Artificial Intelligence for Cyber-Physical Systems

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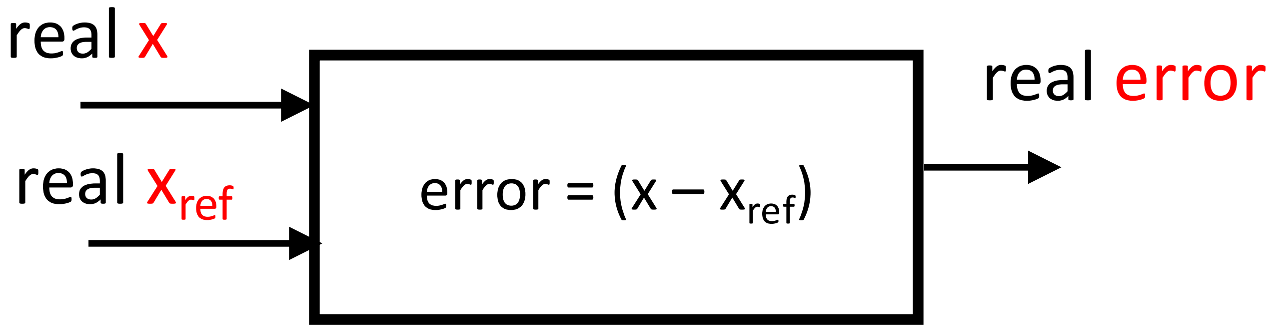
Lecture 3: Dynamical Systems

[Many Slides due to J. Deshmukh, Toyota]

Dynamical Systems

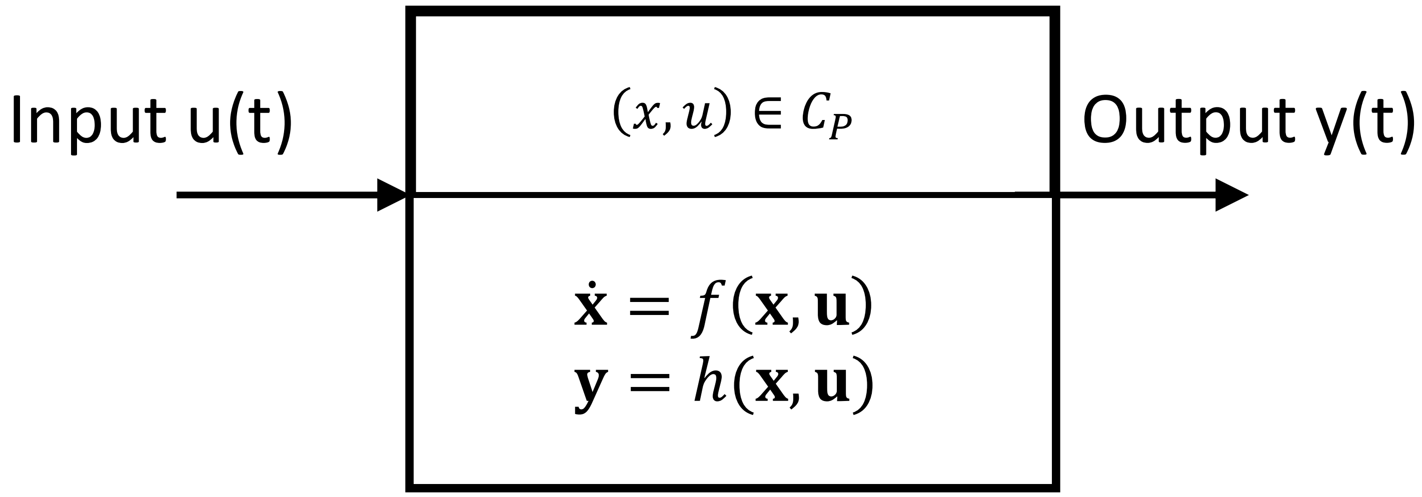
- Most natural model for describing most physical systems
- Continuous/discrete systems that continuously evolve over time
- It is represented by differential equations that involve the rates of change of quantities
- Quantities describe the state of the phenomena, modeled as state variables
  - Pressure, Temperature, Velocity, Acceleration, Current, Voltage, etc.
- Could include algebraic relations between state variables

Continuous-time Component (Algebraic)

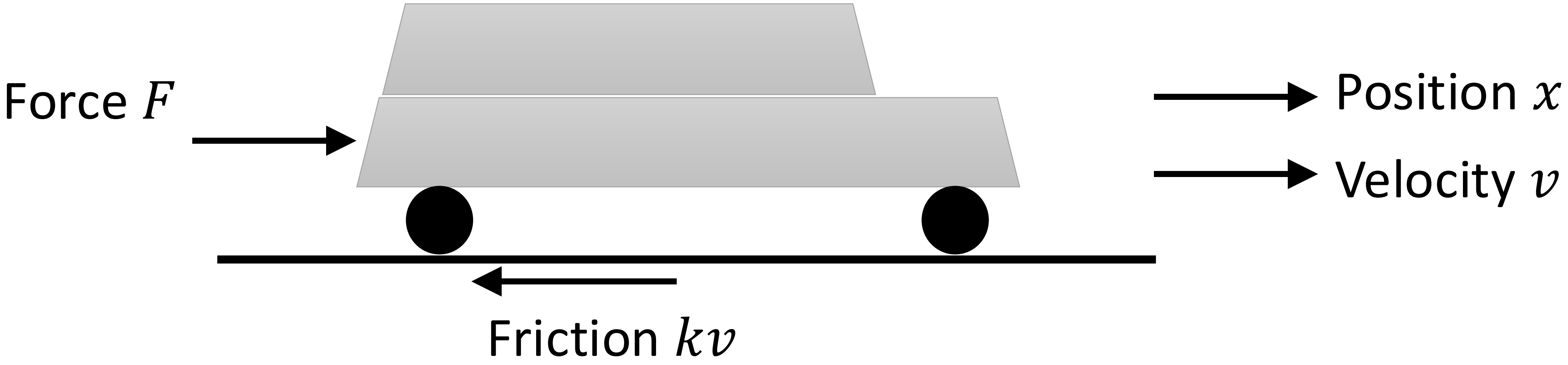


- Input variables: x and x_{ref} of type real, Output variable: error of type real
- No state variables
- Signals: Assignments of values to variables as a function of time
- At each time t, error(t) = x(t) – x_{ref}(t)
- Input/Output relation expressed algebraically instead of as an assignment

Continuous-time component (differential)



Model of a simple car



Newton's law of motion: $F = m\frac{d^2x}{dt^2} + kv\ ;\ v = \frac{dx}{dt}$

State-Space representation

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$$
$$\mathbf{y} = h(\mathbf{x}, \mathbf{u})$$

Example:

Convert

$$\dot{x} = v(t)$$
$$\dot{v} = \frac{F(t) - kv(t)}{m}$$

- It is numerically efficient to solve
- It can handle complex systems
- It allows for a more geometric understanding of dynamic systems
- It forms the basis for much of modern control theory

State-Space representation

All derivatives are with respect to single independent variable, often representing time.

