10 october

$L^p(\mathbb{R}^d)$ $\quad 1 \le p < \infty$

$\lambda > 0$

$$T_\lambda f(x) = \underline{\chi_{D_{\mathbb{R}^d}(0,1)}\left(\frac{x}{\lambda}\right)} f(x) = \underline{\chi_{D_{\mathbb{R}^d}(0,\lambda)}(x)} f(x)$$

$m \in L^\infty(\mathbb{R}^d)$

$$T_m f(x) = m(x) f(x)$$

$$|T_m f(x)|_{L^p(\mathbb{R}^d)} = |m f|_{L^p(\mathbb{R}^d)} \le |m|_{L^\infty} |f|_{L^p(\mathbb{R}^d)}$$

Exercise Prove that $\|T\|_{\mathcal{L}(L^p)} = |m|_{L^\infty}$

We said that $\lim\limits_{\lambda \to +\infty} \chi_{D(0,\lambda)} f = f \quad \forall f \in L^p$

$$\Updownarrow$$

$$\lim_{\lambda \to +\infty} (1 - \chi_{D(0,\lambda)}) f = 0$$

$$|(1-\chi_{D(0,\lambda)}) f|_{L^p}^p = \int_{\mathbb{R}^d} (1 - \chi_{D(0,\lambda)})^p |f|^p dx \xrightarrow{\lambda \to \infty} 0 \quad *$$

Pointwise $\quad (1 - \chi_{D(0,\lambda)}(x)) |f(x)|^p \xrightarrow{\lambda \to \infty} 0$

$$(1 - \chi_{D(0,\lambda)}(x)) |f(x)|^p \le |f(x)|^p \in L^1(\mathbb{R}^d)$$

$*$ follow from dominated convergence.

so $\quad s\text{-}\lim\limits_{\lambda \to +\infty} \chi_{D(0,\lambda)} = 1$

But it is not true that

$$\lim_{\lambda \to +\infty} \| \chi_{D(0,\lambda)} - 1 \|_{\mathcal{L}(L^p)} \cancel{=} 0 \quad = 1 \ne 0$$

so it is not true that $\chi_{D(0,\lambda)} \xrightarrow{\lambda \to +\infty} 1$ in $\mathcal{L}(L^p)$

This because

$$\| \chi_{D(0,\lambda)} - 1 \|_{\mathcal{L}(L^p)} = | \chi_{D(0,\lambda)} - 1 |_{L^\infty(\mathbb{R}^d)} = 1$$

$$\Delta = \partial_1^2 + \cdots + \partial_n^2 \qquad \partial_{\cancel{x}} \to \partial_{x_1}$$

$$\begin{cases} \partial_t u = \Delta u \\ u(0) = f \in L^p \end{cases} \qquad 1 \le p < +\infty$$

$$u(t) = e^{t\Delta} f \qquad t \geq 0$$

$$e^{t\Delta} f(x) = \frac{1}{(2\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} f(y)\, dy$$

$$\forall f \in L^p \qquad \{ t \to e^{t\Delta} f \} \in C^0([0,+\infty), L^p(\mathbb{R}^d))$$

~~its~~ ~~co~~ $t \to e^{t\Delta}$ is strongly continuous

$$\forall\ t_0 \geq 0 \quad s\text{-}\lim_{t \to t_0} e^{t\Delta} = e^{t_0\Delta} \iff \lim_{t \to t_0} e^{t\Delta} f = e^{t_0 \Delta} f \quad \forall f \in L^p(\mathbb{R}^d)$$

It is not true ~~that~~ $\lim_{t \to t_0} e^{t\Delta} = e^{t_0\Delta}$ in norm

$$\| e^{t\Delta} \|_{\mathcal{L}(L^p)} = 1$$

Spectrum

X B - space $\quad T \in \mathcal{L}(X)$ $\qquad \mathbb{K} = \mathbb{C}$

$\rho(T) \cup \sigma(T) = \mathbb{C}$

$\sigma(T) =: \mathbb{C} \setminus \rho(T)$ spectrum of T

$\rho(T) = \{ \lambda \in \mathbb{C} : T - \lambda$ is invertible and $(T-\lambda)^{-1} \in \mathcal{L}(X) \}$

