartheta}}$$

We know $\qquad N_\varepsilon = M_\varepsilon$?

$\forall \varepsilon > 0 \quad \exists\, M_\varepsilon$ st $n, m \geq M_\varepsilon$

$$[f_n - f_m]_{C^{\vartheta}} \leq \varepsilon$$

$n \geq M_\varepsilon$

$$\frac{|(f(x)-f_n(x))-(f(y)-f_n(y))|}{|x-y|^{\vartheta}} =$$

$$= \lim_{m \to +\infty} \frac{|(f_m(x)-f_n(x))-(f_m(y)-f_n(y))|}{|x-y|^{\vartheta}}$$

$\leq \varepsilon$

If $m > M_\varepsilon$,

$(n > M_\varepsilon)$

$$\frac{|(f_m(x) - f_n(x)) - (f_m(y) - f_n(y))|}{|x-y|^{\vartheta}} < \varepsilon$$

$$\Big\downarrow \; m \to +\infty$$

$$\frac{|(f(x) - f_n(x)) - (f(y) - f_n(y))|}{|x-y|^{\vartheta}} \leq \varepsilon$$

$$\forall \; x \neq y$$

$$\Rightarrow \quad \cancel{\forall} \quad \text{If} \quad n > M_{\varepsilon}$$

$$[f - f_n]_{C^{\vartheta}} \leq \varepsilon$$

$$\Rightarrow \quad \lim_{n \to +\infty} [f - f_n]_{C^{\vartheta}} = 0$$

$0 < \vartheta < 1$ $\quad (\vartheta = 1, 2, \ldots)$

$W^{\vartheta, p}(\Omega)$ $\quad 1 \leq p < +\infty$

$$[f]^p_{W^{\vartheta,p}} = \int_{\Omega\times\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^{\vartheta p+d}}\,dx\,dy$$

$\Omega \subseteq \mathbb{R}^d$ open

$$W^{\vartheta,p}(\Omega) = \left\{ f \in L^p(\Omega) : [f]_{W^{\vartheta,p}} < +\infty \right\}$$

$$\|f\|_{W^{\vartheta,p}} := \|f\|_{L^p} + [f]_{W^{\vartheta,p}}$$

$W^{\vartheta,p}$ is a Banach space.

$\{f_n\}$ cauchy in $W^{\vartheta,p}$

$\forall \varepsilon > 0 \;\exists\; M_\varepsilon > 0$ s.t.

$n, m \geq M_\varepsilon$ $\qquad 1 \leq p < +\infty$

$$\varepsilon > |f_n - f_m|_{W^{\vartheta,p}} = \underbrace{|f_n - f_m|_{L^p}} + \underbrace{[f_n - f_m]_{W^{\vartheta,p}}}$$

$\{f_n\}$ is Cauchy in L^p

$f_n \longrightarrow f$ in L^p

Need to show $f \in W^{\vartheta,p}(\Omega)$

$$\left(\frac{|f(x) - f(y)|^p}{|x-y|^{\vartheta p + d}} \right)^{\frac{1}{p}}$$

$$\frac{|f(x) - f(y)|}{|x-y|^{\vartheta + \frac{d}{p}}} \leq$$

$$\leq \frac{|f_n(x) - f_n(y)|}{|x-y|^{\vartheta + \frac{d}{p}}} + \frac{|f(x) - f_n(x)|}{|x-y|^{\vartheta + \frac{d}{p}}} +$$

$$+ \frac{|f(y) - f_n(y)|}{|x-y|^{\vartheta + \frac{d}{p}}} \qquad L^p(\Omega \times \Omega$$

$$\left\| \frac{|f(x)-f(y)|}{|x-y|^{\vartheta+\frac{d}{p}}} \right\|_{L^p(\{(x,y)\in\Omega\times\Omega;\ |x-y|\geq\varepsilon\})}$$

$$\leq \left\| \frac{|f_n(x)-f_n(y)|}{|x-y|^{\vartheta+\frac{d}{p}}} \right\|_{L^p(\{(x,y)\in\Omega\times\Omega;\ \qquad \})}$$

$$+ 2\left\| \frac{|f(x)-f_n(x)|}{|x-y|^{\vartheta+\frac{d}{p}}} \right\|_{L^p(\{(x,y)\in\Omega\times\Omega;\ |x-y|\geq\varepsilon\})}$$