Order of ODE is determined by highest-order derivative of state variable function appearing in ODE

ODE with higher-order derivatives can be transformed into equivalent first-order system.

$$x^{(k)} = f(x, \dots, x^{(k-1)})$$

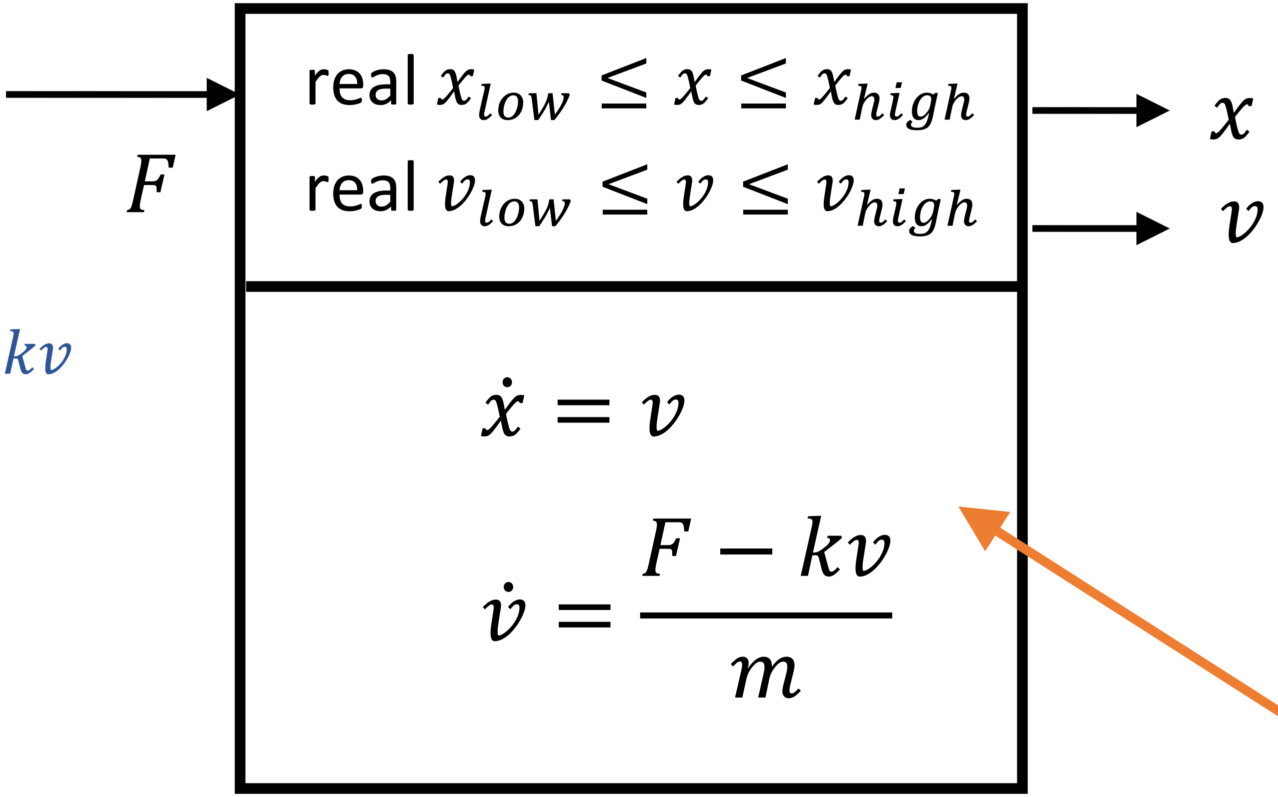
$$z_1 = x,\ z_2 = \dot{x}, \dots, z_k = x^{(k-1)}$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ . \\ . \\ . \\ \dot{z}_k \end{bmatrix} = \begin{bmatrix} z_2 \\ z_3 \\ z_4 \\ . \\ . \\ . \\ f(x, \dots, x^{(k-1)}) \end{bmatrix}$$

Model of a simple car

$$m\frac{d^2x}{dt^2} = F - kv$$

$$v = \frac{dx}{dt}$$



- Rate of change of each state variable and output variables defined using expressions over inputs and states

Expressions, not assignments!

Executions of Car

- Let $\mathbb{T}$ represent a set representing time instants, i.e. $\mathbb{T} \subseteq \mathbb{R}^{\geq 0}$

- Input Signal: Function F from $\mathbb{T} \to \mathbb{R}$
  - Input signal is assumed to be continuous or piecewise-continuous

- Given an initial state (x_0, v_0) and an input signal $F(t)$, the execution of the system is defined by ***state-trajectories*** $x(t)$ and $v(t)$ (from $\mathbb{T}$ to $\mathbb{R}$) that satisfy the ***initial-value problem***:

  - $x(0) = x_0; v(0) = v_0$
  - $\dot{x} = v(t); \dot{v} = \frac{F(t) - kv(t)}{m}$

