4th of October

1) $\left(X, \{p_j\}_{j \in J}\right)$, $\left(Y, \{q_k\}_{k \in K}\right)$

$T : X \longrightarrow Y$ linear operator

is continuous iff

$\forall \; k_0 \in K \quad \exists \; J_{k_0}$, a finite subspace of J s.t. and a constant $C_{k_0} \geq 0$

$$q_{k_0}(Tx) \leq C_{k_0} \sum_{j \in J_{k_0}} p_j(x)$$

$$\forall x \in X.$$

2) $f : [0, +\infty) \longrightarrow [0, +\infty)$ concave with $f(0) = 0$ then

$$f(x+y) \leq f(x) + f(y)$$

$$\forall \; x, y \in [0, +\infty)$$

<u>Example</u> $L^p((0,1))$ $\quad 0 < p < 1$

$= \{ f \text{ measurable} : \int_0^1 |f|^p \, dx < +\infty \}$

$f, g \in L^p$

$$|f+g|^p \leq \big|\, |f| + |g| \,\big|^p = (|f| + |g|)^p$$
$$\leq |f|^p + |g|^p$$

$y \to y^p \quad y \geq 0$ is concave

$$\Rightarrow \int_0^1 |f+g|^p \, dx \leq \int_0^1 |f|^p \, dx + \int_0^1 |g|^p \, dy$$

$$d(f,g) = \int_0^1 |f-g|^p \, dx$$

$$\boxed{d(f,g) \leq d(f,h) + d(h,g)}$$

$$\int_0^1 |f-g|^p \, dx =$$

$$= \int_0^1 |(f-h)+(h-g)|^p \, dx$$

$$\leq \int_0^1 |f-h|^p \, dx + \int_0^1 |h-g|^p \, dx$$

$L^p((0,1))$ becomes a TVS

The only open convex subsets of

L^p are L^p itself ~~ϕ~~.

V open and convex $V \neq \phi$

$V \ni 0$. $\exists$ $\varepsilon_0 > 0$

s.t. $D_{L^p}(0, \varepsilon_0) \subseteq V$

Let $f \in L^p(0,1)$ and

let $n \in \mathbb{N}$ s.t.

$$n^{p-1} \int_0^1 |f|^p \, dx < \varepsilon_0 \qquad (*)$$

$0 < p < 1$

For $n >> 1$ $(*)$ is true

$$0 < t_1 < \cdots < t_n = 0$$
$$\overset{\shortparallel}{t_0}$$

$$\boxed{\int_{t_{j-1}}^{t_j} |f(t)|^p \, dt = \frac{1}{n} \int_0^1 |f(t)|^p \, dt}$$

$$g_j = n \chi_{[t_{j-1}, t_j]} f$$

$$\int_0^1 |g_j(t)|^p \, dt = n^p \int_{t_{j-1}}^{t_j} |f|^p \, dt$$

$$= n^{p-1} \int_0^1 |f(t)|^p \, dt < \varepsilon_0$$

$$\Longrightarrow \quad g_j \in D_{L^p}(0, \varepsilon_0) \subseteq V$$

$$g_j = n \, \chi_{[t_{j-1}, t_j]} f$$

$$\sum_{j=1}^{n} g_j = n \underbrace{\sum_{j=1}^{n} \chi_{[t_{j-1}, t_j]}}_{1} f$$

$$f = \frac{1}{n} \sum_{j=1}^{n} g_j \in V$$

$$\forall f \in L^p, \quad f \in V.$$

$$\boxed{V = L^p}$$

$0 < p < 1$

Exampl L^p, let X be a lc locally convex TVS

$$T \in \mathcal{L}(L^p, X)$$

$$\Longrightarrow \quad T = 0$$

$$(L^p)' = 0$$

$V \subset X$ $\quad V$ open and convex, $V \neq \emptyset$

$\Rightarrow$ $T^{-1}V$ is open in X

and is convex.

$$T^{-1} V = L^p$$

$$T L^p \subseteq V$$

If $\{V_j\}_{j \in J}$ is a basis of open ~~convex~~ neigh. of $0 \in X$

$$T L^p \subseteq \bigcap_{j \in J} V_j = \{0\}$$

$$T L^p = \{0\} \Longleftrightarrow T = 0$$

$0 < p < 1$ $\qquad X'$

$$L^p \longrightarrow \mathbb{R}$$

$(X, \{p_j\}_{j \in J})$ will be

metrizable if there is a

sub-basis of seminorms, let me assume $\{p_j\}_{j\in J}$ will do,

s.t. $\text{card} J = \begin{cases} \text{finite} \\ = \text{card } \mathbb{N} \end{cases}$

There is an explicit metric.