$$+ \cancel{\left\| \frac{|f(y)-f_n(y)|}{|x-y|^{\vartheta+\frac{d}{p}}} \right\|_{L^p(\{(x,y)\in\Omega\times\Omega;\ |x-y|\geq\varepsilon\})}}$$

$$\leq [f_n]_{W^{\vartheta,p}}$$

$$+ 2\left\| \frac{|f(x)-f_n(x)|}{|x-y|^{\vartheta+\frac{d}{p}}} \right\|_{L^p(\{(x,y)\in\Omega\times\Omega;\ |x-y|\geq}$$

$$\left\| \frac{|f(x)-f(y)|}{|x-y|^{\vartheta+\frac{d}{p}}} \right\|_{L^p(\{(x,y)\in\Omega\times\Omega;\ |x-y|\geq\varepsilon\})}$$

$$\leq [f_n]_{W^{\vartheta,p}}$$

$$+ 2\left\| \frac{|f(x)-f_n(\cancel{x})|}{|x-y|^{\vartheta+\frac{d}{p}}} \right\|_{L^p(\{(x,y)\in\Omega\times\Omega;\ |x-y|\geq\varepsilon\})}$$

$$\left(\int_{\substack{|x-y|\geq\varepsilon \\ x,y\in\Omega}} \frac{|f(x)-f_n(x)|^p}{|x-y|^{\vartheta p+d}}\, dx\, dy \right)^{\frac{1}{p}}$$

$$\leq \Big(\underbrace{\int_\Omega dx\ |f(x)-f_n(\cancel{x})|^p}_{\varepsilon^{-\vartheta p}\ |f-f_n|_{L^p}} \underbrace{\int_{\substack{|x-y|\geq\varepsilon \\ \cancel{y\in\Omega}}} dy\ |x-y|^{-\vartheta p-d}}_{C_{d,p,\vartheta}\ \varepsilon^{-d\vartheta p}} \Big)^{\frac{1}{p}}$$

$$\int_{|x-y|\geq\varepsilon} dy\ |x-y|^{-\vartheta p-d} =$$

$$= \int_{|y| \ge \varepsilon} dy \, |y|^{-\vartheta p - d}$$

$$= \int_{S^{d-1}} d\theta \int_{\varepsilon}^{+\infty} r^{\cancel{d-1}} \, r^{-\vartheta p - \cancel{d}} \, dr$$

$$= \mathrm{vol}(S^{d-1}) \, \frac{\varepsilon^{-\vartheta p}}{\vartheta p}$$

$\forall \varepsilon > 0$

$$\left\| \frac{|f(x) - f_n(y)|}{|x-y|^{\vartheta + d/p}} \right\|_{L^p(\{(x,y) \in \Omega \times \Omega \, ; \, \cancel{|x-y| \ge \varepsilon}\})} \qquad \varepsilon = 0$$

$$\le [\cancel{f_n}]_{W^{\vartheta,p}} \, C_0$$

$$\cancel{+ \, C \, \varepsilon^{-\vartheta} \, \| f - f_n \|_{L^p}} \qquad 0$$

$$[f_n]_{W^{\vartheta,p}} \le C_0 \quad \forall n$$

$$\Longrightarrow \quad f \in W^{\vartheta,p}$$

$$\int_{\substack{|x-y|\geq\varepsilon \\ (x,y)\in\Omega\times\Omega}} \frac{|f(x)-f(y)|^p}{|x-y|^{p\theta+d}}\, dx\,dy \leq C_0^p$$

$\forall \varepsilon > 0$ $\qquad \varepsilon_n \longrightarrow 0^+$

$$\int_{\Omega\times\Omega} 1_{|x-y|\geq\varepsilon_n} \frac{|f(x)-f(y)|^p}{|x-y|^{p\theta+d}}\, dx\,dy \leq C_0^p$$

We want

$$\int_{\Omega\times\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^{p\theta+d}}\, dx\,dy \leq C_0^p$$

$$1_{|x-y|\geq\varepsilon_n} \frac{|f(x)-f(y)|^p}{|x-y|^{p\theta+d}} \xrightarrow{n\to+\infty} \frac{|f(x)-f(y)|^p}{|x-y|^{p\theta+d}}$$

a.a.

