The structure and evolution of stars

Lecture 2: The equations of stellar structure

Learning Outcomes

- The student will learn
  - There are 4 basic equations of stellar structure, their solution provides description of models and evolution
  - Derivation of the first two of these equations
  - How to derive the equation of hydrostatic support
  - How to show that the assumption of hydrostatic equilibrium is valid
  - How to derive the equation of mass conservation
  - How to show that the assumption of spherical symmetry is valid

Introduction

What are the main physical processes which determine the structure of stars?

- Stars are held together by gravitation – attraction exerted on each part of the star by all other parts
- Collapse is resisted by internal thermal pressure.
- These two forces play the principal role in determining stellar structure – they must be (at least almost) in balance
- Thermal properties of stars – continually radiating into space. If thermal properties are constant, continual energy source must exist
- Theory must describe - origin of energy and transport to surface

We make two fundamental assumptions :

1) Neglect the rate of change of properties – assume constant with time
2) All stars are spherical and symmetric about their centres

We will start with these assumptions and later reconsider their validity

For our stars – which are isolated, static, and spherically symmetric – there are four basic equations to describe structure. All physical quantities depend on the distance from the centre of the star alone

1) **Equation of hydrostatic equilibrium:** at each radius, forces due to pressure differences balance gravity
2) **Conservation of mass**
3) **Conservation of energy :** at each radius, the change in the energy flux = local rate of energy release
4) **Equation of energy transport :** relation between the energy flux and the local gradient of temperature

These basic equations supplemented with

- Equation of state (pressure of a gas as a function of its density and temperature)
- Opacity (how opaque the gas is to the radiation field)
- Core nuclear energy generation rate

Equation of hydrostatic support

Balance between gravity and internal pressure is known as *hydrostatic equilibrium*

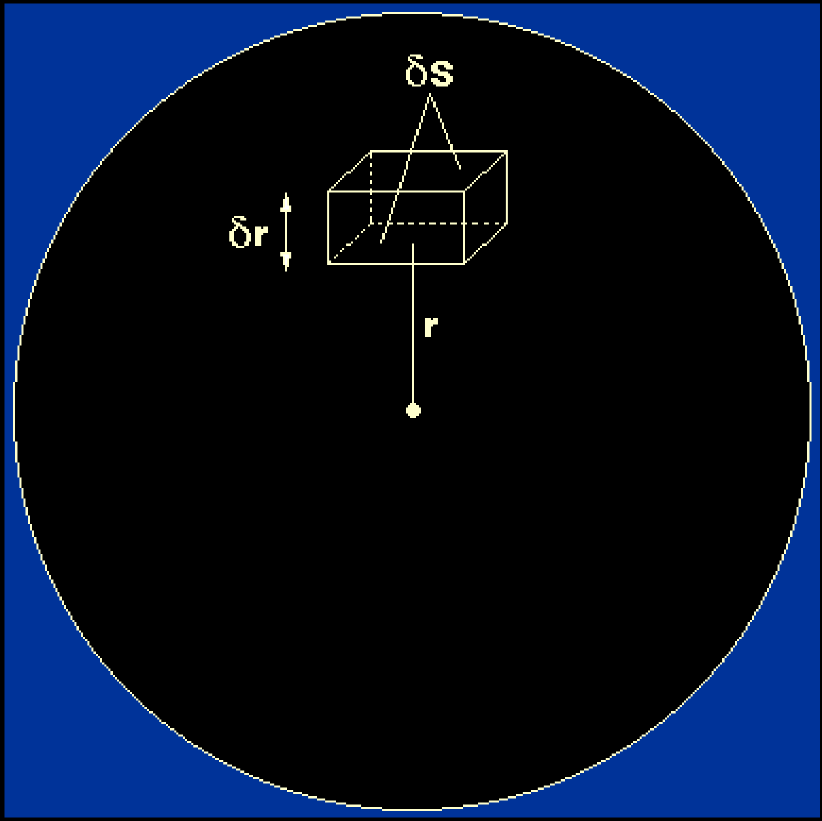
Mass of element

$\rho(r)\delta S\delta r$ where $\rho(r)$=density at r

Consider forces acting in radial direction

1. Outward force: pressure exerted by stellar material on the lower face: $P(r)\delta S$

2. Inward force: pressure exerted by stellar material on the upper face, and gravitational attraction of all stellar material lying within *r*



$$P(r+\delta r)\delta S + \frac{GM(r)}{r^2}\delta m =$$

$$P(r+\delta r)\delta S + \frac{GM(r)}{r^2}\rho(r)\delta S\delta r$$

In hydrostatic equilibrium:

$$P(r)\delta S = P(r+\delta r)\delta S + \frac{GM(r)}{r^2}\rho(r)\delta S\delta r$$

$$\rightarrow P(r+\delta r) - P(r) = -\frac{GM(r)}{r^2}\rho(r)\delta r$$

If we consider an infinitesimal element, we can write

$$\frac{P(r+dr)-P(r)}{dr} = -\frac{GM(r)}{r^2}\rho(r) \quad \text{for } \delta r \rightarrow 0$$