$\uparrow$ resolvent set

$R_T(z) = (T - z)^{-1}$ is the resolvent of T

<u>Lemma</u> $\rho(T)$ is open in $\mathbb{C}$, $\sigma(T) \subseteq \overline{D_{\mathbb{C}}(0, \|T\|)}$

Examples 1) If λ is an eigenvalue $\ker(T-\lambda) \neq 0$
$\Rightarrow$ $T - \lambda$ is not invertible. So the eigenvalues of T are elements of ~~the spectrum~~ $\sigma(T)$.

2) $\Omega \subseteq \mathbb{R}^d$ $\quad m \in BC^0(\Omega, \mathbb{C})$, in $L^p(\mathbb{R}^d)$ $\quad 1 \leq p < +\infty$

$T_m f(x) = m(x) f(x)$

Then $\sigma(T_m) = \overline{m(\Omega)} \subseteq \mathbb{C}$

T $\quad z \notin \overline{m(\Omega)} \Rightarrow (T_m - z)^{-1}$

while if $z \in \overline{m(\Omega)}$ $\quad (T_m - z)^{-1} \notin \mathcal{L}(L^p(\Omega))$

Lemma $\rho(T)$ is open
$\sigma(T) \subseteq \overline{D_{\mathbb{C}}(0, \|T\|)}$

Lemma If $A \in \mathcal{L}(X)$ with $\|A\| < 1$
then $(1-A)^{-1} \in \mathcal{L}(X)$

Proof We want a candidate for, in fact we wante to show

$$(1-A)^{-1} = \sum_{n=0}^{+\infty} A^n$$

We know $\sum_{n=0}^{+\infty} \|A\|^n = \frac{1}{1-\|A\|}$

Then $\sum_{n=0}^{+\infty} A^n$ is Cauchy

$s_n = \sum_{j=0}^{n} A^j$ $\{s_n\}$ Cauchy in $\mathcal{L}(X)$

$n > m$

$$s_n - s_m = \sum_{j=m+1}^{n} A^j$$

$$\|s_n - s_m\| \leq \sum_{j=m+1}^{n} \|A^j\| \leq \sum_{j=m+1}^{n} \|A\|^j$$

we conclude that $\forall \varepsilon > 0$ $\exists N_\varepsilon$ s.t. $n > m > N_\varepsilon$
$\|s_n - s_m\| < \varepsilon$. Therefore there exists a limit in $\mathcal{L}(X)$

So $\sum_{n=0}^{\infty} A^n \in \mathcal{L}(X)$

Want to show that $(1-A)\sum_{n=0}^{\infty} A^n = 1$

$$\boxed{\lim_{n\to+\infty} (1-A)\, s_n = 1}$$

$\mathcal{L}(X)$

$\mathcal{L}(X) \times \mathcal{L}(X) \longrightarrow \mathcal{L}(X)$
$(T, S) \longrightarrow TS$

$$(1-A)\sum_{j=0}^{n} A^j = \sum_{j=0}^{n} (1-A)A^j = \sum_{j=0}^{n} (A^j - A^{j+1})$$

$$= A^0 - A^{n+1} = 1 - A^{n+1}$$

$$(1-A)\, s_n = 1 - A^{n+1} \xrightarrow{n\to+\infty} 0$$

$\|A\| < 1$

$$\lim_{n\to+\infty} \|A^n\| = 0$$

$$\underbrace{\|A^n\|}_{\downarrow\, 0} \leq \|A\|^n \xrightarrow{n\to+\infty} 0$$