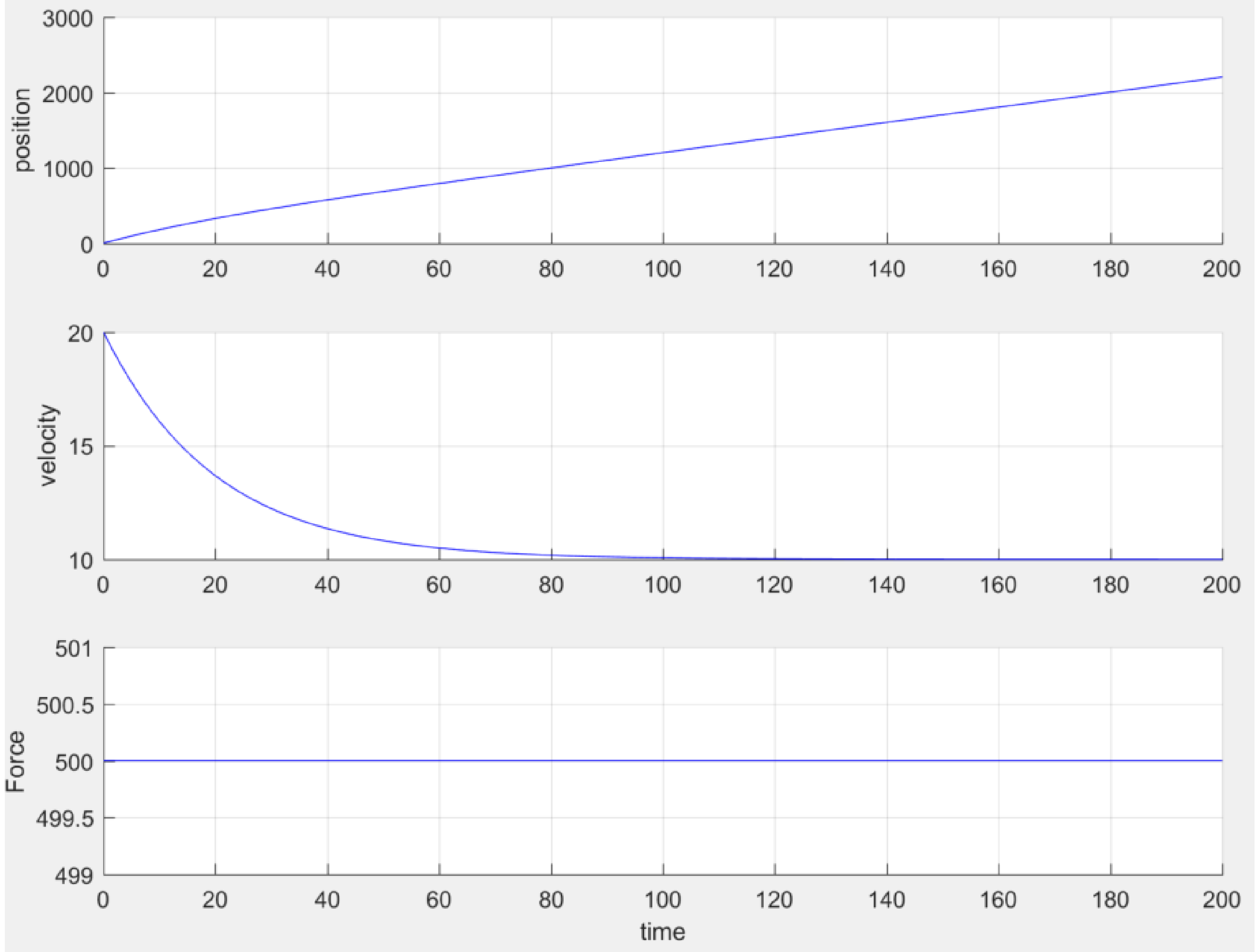
Sample Execution of Car

Suppose $\forall t: F(t) = 0, x_0 = 5\text{ m}, v_0 = 20\text{ m/s}, m = 1000\text{kg}, k = 50Ns/m$

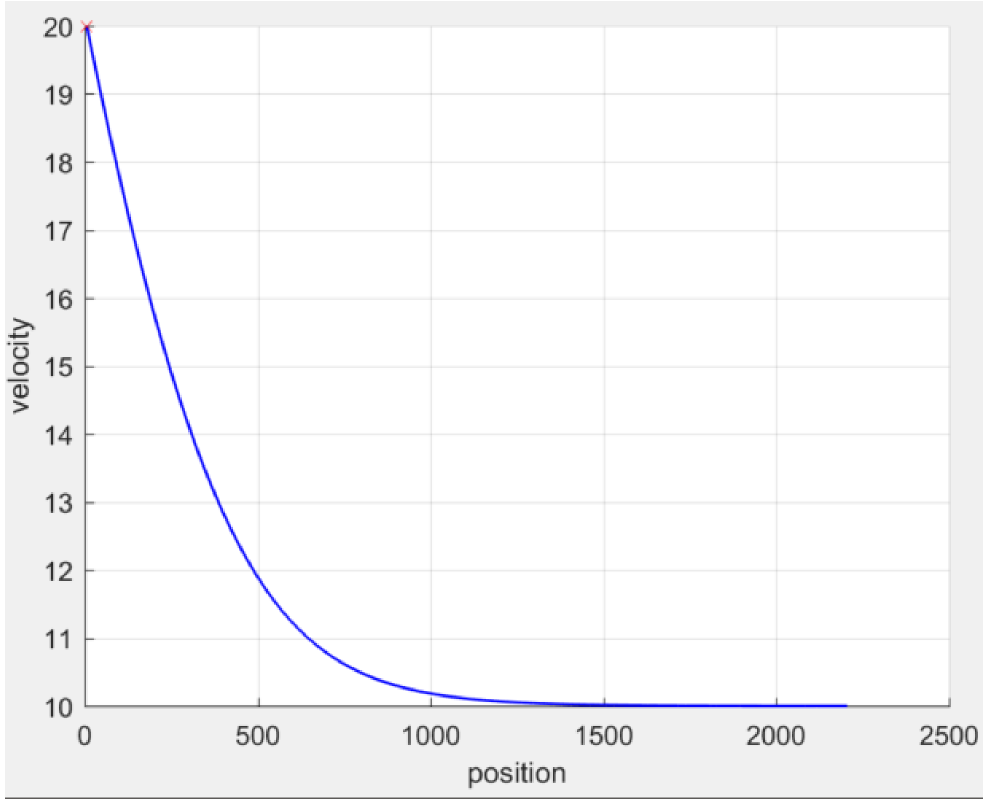
- Then, we need to solve:
  - $x(0) = 5; v(0) = 20$
  - $\dot{x} = v; \dot{v} = -\frac{kv}{m}$

- Solution to above differential equation (solve for v first, then x):
- $v(t) = v_0 e^{-\frac{kt}{m}}; x(t) = \frac{mv_0}{k}\left(1 - e^{-\frac{kt}{m}}\right)$
- Note, as $t \to \infty$, $v(t) \to 0$, and $x(t) \to \frac{mv_0}{k}$

Plots



position
3000
2000
1000
0
0 20 40 60 80 100 120 140 160 180 200
velocity
20
15
10
0 20 40 60 80 100 120 140 160 180 200
Force
501
500.5
500
499.5
499
0 20 40 60 80 100 120 140 160 180 200
time



velocity
20
19
18
17
16
15
14
13
12
11
10
0 500 1000 1500 2000 2500
position

Differential Equation

The state of the system is characterized by state variables, which describe the system. The rate of change is (usually) expressed with respect to time