1) $p(x+y) \le p(x) + p(y) \quad \forall x, y$

2') $p(\lambda x) = |\lambda| \, p(x) \quad \forall \lambda, x \in X$
$\quad \lambda \in \mathbb{K}$

p_j satisfy 1) and 2')

$d(x,y) = p_1(x-y) + \cdots + p_n(x-y)$ ✓

$$d(x,y) = \sum_{j=1}^{n} 2^{-j} \frac{p_j(x-y)}{1+p_j(x-y)}$$

$$d(x,y) = \sum_{j=1}^{\infty} 2^{-j} \frac{p_j(x-y)}{1+p_j(x-y)}$$

X $\quad d_1 \quad d_2$

$\exists\, C > 0$

$$\boxed{\frac{1}{C}\, d_2(x,y) \le d_1(x,y) \le C\, d_2(x,y)}$$

$\forall$

$(X, \|\cdot\|)$

$$d_1(x,y) = \|x-y\|$$

$$d_2(x,y) = \frac{\|x-y\|}{1+\|x-y\|}$$

X $\quad \|\cdot\|_1 \quad \|\cdot\|_2$

are equivalent if $\exists\, C \geq 1$

s.t.

$$\frac{1}{C}|x|_2 \le |x|_1 \le C\,\|x|_2 \qquad \forall x \in X$$

<u>Lemma</u> $\quad (X, \|\cdot\|_X) \quad (Y, \|\cdot\|_Y)$

and $T : X \to Y$ is linear the following are equivalent

1) T is continuous

2) T is bounded

$$\|T\|_{\mathcal{L}(X,Y)} := \sup_{x \in D_X(0,1)\setminus\{0\}} \frac{\|Tx\|_Y}{\|x\|_X} < \infty$$

$$\parallel$$

$$\sup_{x \in X\setminus\{0\}} \frac{\|Tx\|_Y}{\|x\|_X}$$

$$= \sup_{\|x\|_X = 1} \|Tx\|_Y$$

2) is equivalent to saying that T is a bounded operator

2) $\Rightarrow$ 1)

We have to show that

$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad$ st

$$|x|_X < \delta \implies |Tx|_Y < \varepsilon.$$

$$T D_X(0,\delta) \subseteq D_Y(0,\varepsilon)$$

$$\sup_{x \in D_X(0,1)\setminus\{0\}} \frac{|Tx|_Y}{|x|_X} = C \geq 0$$

$$\implies \frac{|Tx|_Y}{|x|_X} \leq C \quad \forall \, x \in D_X(0,1), \; x \neq 0$$

$$|Tx|_Y \leq C \, |x|_X \quad \forall \, \cancel{x \in D_X(0,1)} \quad \forall x \in X$$

$$|Tx|_Y \leq C|x|_X < \varepsilon$$

$$|x|_X < \frac{\varepsilon}{C} =: \delta_\varepsilon$$

$$|x|_X < \delta_\varepsilon \implies |Tx|_Y < \varepsilon$$

$$X \xrightarrow{T} Y \xrightarrow{S} Z$$

$$\|ST\|_{\mathcal{L}(X,Z)} \le \|S\|_{\mathcal{L}(Y,Z)} \, \|T\|_{\mathcal{L}(X,Y)}$$

$$\mathcal{L}(X, Y)$$

Th $\|\cdot\|_{\mathcal{L}(X,Y)}$ is a norm

Y B - space

X normed space

$\mathcal{L}(X, Y)$ is a B - space.

$$\Updownarrow$$

$$\mathcal{L}(\bar{X}, \bar{Y})$$

$$\mathcal{L}(X)$$

$$|TS|_{\mathcal{L}(X)} \le |T|_{\mathcal{L}(X)} \quad |S|_{\mathcal{L}(Y)}$$

$X \quad \|\cdot\|$

$$X' = \mathcal{L}(X, K)$$

$f \in X'$

$$|f|_{X'} = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|}$$

$$f(x) = \langle f, x \rangle_{X' \times X} =$$

$$= \langle x, f \rangle_{X \times X'}$$

<u>Def</u> $\{T_n\}$ a sequence in $\mathcal{L}(X, Y)$

$T \in \mathcal{L}(X, Y)$ we say that

$T_n \xrightarrow{n \to +\infty} T$ uniformly or in norm

if $\lim_{n \to +\infty} |T - T_n|_{\mathcal{L}(X,Y)} = 0$

We say that $\{T_n\}$ converges strongly to T if

$$\lim_{n\to+\infty} T_n x = Tx \quad \forall x \in X$$

$$s-\lim_{n\to+\infty} T_n = T$$

$p = +\infty$

Exemple $f \in L^p(\mathbb{R}^d)$ $+\infty > p \geq 1$

then ~~for any~~

we have $\lim_{\lambda\to+\infty} \chi_{D_{\mathbb{R}^d}(0,1)}\left(\frac{\cdot}{\lambda}\right) f = f$

in $L^p(\mathbb{R}^d)$

$f \to \chi_{D_{\mathbb{R}^d}(0,1)} f \quad \mathcal{L}(L^p)$

$$\chi_{D_{\mathbb{R}^d}(0,1)}\left(\frac{\cdot}{\lambda}\right) \xrightarrow{\lambda\to+\infty} \text{identity } 1$$

$$m \in L^\infty(\mathbb{R}^d)$$

$$T_m f = m f$$