Hence rearranging the above we get

$$\frac{dP}{dr} = -\frac{GM(r)}{r^2}\rho(r)$$

The equation of hydrostatic support

Equation of mass conservation

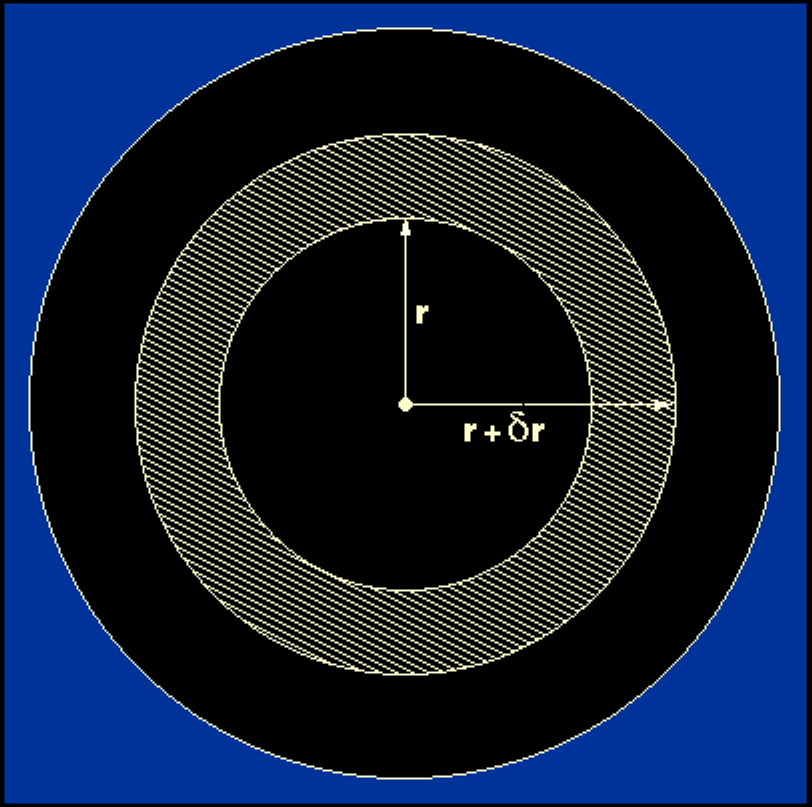
Mass $M(r)$ contained within a star of radius r is determined by the density of the gas $\rho(r)$.

Consider a thin shell inside the star with radius r and outer radius $r+\delta r$

$$\delta V = 4\pi r^2 \delta r$$

$$\delta M = 4\pi r^2 \rho(r) \delta r$$

$$\frac{dM}{dr} = 4\pi r^2 \rho(r)$$



In the limit where $\delta r \rightarrow 0$

This the equation of mass conservation

Accuracy of hydrostatic assumption

We have assumed that the gravity and pressure forces are balanced - how valid is that ?

Consider the case where the outward and inward forces are not equal, there will be a resultant force acting on the element which will give rise to an acceleration a

$$P(r+\delta r)\delta S - P(r)\delta S + \frac{GM(r)}{r^2}\rho(r)\delta S\delta r = \mathrm{a}\,\rho(r)\delta S\delta r$$

$$\frac{dP(r)}{dr} + \frac{GM(r)}{r^2}\rho(r) = \mathrm{a}\,\rho(r)$$

Now acceleration due to gravity is $g=GM(r)/r^2$

$$\frac{dP(r)}{dr} + g\rho(r) = \mathrm{a}\,\rho(r)$$

Which is the generalised form of the equation of hydrostatic support

Accuracy of hydrostatic assumption

Now suppose there is a resultant force on the element (LHS $\neq$0).
Suppose their sum is small fraction of gravitational term (*β)*

$$\beta\rho(r)g = \text{a}\ \rho(r)$$

Hence there is an inward acceleration of

$$a = \beta g$$

Assuming it begins at rest, the spatial displacement *d* after a time *t is*

$$d = \frac{1}{2}at^2 = \frac{1}{2}\beta g t^2$$

<u>Class Tasks</u>

1. Estimate the timescale for the Sun's radius to change by an observable amount (as a function of *β)*. Assume β is small, is the timescale likely ? (r=7x10^8 m ; g=2.5x10^2 ms^{-2})

2. We know from geological and fossil records that it is unlikely to have changed its flux output significantly over the last 10^9 . Hence find an upper limit for *β.* What does this imply about the assumption of hydrostatic equilibrium ?