Example: Temperature equations

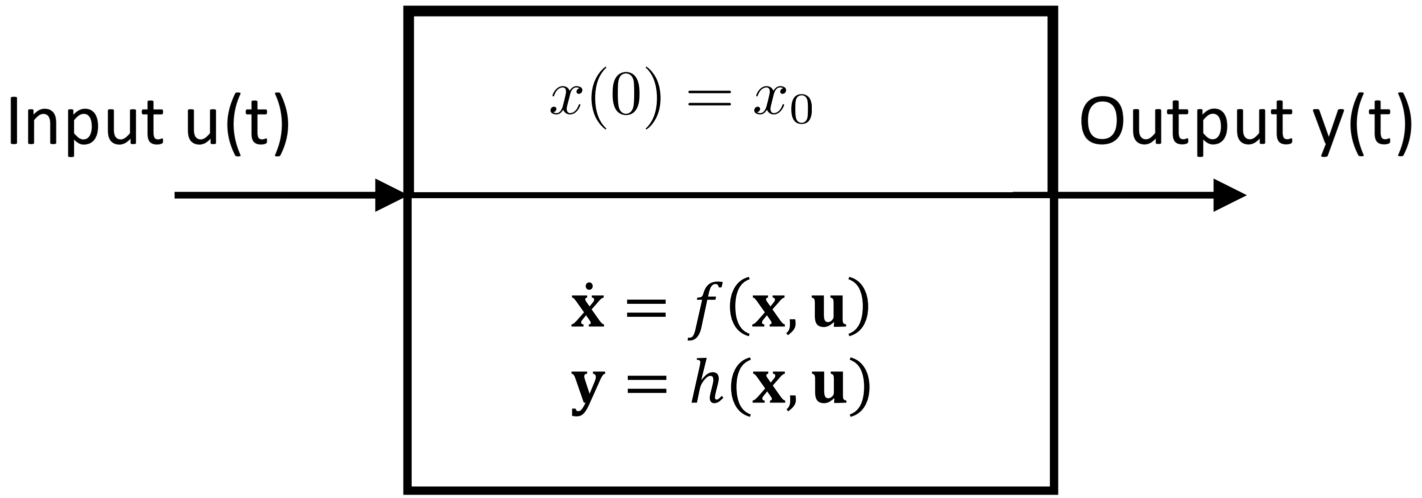
$$\frac{dT}{dt} = -aT + T_{ext} + K_H u$$

Continuous-Time Component Definition

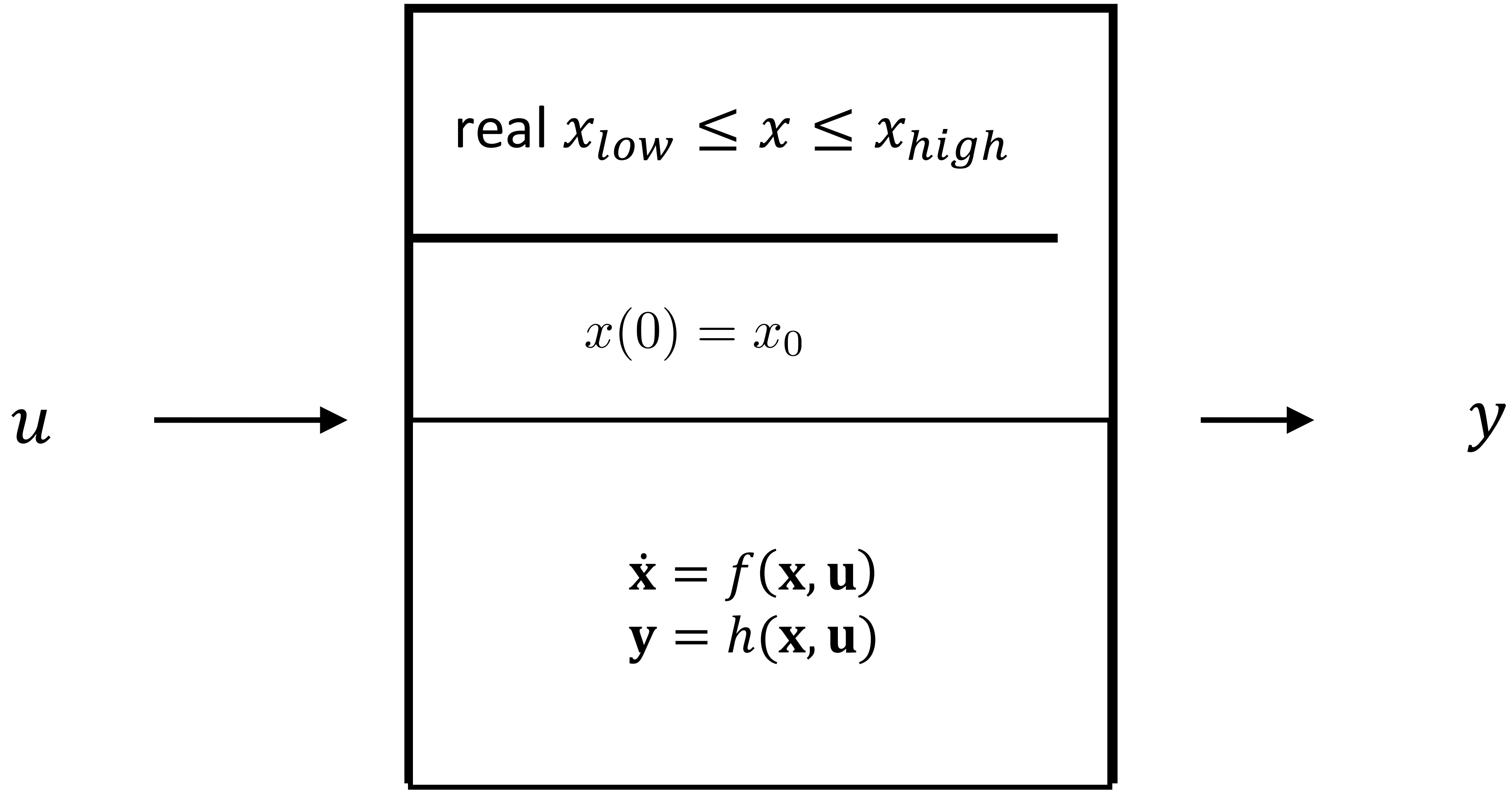
- Set I of real-valued input variables
- Set O or real-valued output variables
- Set X of real-valued (continuous) state variables
- Initialization $Init$ specifying a set X_0 of initial values for states
- Dynamics: for each state variable, x, a real valued expression f over I and X
- Output Function: for each output variable, y, a real valued expression h over I and X.

Execution Definition

- Convention: $\mathbf{x} = (x_1, x_2, \dots x_n), \mathbf{y} = (y_1, y_2, \dots, y_m)$
- Given an input signal $u: \mathbb{T} \to \mathbb{R}$, an execution consists of a *differentiable* state signal $\mathbf{x}(\mathrm{t})$, and an output signal $\mathbf{y}(t)$, such that:
  1. $\mathbf{x}(0) \in X_0$
  2. For each output variable y and time t, $y(t) = h\big(u(t), x(t)\big)$
  3. For each state variable x, $\frac{d}{dt}x(t) = f(u(t), x(t))$



Order Differential Equation



Existence and Uniqueness of Solutions

- Given an input signal $u(t)$, when are we guaranteed that the system has at least one execution? Is there nondeterminism in continuous-time components?