Fatou lemma

$$\int_{\Omega\times\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^{p\vartheta+d}}\,dx\,dy \leq$$

$$\leq \liminf_{n\to+\infty} \int_{\Omega\times\Omega} 1_{|x-y|\geq\varepsilon_n} \frac{|f(x)-f(y)|^p}{|x-y|^{p\vartheta+d}}\,dx\,dy$$

$$\leq C_0^n$$

$$[f]_{W^{\vartheta,p}} < +\infty \Rightarrow f \in W^{\vartheta,p}$$

We know $\|f-f_n\|_{L^p(\Omega)} \xrightarrow{n\to\infty} 0$

We need to show that

$$[f-f_n]_{W^{\vartheta,p}} \xrightarrow{n\to+\infty} 0$$

$$\forall \varepsilon > 0 \quad \exists N_\varepsilon \quad \text{s.t.} \quad n \geq N_\varepsilon$$

$$\Rightarrow [f-f_n]_{W^{\vartheta,p}} \leq \varepsilon.$$

$$\forall \varepsilon > 0 \quad \exists M_\varepsilon \quad \text{s.t.} \quad m, n \geq M_\varepsilon$$

$$\Rightarrow [f_m - f_n]_{W^{\vartheta,p}} < \varepsilon$$

$$f_n \longrightarrow f \quad \text{in } L^p(\Omega) \qquad 1 \le p < +\infty$$

Remark It is not necessarily true that

$$f_n(x) \longrightarrow f(x) \quad \text{a.a. in } \Omega.$$

$[0,1]$ $\quad \forall n$

$$\left[\frac{j-1}{n}, \frac{j}{n}\right] \qquad \phi \le j \le n$$

$$\{ I_\ell \}$$

$$1_{I_\ell} = \chi_{I_\ell}$$

$$\{ 1_{I_\ell} \} \qquad 1_{I_\ell} \xrightarrow{\ell \to +\infty} 0$$

$$\ell \to +\infty$$

$\forall \varepsilon > 0$ there are finitely intervals with

$$|I_\ell| \geq \varepsilon$$

$$\Rightarrow \quad \forall \varepsilon > 0 \quad \exists\, M_\varepsilon \text{ s.t. } \ell \geq M_\varepsilon \Rightarrow$$

$$\Rightarrow \left| 1_{I_\ell} \right|_{L^1} = |I_\ell| < \varepsilon$$

n

$n=1 \qquad I_1 = [0,1]$

$n=2 \qquad I_2 = [0, \tfrac{1}{2}] \,,\; I_3 = [\tfrac{1}{2}, 1]$

$n=3 \qquad I_4 = [0, \tfrac{1}{3}] \,,\; I_5 = [\tfrac{1}{3}, \tfrac{2}{3}] \,,\; I_6 = [\tfrac{2}{3}, 1]$

⋮

It ~~is~~ is not true $1_{I_\ell}(x) \longrightarrow 0$ for a.a. x

In fact for any $x \in [0,1]$

$\lim_{\ell \to +\infty} 1_{I_\ell}(x)$ does not exist

Obviously there is a $\{I_{\ell_k}\}_{k \in \mathbb{N}}$

s.t. $\lim_{k \to +\infty} 1_{\ell_k}(x) = 0$

Just take intervals with $x \notin I_{\ell_k}$

$\forall \varepsilon > 0 \quad \exists \, M_\varepsilon \quad$ s.t. $\quad \ell > M_\varepsilon$

$$\Rightarrow \quad 1_{I_{\ell_k}}(x) = 0$$

But for any M_ε $\exists$

n s.t. all the intervals

$\cancel{[0, \tfrac{1}{n}]}$, $\left[\frac{j-1}{n}, \frac{j}{n}\right] \quad 1 \le j \le n$

appear in the sequence of I_ℓ with

$\ell > M_\varepsilon$.