The dynamical timescale

If we allowed the star to collapse i.e. set $d=r$ and substitute $g=GM/r^2$

$$t=\sqrt{\frac{2R^3}{\beta GM}}$$

Assuming $\beta \sim 1$

$$t_d=\sqrt{\frac{2R^3}{GM}}$$

t_d is known as the dynamical time. What is it a measure of ?

$r_\odot = 7 \times 10^8$ m

$M_\odot = 1.99 \times 10^{30}$ kg

Accuracy of spherical symmetry assumption

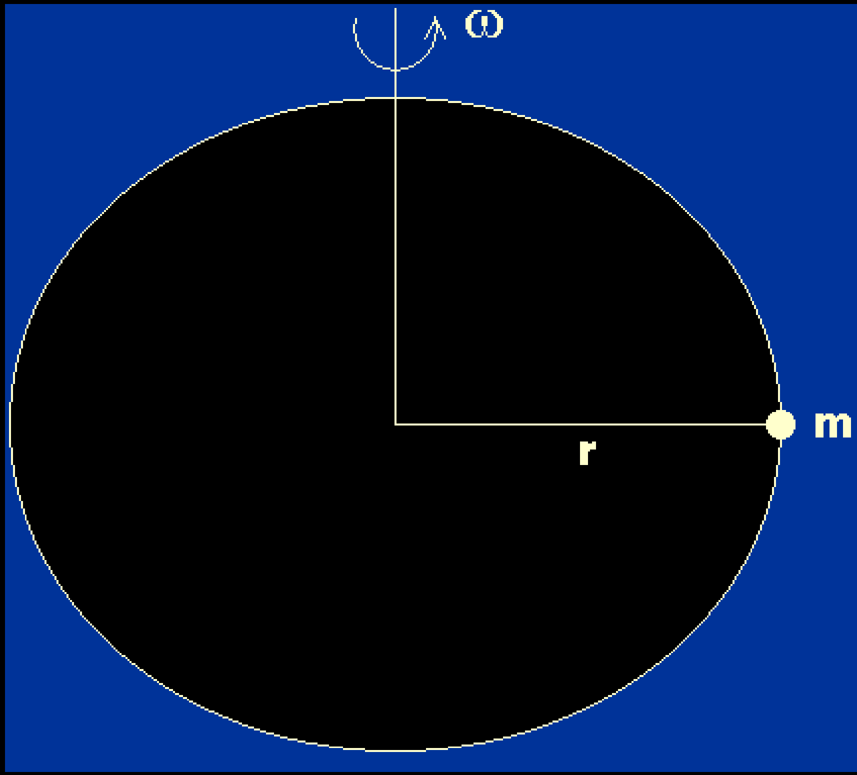
Stars are rotating gaseous bodies – to what extent are they flattened at the poles ?

If so, departures from spherical symmetry must be accounted for

Consider mass δm near the surface of star of mass M and radius r

Element will be acted on by additional inwardly acting force to provide circular motion.

Centripetal force is given by: $m\omega^2 r$



Where ω = *angular velocity of star*

There will be no departure from spherical symmetry provided that

$$m\omega^2 r \ll \frac{GMm}{r^2} \rightarrow \omega^2 \ll \frac{GM}{r^3}$$

Accuracy of spherical symmetry assumption

Note the RHS of this equation is similar to t_d

$$t_d = \sqrt{\frac{R^3}{2GM}} \quad \text{or} \quad 2t_d^{\,2} = \frac{R^3}{GM}$$

$$\omega^2 \ll \frac{1}{2t_d^{\,2}}$$

And as $\omega = 2\pi/P$; *where P=rotation period*

If spherical symmetry is to hold then $P \gg t_d$

For example, t_d(sun)~2000s and P~1 month

$\Rightarrow$For the majority of stars, departures from spherical symmetry can be ignored.

Some stars do rotate rapidly and rotational effects must be included in the structure equations - can change the output of models

Summary

There are 4 equations of stellar structure that we need to derive

- Have covered the first 2 (hydrostatic support and mass conservation)
- Have shown that the assumption of hydrostatic equilibrium is valid
- Have derived the dynamical timescale for the Sun as an example
- Have shown that the assumption of spherical symmetry is valid, if the star does not rapidly rotate