- Input signal should be piecewise-continuous, and additional conditions need to be imposed on the RHS of dynamics (f) and output functions (h)

- Related to solutions for the initial value problem in the classical theory of ODEs

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$$
$$\mathbf{y} = h(\mathbf{x}, \mathbf{u})$$

Existence

- There exists at least one solution $\mathbf{x}(t)$ if the function f is continuous

- Definition of continuity uses notion of distance between points
  - Euclidean distance: $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_2 = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$

- f is continuous if for all $\mathbf{x} \in \mathbb{R}^n$, for all $\epsilon > 0$, there exists a $\delta > 0$, such that for all $\mathbf{y} \in \mathbb{R}^n$, if $\|\mathbf{x} - \mathbf{y}\|_2 < \delta$, then $\|f(\mathbf{x}) - f(\mathbf{y})\|_2 < \epsilon$.

- Example when solution does not *globally* exist:
  - $\frac{dx}{dt} = \begin{cases} 1 \text{ if x} = 0 \\ 0 \text{ otherwise} \end{cases}$
  - $\frac{dx}{dt} = 1/t$

Uniqueness

- Solution to initial value problem is unique if f is Lipschitz continuous
- Lipschitz-continuity is a stronger version of continuity: upper bounds how fast a function can change
- Function f is **Lipschitz-continuous** if there exists a constant L (called the Lipschitz constant) such that:
$$\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n : \|f(\mathbf{x}) - f(\mathbf{y})\| \leq L\|\mathbf{x} - \mathbf{y}\|$$
- Examples:
  - Linear functions (e.g. $x_1 - 3x_2$) are Lipschitz continuous
  - Functions: x^2, $\sqrt{x}$ are not Lipschitz continuous over $\mathbb{R}^n$
- Can restrict $\mathbb{T}$ and X to some bounded and closed set such that f is piecewise-continuous and Lipschitz to get unique solutions over such compact domains

What do numeric solvers/simulators do?

- Allow modeling arbitrarily complex functions: even functions with unbounded discontinuities
- May not be even possible to check for Lipschitz conditions for what's implemented in a Matlab function/Simulink model
- Rely on numerical integration schemes/solvers to obtain solutions
  - ode45, ode23, ode15, etc.

Linear Systems

- Equation of simple car dynamics can be written compactly as:

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -k/m \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [F]$$

- Letting $A = \begin{bmatrix} 0 & 1 \\ 0 & -k/m \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we can re-write above equation in the form:

- $\dot{\mathbf{x}} = A\mathbf{x} + \mathrm{B}\mathbf{u}$, where $\mathbf{x} = [x \quad v]$, and $\mathbf{u} = [F]$

Linear Dynamical Systems

- Special kind of dynamical system

$$\dot{\mathbf{x}} = \boldsymbol{f}(\mathbf{x}, \mathbf{u})$$
$$\boldsymbol{y} = \boldsymbol{h}(\mathbf{x}, \mathbf{u})$$

  - f is of the form $a_1 x_1 + \cdots + a_n x_n + b_1 u_1 + \cdots + b_m u_m$ or compactly, $f = A\mathbf{x} + B\mathbf{u}$
  - h is of the form $c_1 x_1 + \cdots + c_n x_n + d_1 u_1 + \cdots + d_m u_m$ or compactly, $h = C\mathbf{x} + D\mathbf{u}$
- Linear algebra was invented to reason about linear systems!
- Linear systems have many nice properties:
  - Many analysis methods in the frequency domain (using Fourier/Laplace transform methods)
  - Superposition principle (net response to two or more stimuli is the sum of responses to each stimulus)

Solutions to Linear Systems

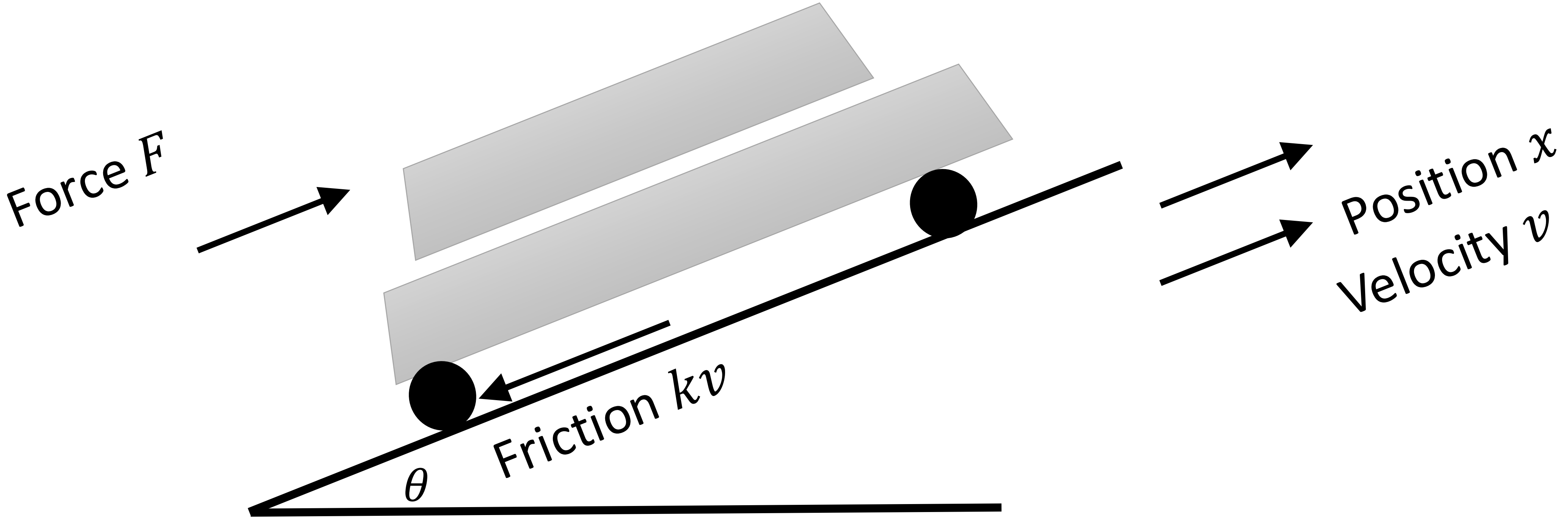
- ***Autonomous*** linear system has no inputs: $\dot{\mathbf{x}} = A\mathbf{x}$