At least one of the contains x.

so $\exists \; \ell > M_\varepsilon$

$$1_{I_\ell}(x) = 1 .$$

$$f_n \longrightarrow f \quad \text{in } L^p(\Omega) \qquad 1 \le p < +\infty$$

<u>Remark</u> It is not necessarily true that

$$f_n(x) \xrightarrow{n \to +\infty} f(x) \quad \text{a.e. in } \Omega .$$

However there exists a subsequence s.t.

$$f_{n_k}(x) \xrightarrow{k\to+\infty} f(x) \quad \text{a.a. } \Omega.$$

$$[f - f_n]^p_{W^{\vartheta,p}} =$$

$$= \int_{\Omega\times\Omega} \frac{\left| f(x) - f_n(x) - (f(y) - f_n(y)) \right|^p}{|x-y|^{\vartheta p + d}} \, dx\,dy$$

$$\leq \liminf_{k\to+\infty} \int_{\Omega\times\Omega} \frac{|f_{n_k}(x) - f_n(x) - f_{n_k}(y) + f_n(y)|^p}{|x-y|^{\vartheta p+d}} \, dx\,dy$$

$$\leq \varepsilon^p$$

$\forall \varepsilon > 0 \quad \exists M_\varepsilon$ s.t. $m, n > M_\varepsilon$

$$\Rightarrow \quad [f_m - f_n]^p_{W^{\vartheta,p}} < \varepsilon^p$$

So if $n > M_\varepsilon$

$$[f - f_n]_{W^{\vartheta,p}} \leq \varepsilon.$$

Locally convex space

<u>Def</u> Let X be a vector space.

$\Omega \subseteq X$ is convex if

$\forall\, x, y \in \Omega$

$$(1-t)x + ty \in \Omega \quad \forall\ 0 \leq t \leq 1$$

Exercise If X is a T.V.S and $\Omega \subseteq X$ is convex then

1) $\overline{\Omega}$ is convex

2) $\mathring{\Omega}$ is convex.

Def A TVS X is locally convex if there exists a basis of neigh. of 0 which are convex.

Remark If X is locally convex then there exists a basis of neigh. of 0 which are convex and balanced

Def X vector space. A function $p : X \to [0, +\infty)$ is a seminorm if

1) $p(x+y) \leq p(x) + p(y)$ $\quad \forall x, y \in X$

2) $p(\lambda x) = \lambda p(x)$ $\quad \forall \lambda > 0$ $\quad \forall x \in X$

later

1) $p(x+y) \leq p(x) + p(y)$

2') $p(\lambda x) = |\lambda| \, p(x)$ $\quad \forall \lambda \in K$ $\quad \forall x \in X$

Exercise X vector space and $p : \cancel{[0,} X \to [0, +\infty)$ a seminorm. Then

$$C = \{ x \in X : \; p(x) < 1 \} \quad (X)$$

is convex

Lemma let X be a TVS, C an open convex set with $0 \in C$.

Then $\exists$ a seminorm $\left(p : \cancel{X} \to [0, +\infty) \right.$ satisfying 1) and 2) $\left. \right)$ s.t. $*$ holds

Proof

$$p(x) = \inf \left\{ a > 0 : \frac{x}{a} \in C \right\}$$

$\forall x \in X$ the set $\nearrow$ is not empty

Pick $x \in X$

We know that for $0 < \lambda << 1$

$\lambda x \in C$

$\lambda \longrightarrow \lambda x$ is continuous $C^0(\mathbb{R}, X)$

$K \times X \longrightarrow X$

$(\lambda, x) \longrightarrow \lambda x$

λ

$$x \in C \iff p(x) < 1$$

$$\Longrightarrow$$

$x \in C$

$$p(x) = \inf \left\{ a > 0 : \frac{x}{a} \in C \right\}$$

$$\frac{x}{1} = x \in C \Rightarrow 1 \in \left\{ a > 0 : \frac{x}{a} \in C \right\}$$

$$\Rightarrow 1 \geq \inf \left\{ a > 0 : \frac{x}{a} \in C \right\} =: p(x)$$

Since C open $\exists$ $\varepsilon > 0$ s.t.

$$(1+\varepsilon) x \in C$$

$$\cancel{\Rightarrow} \quad (1+\varepsilon) x = \frac{x}{\frac{1}{1+\varepsilon}}$$

$$\Rightarrow \frac{1}{1+\varepsilon} \in \left\{ a > 0 : \frac{x}{a} \in C \right\}$$