$\sigma(T) \subseteq \overline{D_{\mathbb{C}}(0, \|T\|)}$

$|z| > \|T\| \geq 0$

$T - z = z\left(\frac{T}{z} - 1\right)$

$(AB)^{-1} = B^{-1}A^{-1}$

$(T-z)^{-1} = \frac{1}{z}\left(\frac{T}{z} - 1\right)^{-1}$

$\frac{T}{z} - 1 = -1\left(1 - \frac{T}{z}\right)$

$\left(1 - \frac{T}{z}\right)^{-1} = ?$ ✓

$|z| > \|T\|$

$\left\|\frac{T}{z}\right\| = \frac{1}{|z|}\|T\| < 1$

Neumann series. $\Rightarrow$ $\sigma(T) \subseteq \overline{D_{\mathbb{C}}(0, \|T\|)}$

We show now that $\rho(T)$ is open

$z_0 \in \rho(T)$. We have to show $\exists\ \varepsilon > 0$

s.t. $D_{\mathbb{C}}(z_0, \varepsilon) \subseteq \rho(T)$

$z \in D_{\mathbb{C}}(z_0, \varepsilon)$

$$T - z = T - z_0 + z - z_0 =$$

$$= (T - z_0)\left(1 + (z - z_0)(T - z_0)^{-1}\right)$$

$$(T - z)^{-1} = \left(1 + (z - z_0)(T - z_0)^{-1}\right)^{-1} (T - z_0)^{-1}$$

Want to make

$1 + A$ invertible if $\cancel{A}$

$A = (z - z_0) R_T(z_0)$

If $\|A\| < 1$ then $1 + A$ is invertible

$$\|A\| = \|(z - z_0) R_T(z_0)\| = |z - z_0| \quad \|R_T(z_0)\| < 1$$

$T - z_0$

$$|z - z_0| < \frac{1}{\|R_T(z_0)\|} = \varepsilon$$

$R_T(z_0)$

Lemma $\quad C^{\omega}(\rho(T), \mathcal{L}(X))$

$$z \to R_T(z)$$

Pf $\quad z_0 \in \rho(T)$

We saw that if ~~take~~ $\varepsilon = \dfrac{1}{\|R_T(z_0)\|} > 0$

~~take~~ and if $|z - z_0| < \varepsilon$

$$R_T(z) = \left(1 + (z-z_0) R_T(z_0)\right)^{-1} R_T(z_0)$$

$$\|(z - z_0) R_T(z_0)\| < 1$$

$$= \sum_{n=0}^{\infty} (z-z_0)^n R_T^n(z_0) \quad R_T(z_0)$$

$$= \sum_{n=0}^{\infty} R_T^{n+1}(z_0) \, (z-z_0)^n \qquad \mathcal{L}(X)$$

in $D_{\mathbb{C}}(z_0, \varepsilon)$

Lemma $\sigma(T) \neq \emptyset$. Otherwise $R_T(\cdot) \in C^\omega(\mathbb{C}, \mathcal{L}(X))$

$$R_T(\cdot) \in C^0(\mathbb{C}, \mathcal{L}(X))$$

$$\lim_{z \to \infty} R_T(z) = 0$$

$$R_T(z) = (T-z)^{-1} = \frac{1}{T-z} = \frac{1}{z} \cdot \frac{1}{\frac{T}{z} - 1}$$

$z \to \infty$: $\frac{1}{z} \to 0$, $\frac{1}{\frac{T}{z}-1} \to -1$

$$\Rightarrow R_T(\cdot) \in BC^0(\mathbb{C}, \mathcal{L}(X)) \cap C^\infty(\mathbb{C}, \mathcal{L}(X)) \cap$$

By Liouville theorem $R_T(z) \equiv C_0 = 0$ contradiction

$\sigma_d(T) = \{ \lambda \in \mathbb{C} ;$ λ is an eigenvalue of finite algebraic dimension and λ is an isolated point of $\sigma(T) \}$

$$\dim \operatorname{Ker}(T - \lambda)$$

$N_g(T-\lambda) = \bigcup_{n=1}^{\infty} \operatorname{Ker}(T-\lambda)^n$ $\qquad \dim N_g(T-\lambda) < +\infty$