October 11

$\Omega \subseteq \mathbb{R}^d$ $\qquad 1 \le p < \infty$

$m \in \cancel{B}C^0(\Omega)$

$T_m : L^p(\Omega) \supseteq$

$\sigma(T_m) = \overline{m(\Omega)}$ $\qquad \|T_m\| = \|m\|_{L^\infty(\Omega)}$

$T_m f(x) = m(x) f(x)$

$$|T_m f|_{L^p(\Omega)} = |m f|_{L^p(\Omega)} \le |m|_{L^\infty(\Omega)} |f|_{L^p(\Omega)}$$

$$\Rightarrow \|T_m\| \le \|m\|_{L^\infty(\Omega)}$$

$\forall\ 0 < \alpha < \|m\|_{L^\infty(\Omega)}$

$$|\underbrace{\{x : |m(x)| > \alpha\}}_{\supseteq E}| > 0$$

E bounded $\quad |E| > 0$

$f = c\,\chi_E \qquad c > 0$

$$1 = \|f\|_{L^p} = \|c\chi_E\|_{L^p} = c\,\|\chi_E\|_{L^p} = c\,|E|^{\frac{1}{p}}$$

$$c = |E|^{-\frac{1}{p}}$$

$$\|T_m\| \ge \|T_m f\|_{L^p} = \|\,|m|\,\overbrace{|E|^{-\frac{1}{p}}\chi_E}^{f}\|_{L^p} > \alpha\,\|f\|_{L^p} = \alpha$$

$\|T_m\| > \alpha \quad \forall\ 0 < \alpha < \|m\|_{L^\infty(\Omega)}$

$\sigma(T_m) = \overline{m(\Omega)} \subseteq \overline{D_{\mathbb{C}}(0, \|m\|_{L^\infty})}$

$z \in \mathbb{C} \setminus \overline{m(\Omega)}$ is open

$f \to (m(x) - z)\, f$ is invertible

$\exists\ \varepsilon > 0$ s.t. $D_{\mathbb{C}}(z,\varepsilon) \subseteq \mathbb{C} \setminus \overline{m(\Omega)}$

$f \to \dfrac{1}{m(x) - z} f$

$$\sup\left\{ \frac{1}{|m(x) - z|} ; x \in \Omega \right\} = \frac{1}{\inf\{\underbrace{|m(x)-z|}_{\geq \varepsilon} : x \in \Omega\}} \leq$$

$|m(x) - z| \geq \varepsilon \qquad \leq \dfrac{1}{\varepsilon} < +\infty$

Hence $\varrho(T_m) \supseteq \mathbb{C} \setminus \overline{m(\Omega)}$.

Let us show that if $\hat{z} \in \overline{m(\Omega)} \Rightarrow \hat{z} \notin \varrho(T_m)$

$f \to (m(x) - \hat{z})\, f$

If there is an inverse is the operator

$\dfrac{1}{m(x) - \hat{z}} f$ and this has $\left| \dfrac{1}{m(x) - \hat{z}} \right|_{L^\infty(\Omega)} = +\infty$

Suppose S is the inverse operator

$(m(x) - \hat{z})\, S f(x) = f(x)$

In the points where $m(x) \neq \hat{z}$ we get

$$\boxed{S f(x) = \frac{f(x)}{m(x) - \hat{z}}}$$

Suppose that $m(x) \neq \hat{z}$ a.e.

What if $|\{x : m(x) = \hat{z}\}| > 0$ $\qquad$ E bounded

then $f \to (m(x) - \hat{z})\, f$ has no inverse $\qquad |E| > 0$

$(m(x) - \hat{z})\, \chi_E = 0 \qquad \chi_E \in \ker(m - \hat{z})$

$\hat{z} \in \overline{m(\Omega)}$

$\hat{z} = \lim_{n \to +\infty} m(x_n)$ $\quad$ $\{x_n\}$ a sequence in Ω

$$\left\| \frac{1}{|m(x) - \hat{z}|} \right\|_{L^\infty} = \sup_x \frac{1}{|m(x) - \hat{z}|} = +\infty$$

$$\geq \lim_{n \to +\infty} \frac{1}{|m(x_n) - \hat{z}|} = +\infty$$

$i\partial_1$

$H \xrightarrow{U} L^2(X, d\mu)$

$A \rightsquigarrow m(x)$

$$\boxed{UAf = \underline{m(x)}\,Uf}$$

Thm (Analytic Hahn Banach)

X a vector space on $\mathbb{R}$, $p : X \to \mathbb{R}$ is a seminorm

$Y \subseteq X$ a subspace $\quad g : Y \to \mathbb{R}$ linear

with

$$g(y) \le p(y) \quad \forall\, y \in Y.$$

Then $\exists\, f : X \to \mathbb{R}$ linear s.t. $f|_Y = g$

and

$$f(x) \le p(x) \quad \forall x \in X.$$

Pf Let $x_0 \notin Y$ and consider $\mathbb{R}x_0 + Y \supsetneq Y$

$$f(tx_0 + y) = t f(x_0) + f(y) = t\alpha + f(y)$$

$f(x_0) = \alpha \qquad$ We want $\alpha \in \mathbb{R}$ s.t.