- Solution of autonomous linear system can be fully characterized:
  - $\mathbf{x}(t) = e^{At}\mathbf{x}_0$
  - Computing e^{A} is easy if A is a diagonal matrix (non-zero elements are only on the diagonal)

- For a linear system with ***exogenous*** inputs?
  - $x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$

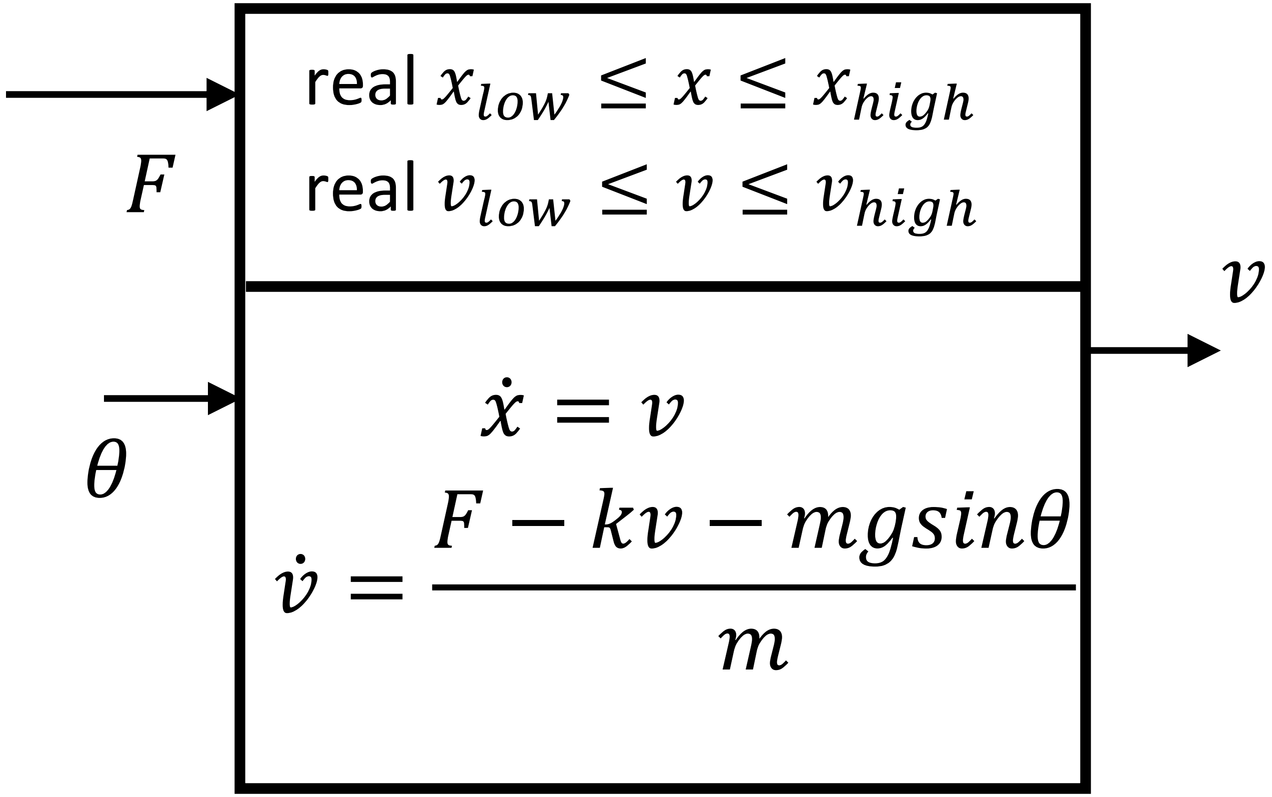
- In practice, numerical integration methods outperform matrix exponential

Model with disturbance



Newton's law of motion: $F \ = m\frac{d^2x}{dt^2} + kv + mg\sin(\theta)$

Model with disturbance



Time Invariant System

The system is time invariant because the output does not depend on the particular time the input is applied.

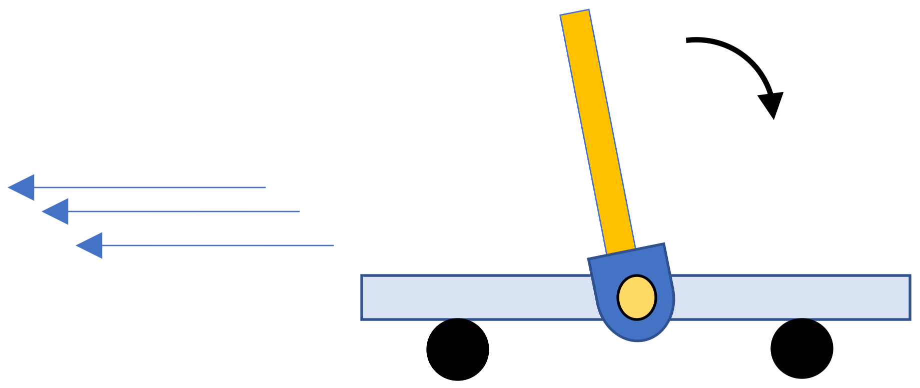
f does not depend on time

$$\frac{dx}{dt} = \dot{x} = f(x, u)$$

The underlying physical laws themselves do not typically depend on time.

Stability of Systems

- Property capturing the ability of a system to return to a quiescent state after perturbation
  - Stable systems recover after disturbances, unstable systems may not
  - Almost always a desirable property for a system design
- Fundamental problem in control: design controllers to *stabilize* a system