$$\Rightarrow 1 > \frac{1}{1+\varepsilon} \geq p(x)$$

Therefore $x \in C \Rightarrow p(x) < 1$

Now

$$p(x) < 1 \Rightarrow x \in C$$

Let $x \in X$ with $p(x) < 1$

$$p(x) = \inf \left\{ a > 0 : \frac{x}{a} \in C \right\}$$

Then $\exists$ $0 < a < 1$ s.t. $\frac{x}{a} \in C$

$$x = a \frac{x}{a} + (1-a)\, 0 \qquad 0<a<1$$

$$\frac{x}{a}, 0 \in C$$

Since C is convex $\Longrightarrow$ $x \in C$

So $p(x) < 1 \Longleftrightarrow x \in C$

We will prove

$$p(\lambda x) = \lambda p(x) \qquad \forall \lambda > 0 \quad x \in X$$

$$p(\lambda x) = \inf\{ a > 0 : \frac{\lambda x}{a} \in C \} =$$

$$\alpha = \frac{a}{\lambda} \qquad a = \lambda \alpha$$

$$= \inf\{ \lambda \alpha \,;\, \alpha > 0, \frac{x}{\alpha} \in C \}$$

$$= \lambda \inf\{ \alpha : \alpha > 0, \frac{x}{\alpha} \in C \}$$

$$= \lambda \, p(x)$$

$$p(x+y) \le p(x) + p(y)$$

$$x, y \in X$$

$\varepsilon > 0$

$$\frac{x}{p(x)+\varepsilon} \qquad p\left(\frac{x}{p(x)+\varepsilon}\right) = \frac{1}{p(x)+\varepsilon} p(x) < 1$$

$$\frac{y}{p(y)+\varepsilon} \qquad p\left(\frac{y}{p(y)+\varepsilon}\right) < 1$$

$$\frac{x}{p(x)+\varepsilon}, \quad \frac{y}{p(y)+\varepsilon} \in C$$

$$t \in [0, 1]$$

$$t \frac{x}{p(x)+\varepsilon} + (1-t) \frac{y}{p(y)+\varepsilon} \in C$$

$$t = \frac{p(x) + \varepsilon}{p(x)+p(y)+2\varepsilon}$$

$$\frac{(\cancel{p(x)+\varepsilon})}{p(x)+p(y)+2\varepsilon} \frac{x}{\cancel{p(x)+\varepsilon}} +$$

$$+ \left(1 - \frac{p(x)+\varepsilon)}{p(x)+p(y)+2\varepsilon}\right) \frac{y}{p(y)+\varepsilon}$$

$$= \frac{x}{p(x)+p(y)+2\varepsilon} +$$

$$+ \frac{\cancel{p(x)} + p(y) + \cancel{2\varepsilon} - \cancel{p(x)} - \cancel{\varepsilon}}{p(x)+p(y)+2\varepsilon} \frac{y}{p(y)+\varepsilon}$$

$$= \frac{x+y}{p(x)+p(y)+2\varepsilon} \in C$$

$$p\left(\frac{x+y}{p(x)+p(y)+2\varepsilon}\right) < 1$$

$$p(x+y) < p(x)+p(y)+2\varepsilon \qquad \forall \varepsilon >$$

$$\Longrightarrow \quad p(x+y) \leq p(x)+p(y)$$

$$\lambda V_1 \subseteq V \qquad \forall\ |\lambda| \leq \delta$$

$$\tilde{V} = \bigcup_{|\lambda| \leq \delta} \lambda V_1 \subseteq V$$

TVS

$X \qquad \{p_j\}_{j \in J}$ is a subbasis of seminorms if

$\forall$ neigh. U of 0 $\exists$ $r > 0$ and

~~s.t.~~ $\exists$ a finite set $j_1, \dots, j_n \in J$ and

$\{ x \in X : \quad P_{j_1}(x) < r, \dots, P_{j_n}(x) < r \} \subseteq U$

$$\bigcap_{l=1}^{n} P_{j_l}^{-1}([0, r))$$

$$\left(X, \{ P_j \}_{j \in J} \right)$$

$\mathbb{N} = \cancel{\{1\}}$

$\mathbb{N}_0 = \cancel{\{0\}}$

Space of Schwartz $\qquad \alpha, \beta \in \mathbb{N}_0^d$

$$S(\mathbb{R}^d) = \{ \phi \in C^\infty(\mathbb{R}^d, \mathbb{C}) :$$

$$\underline{P_{\alpha,\beta}(\phi) := \sup_{x \in \mathbb{R}^d} |x^\alpha \partial_x^\beta \phi(x)| < +\infty \}}$$

$$S(\mathbb{R}^d) \longrightarrow \mathbb{C}$$

$S'(\mathbb{R}^d)$ space of tempered distribution