$\mathbb{R}x_0 + Y$

$$f(tx_0 + y) \overset{t\in\mathbb{R}}{\le} p(tx_0 + y)$$

(1) $\quad t\alpha + g(y) \le p(tx_0 + y) \quad \forall\, t \in \mathbb{R} \;\; \forall y \in Y$

$$t\alpha + g(y) \le p(tx_0 + y) \quad \forall t > 0 \quad \forall y \in Y$$

$$\Updownarrow$$

$$\alpha + g\left(\tfrac{y}{t}\right) \le \tfrac{1}{t}\, p(tx_0 + y) = p\left(x_0 + \tfrac{y}{t}\right) \quad \forall y \in Y \;\; \forall t > 0$$

$$\Updownarrow$$

$$\boxed{\alpha + g(y) \le p(x_0 + y) \quad \forall y \in Y}$$

$$t\alpha + g(y) \le p(tx_0 + y) \quad \forall\, t < 0 \quad \forall y \in Y$$

$\frac{1}{|t|}\cdot$

$$-\alpha + g\left(\tfrac{y}{|t|}\right) \le p\left(-x_0 + \tfrac{y}{|t|}\right) \quad \forall\, t < 0 \quad \forall y \in Y$$

$$\boxed{-\alpha + g(y) \le p(-x_0 + y) \quad \forall y \in Y}$$

$$-p(-x_0 + y) + g(y) \le \alpha \le p(x_0 + y) - g(y) \quad \forall y \in Y$$

$$\sup\{-p(-x_0+y) + g(y) : y \in Y\} \le \alpha$$

$$\le \inf\{p(x_0+y) - g(y) : y \in Y\}$$

This holds if

$$\to \quad -p(-x_0 + y_1) + g(y_1) \overset{\le\alpha}{\le} p(x_0 + y_2) - g(y_2) \quad \forall y_1, y_2 \in Y$$

$$g(y_1) + g(y_2) \le p(-x_0 + y_1) + p(x_0 + y_2) \quad \forall y_1, y_2 \in Y \quad \forall x_0$$

$$g(y_1) + g(y_2) = g(y_1 + y_2) \le p(y_1 + y_2) =$$

$$= p((y_1 - x_0) + (x_0 + y_2)) \le p(y_1 - x_0) + p(x_0 + y_2)$$

$\mathcal{P} = \{ (h, D) :$ $Y \subseteq D \subseteq X$ and
$h : D \to \mathbb{R}$ linear s.t.
$h|_Y = g$ and s.t.
$h(x) \le p(x)$ $\forall x \in D \}$

I write
$(h_1, D_1) \le (h_2, D_2)$ if

$D_2 \supseteq D_1$ and $h_2|_{D_1} = h_1$

$\mathcal{P}$ is an inductive set that is, if
$\{(h_q, D_q)\}_{q \in \Gamma}$ is a totally ordered subset of $\mathcal{P}$
then $\exists$ $(\hat{h}, \hat{D})$ s.t. $(h_q, D_q) \le (\hat{h}, \hat{D})$

Just take $\hat{D} = \bigcup_{q \in \Gamma} D_q$

$x_1, x_2 \in \hat{D}$ want $x_1 + x_2 \in \hat{D}$

$x_1 \in D_{q_1}$ $x_2 \in D_{q_2}$. It is not restrictive
to assume $D_{q_1} \subseteq D_{q_2} \Rightarrow x_1, x_2 \in D_{q_2} \Rightarrow$
$\Rightarrow x_1 + x_2 \in D_{q_2} \subseteq \hat{D}$

$\forall x \in \hat{D}$ let D_q be s.t. $x \in D_q$ then
$\hat{h}(x) = h_q(x)$

By Zorn's lemma $\mathcal{P}$ has
maximal elements. $\exists$ $(f, \mathcal{D})$
s.t. $(h, D) \ge (f, \mathcal{D}) \Rightarrow (h, D) = (f, \mathcal{D})$

$f(x) \le p(x)$ $\forall x \in \mathcal{D}$ script D
$\qquad\qquad\qquad\qquad \cup|$
$f|_Y = g$ $\qquad\qquad Y$

We need to show $\mathcal{D} = X$
If not $\exists$ $x_0 \notin \mathcal{D}$ and by proceeding
like above there is a further extension
$(\tilde{f}, \mathbb{R}x_0 + \mathcal{D}) \gneq (f, \mathcal{D})$

Corollary $(X, \|\cdot\|)$ $Y \leq X$ a vect. subspace.

$\mathbb{K} = \mathbb{R}, \mathbb{C}$

$g \in Y'$. $\exists f \in X'$ s.t. $f|_Y = g$

$\|f\|_{X'} = \|g\|_{Y'}$

Brothier

$X' = \underline{\mathcal{L}(X, K)}$

Example $L^p(I)$ $0<p<1$

$(L^p(I))' = 0$. Let $f \neq 0$ in $L^p(I)$

$Y = \operatorname{Span}\{f\} \approx \mathbb{R}$

$\exists g \neq 0$ $g \in Y'$ but $\cancel{\exists}$ $f \in (L^p(I))'$
f extension of g