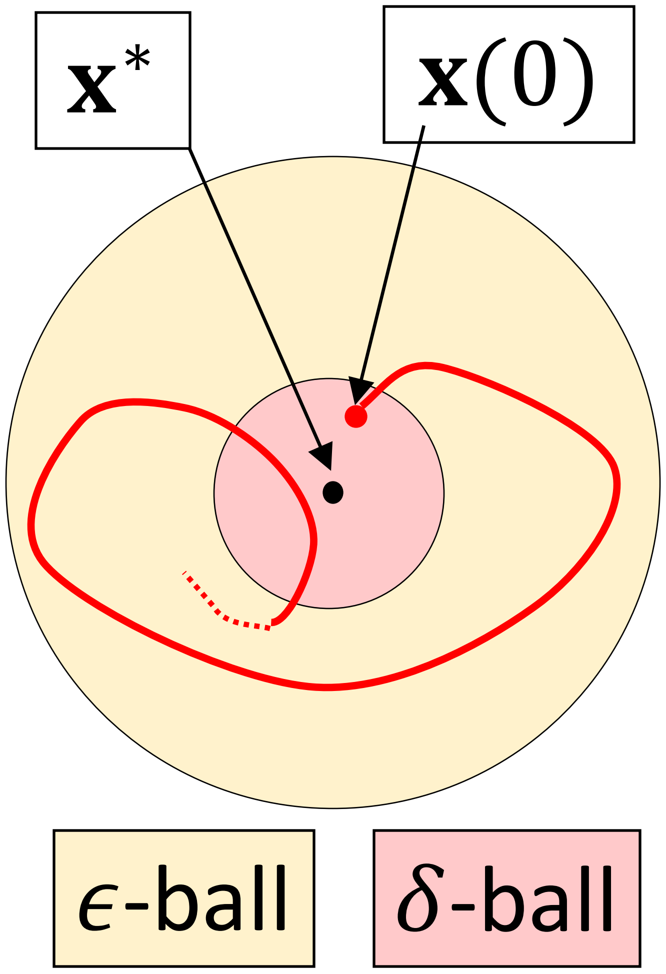


- Problem: Cart-pole is inherently unstable, aim: keep it upright
- Solution Strategy: Move cart in direction in the same direction as the pendulum’s falling direction
- Design problem: Design a controller to stabilize the system by computing velocity and direction for cart travel

Lyapunov stability

Solutions starting δclose from equilibrium point must remain close (within ϵ) forever

- System $\dot{\mathbf{x}} = f(\mathbf{x})$ with f Lipschitz continuous
- Equilibrium point when $f(\mathbf{x})$ is zero (say $\mathbf{x}^*$)
- Equilibrium point $\mathbf{x}^*$ is Lyapunov-stable if:
  - For every $\epsilon > 0$,
    - There exists a $\delta > 0$, such that
      - if $\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$, then,
      - for every $t \geq 0$, we have $\|\mathbf{x}(t) - \mathbf{x}^*\| < \epsilon$



Asymptotic Stability

Solutions not only remain close, but also converge to the equilibrium

- System $\dot{\mathbf{x}} = f(\mathbf{x})$
- Equilibrium point $\mathbf{x}^*$ is asymptotically-stable if:
  - $\mathbf{x}^*$ is Lyapunov-stable +
  - There exists $\delta > 0$ s.t. if $||\mathbf{x}(0) - \mathbf{x}^*|| < \delta$, then $\lim_{t \to \infty} ||\mathbf{x}(t) - \mathbf{x}^*|| = 0$

Exponential Stability

Solutions not only converge to the equilibrium, but in fact converge at least as fast as a known exponential rate

- All stable linear systems are exponentially stable
- This need not be true for nonlinear systems!

- System $\dot{\mathbf{x}} = f(\mathbf{x})$
- Equilibrium point $\mathbf{x}^*$ is exponentially-stable if:
  - $\mathbf{x}^*$ is asymptotically stable +
  - There exist $\alpha > 0, \beta > 0$ s.t. if $\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$, then for all $t \geq 0$:

$$\|\mathbf{x}(t) - \mathbf{x}^*\| \leq \alpha \|\mathbf{x}(0) - \mathbf{x}^*\| e^{-\beta t}$$

Analyzing stability for linear systems

- Eigenvalues of a matrix A:
  - Value λ satisfying the equation $A\mathbf{v} = \lambda\mathbf{v}$. $\mathbf{v}$ is called the eigenvector
  - Equivalent to saying: values satisfying $|A - \lambda I| = 0$, where I is the identity matrix
- Interesting result for linear systems: System $\dot{\mathbf{x}} = A\mathbf{x}$ is asymptotically stable if and only if every eigenvalue of A has a negative real part
- Lyapunov stable if and only if every eigenvalue has non-positive real part
- Nonlinear systems: no simple analysis technique exists
  - Lyapunov's methods allow reasoning about stability of nonlinear systems

Stability analysis example for linear systems

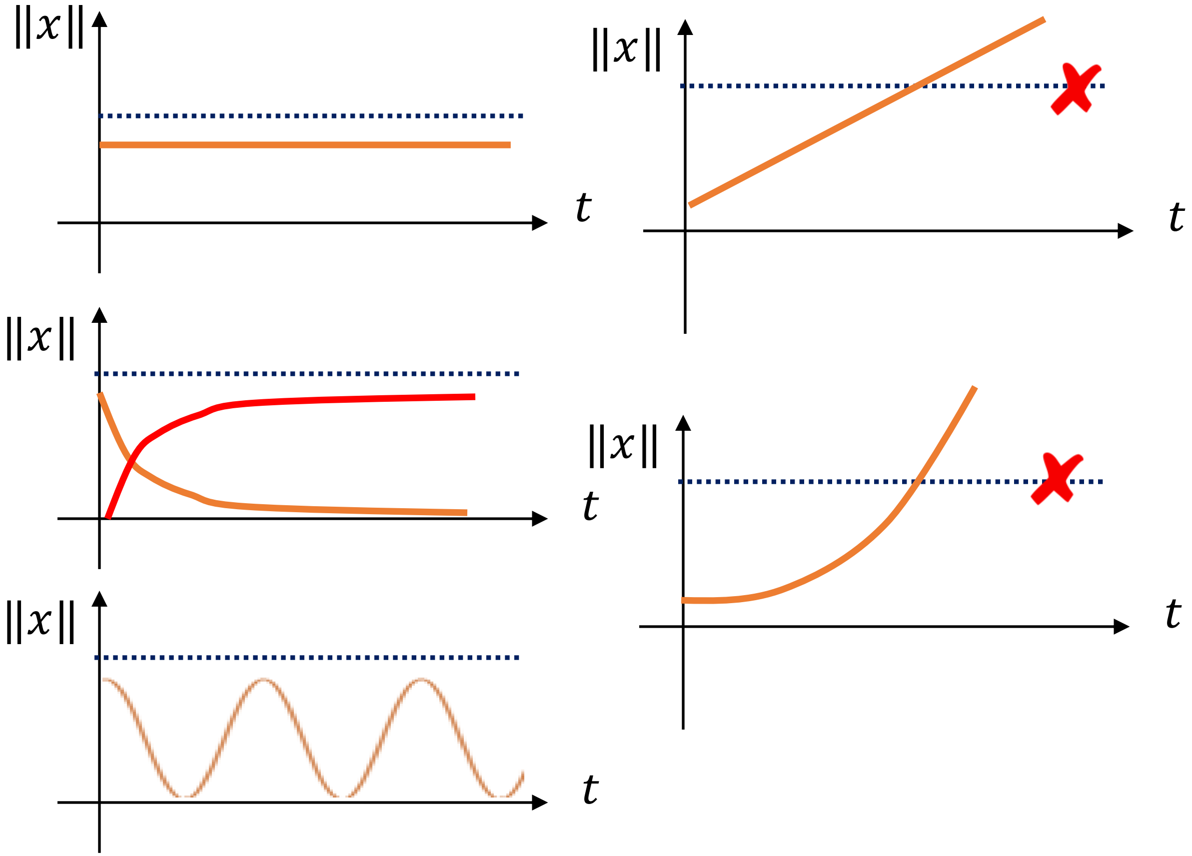
- Manual way: solve the characteristic equation of the matrix A
- $A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}$
- Characteristic equation: $|A \ - \lambda I| = 0$, i.e.
- $\begin{vmatrix} 1-\lambda & -1 \\ 3 & 2-\lambda \end{vmatrix} = 0$, or $(1-\lambda)(2-\lambda)+3=0$
- $(\lambda^2 - 3\lambda + 2 + 3) = 0$
- i.e., $\lambda = \frac{(3 \pm \sqrt{9-4\times 5})}{2} = 1.5 \pm 1.65i$
- Real part is positive $\Rightarrow$ A represents an unstable linear system

Stability analysis example for linear systems

- $A = \begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix}$
- Characteristic equation: $|A \; - \lambda I| = 0$, i.e.
- $\begin{vmatrix} 1-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = 0$, or $(1-\lambda)(-2-\lambda) + 3 = 0$
- $(\lambda^2 + \lambda - 2 + 3) = 0$
- i.e., $\lambda = \frac{(-1 \pm \sqrt{-3})}{2} = -0.5 \pm i\sqrt{3}$
- Real part is negative $\Rightarrow A$ represents a stable linear system

Bounded signals

- A signal $\mathbf{x}$ is bounded if there is a constant c, s.t. $\forall t$: $\|\mathbf{x}(\mathrm{t})\| < \mathrm{c}$
- Bounded signals:
  - Constant signal : $x(t) = 1$
  - Exponential signal: $x(t) = ae^{bt}$, for $b \leq 0$
  - Sinusoidal signals: $x(t) = a\sin \omega t$
- Not bounded:
  - $x(t) = a + bt$ for any $b \neq 0$
  - Exponential signal: $x(t) = ae^{bt}$, for $b > 0$

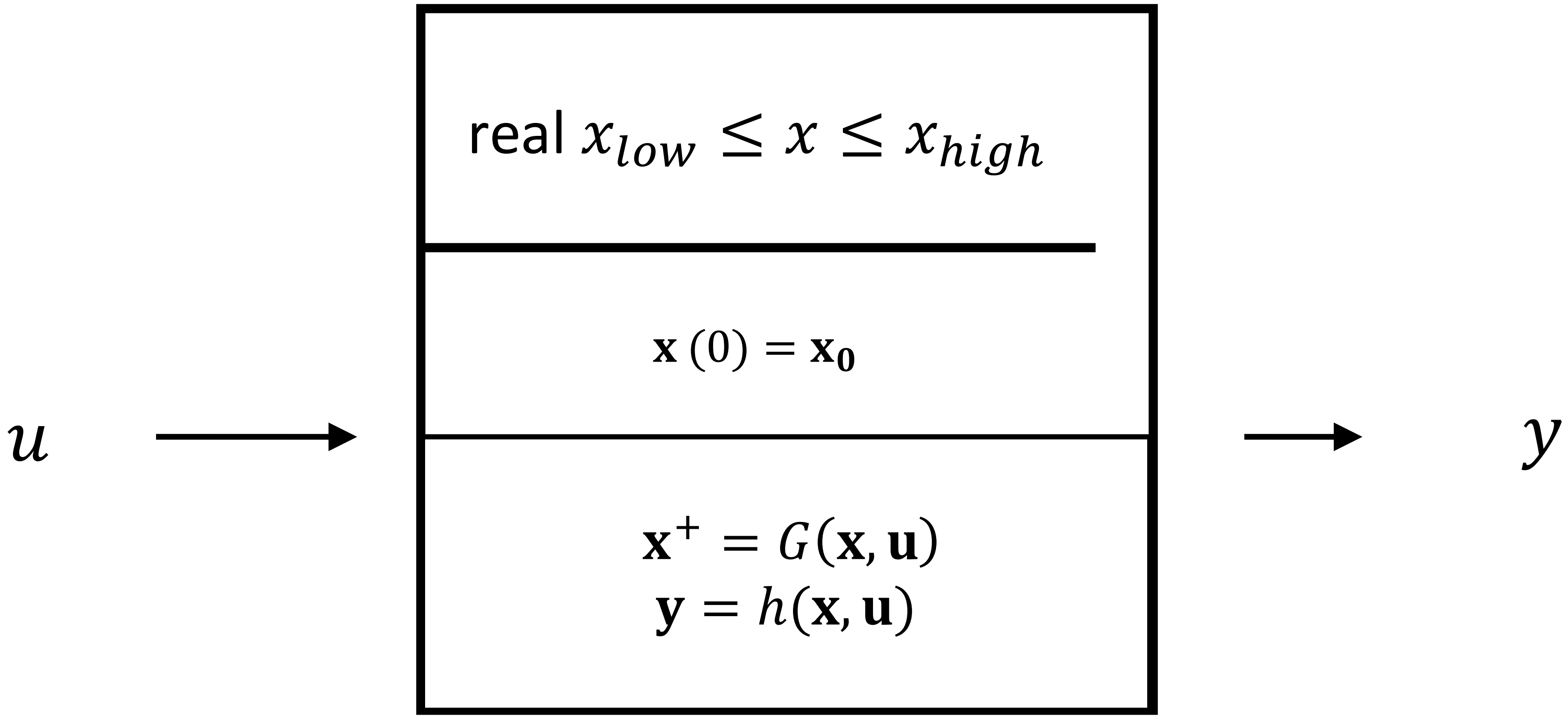


Bounded-Input-Bounded-Output (BIBO) stability

The dynamical system is seen as a transformer, mapping input signals to output signals, and demands that a small change to the input signal should cause only a small change to the output signal.

- A system with Lipschitz-continuous dynamics is BIBO-stable if:
  - For every bounded input $\mathbf{u}(t)$, the output $\mathbf{y}(t)$ from initial state $\mathbf{x}(0) = 0$ is bounded

Difference Equation



$$\mathbf{x}(\mathrm{k}+1) = \mathrm{G}(\mathrm{x}(\mathrm{k}), \mathrm{u}(\mathrm{k}))$$

Difference Equation

u_1, u_2 →

force, angular speed

real $-\pi \leq \theta \leq \pi$

$$x_1(0) = 0$$
$$x_2(0) = 0$$
$$\theta(0) = 0$$

$$x_1^+ = x_1 + d\sin(\theta)u_1$$
$$x_2^+ = x_2 + d\,cos(\theta)u_1$$
$$\theta^+ = \theta + c\,u_2$$
$$y = \theta$$

→ y