October 17

Corollary X normed space, $Y \subset X$ a vector subspace. $g : Y \to K$ continuous linear function

Then $\exists\ f \in X'$ s.t. $f|_Y = g$ and

$$\|f\|_{X'} = \sup_{x \in D_X(0,1)} |f(x)| = |g|_{Y'} = \sup_{y \in D_Y(0,1)} |g(y)|$$

Pf $K = \mathbb{R}$ $p : X \to [0,+\infty)$

$p(x) = \|g\|_{Y'} \|x\|_X$

Then by $|g(y)| \le \|g\|_{Y'} \underbrace{\|y\|_Y}_{=\|y\|_X}$ $\forall y \in Y$

we get also $g(y) \le \|g\|_{Y'} \|y\|_Y = p(y)$ $\forall y \in Y$

$$g(y) \le p(y) \quad \forall y \in Y.$$

$\exists\ f : X \to \mathbb{R}$ s.t. $f|_Y = g$ and

$$f(x) \le p(x) \quad \forall x \in X$$

$$(1) \quad f(x) \le \|g\|_{Y'} \|x\|_X \quad \forall x \in X$$

This implies $|f(x)| \le \|g\|_{Y'} \|x\|_X$ (2)

(1) $\Leftrightarrow$ (2) if $f(x) \ge 0$

If $f(x) < 0$ we know that 1) holds

for $-x$ $\quad \underset{-f(x) = |f(x)|}{f(-x)} \le \|g\|_{Y'} \|\cancel{-}x\|_X$

$K = \mathbb{C}$ $\quad g : Y \to \mathbb{C}$ $\quad u(y) = \operatorname{Re} g(y)$

$u : Y \to \mathbb{R}$

If $U : X \to \mathbb{R}$ is the extension provided by first part of the proof then

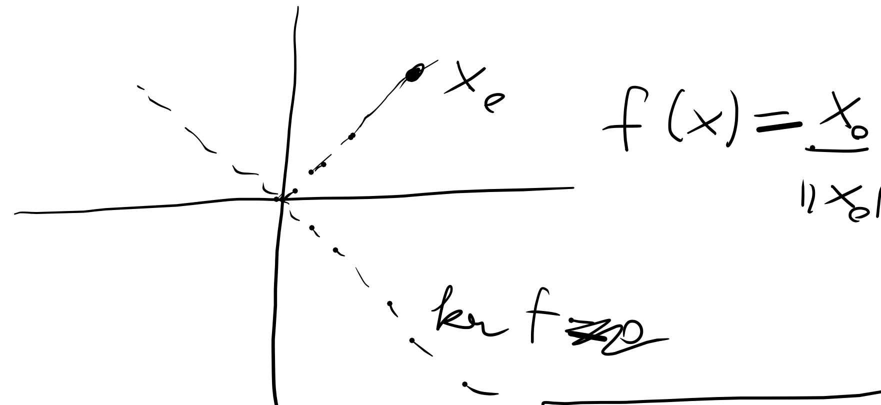
$$f(x) = U(x) - i\,U(ix)$$

Corollary ~~$\forall$~~ normed space $\|\cdot\|$, $x_0 \neq 0$ $x_0 \in X$.

$\exists$ $f \in X'$ $\|f\|_{X'} = 1$ s.t. $f(x_0) = \|x_0\|$.

Pf $Y = \mathbb{R}x_0$

$g(t x_0) = t\,\|x_0\|$



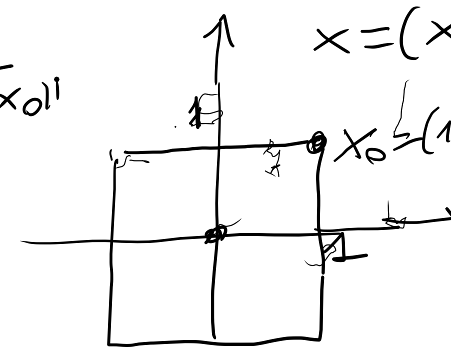
$f(x) = \dfrac{x_0 \cdot x}{\|x_0\|_{\mathbb{R}^2}}$

$\|x\|_{\mathbb{R}^2} = \sqrt{x_1^2 + x_2^2}$

$$\|g\|_{Y'} = \sup_{t x_0 \in D_{Y'}(0,1)} |g(t x_0)| =$$

$$= \sup_{t x_0 \in D_{Y'}(0,1)} |t|\,\|x_0\| = \sup_{|t| \le \frac{1}{\|x_0\|}} t\|x_0\| = 1$$

$x = (x_1, x_2)$ $\mathbb{R}^2$



$\|x\| = \max\{|x_1|, |x_2|\}$

$\|t x_0\| \le 1$

$|t| \le \dfrac{1}{\|x_0\|}$

We know $\exists$ $f \in X'$ s.t. $f|_Y = g$

$\|f\|_{X'} = \|g\|_{Y'} = 1$

$f(x_0) = g(x_0) = 1.$

Example Let $T: X \to Y$ continuous between normed spaces. Then $T': Y' \to X'$ remains defined

T^*

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[dr, "x'"'] & Y \arrow[d, "y'"] \\ & K \end{tikzcd}$$

$x' = y' \circ T = T'y'$

$$y'(Tx) = \langle Tx, y' \rangle_{Y \times Y'} = \langle x, T'y' \rangle_{X \times X'} = T'y'(x)$$

$(y' \circ T(x))$

$$\|T\|_{\mathcal{L}(X,Y)} = \|T'\|_{\mathcal{L}(Y',X')}$$

$\|y'\|_{Y'} = 1$

$$\|T'y'\|_{X'} = \sup\{ \langle T'y', x \rangle_{X' \times X} : \|x\|_X = 1 \} =$$

X'

$$= \sup\{ \langle y', Tx \rangle_{Y' \times Y} : \|x\|_X = 1 \}$$

$$\leq \sup\{ \underbrace{|y'|_{Y'}}_{1} \, |Tx|_Y : \|x\|_X = 1 \} = \|T\|_{\mathcal{L}(X,Y)}$$

$|T'y'|_{X'} \leq \|T\|_{\mathcal{L}(X,Y)}$ $\forall$ $y' \in Y'$ s.t. $\|y'\|_{Y'} = 1$

$$\Rightarrow \|T'\|_{\mathcal{L}(Y',X')} \leq \|T\|_{\mathcal{L}(X,Y)}$$

Take $\|x\|_X = 1$

$$\|Tx\|_Y = \sup\{ \langle Tx, y' \rangle_{Y \times Y'} : \|y'\|_{Y'} = 1 \} =$$

$$\langle Tx, y' \rangle_{Y \times Y'} \leq \|Tx\|_Y \, \|y'\|_{Y'}$$

$$\sup\{ \langle x, T'y' \rangle_{X \times X'} : \|y'\|_{Y'} = 1 \}$$

$$\leq \sup\{ \overbrace{\|x\|_X}^{1} \, \|T'\|_{\mathcal{L}(Y',X')} \overbrace{\|y'\|_{Y'}}^{1} : \|y'\|_{Y'} = 1 \}$$

$$= \|T'\|_{\mathcal{L}(Y',X')}$$

$$\|Tx\|_Y \leq \underbrace{\|T'\|_{\mathcal{L}(Y',X')}}$$ $\forall$ $\|x\|_X = 1$

$$\|T\|_{\mathcal{L}(X,Y)} \leq$$

Rudin
Real and Complex Analysis

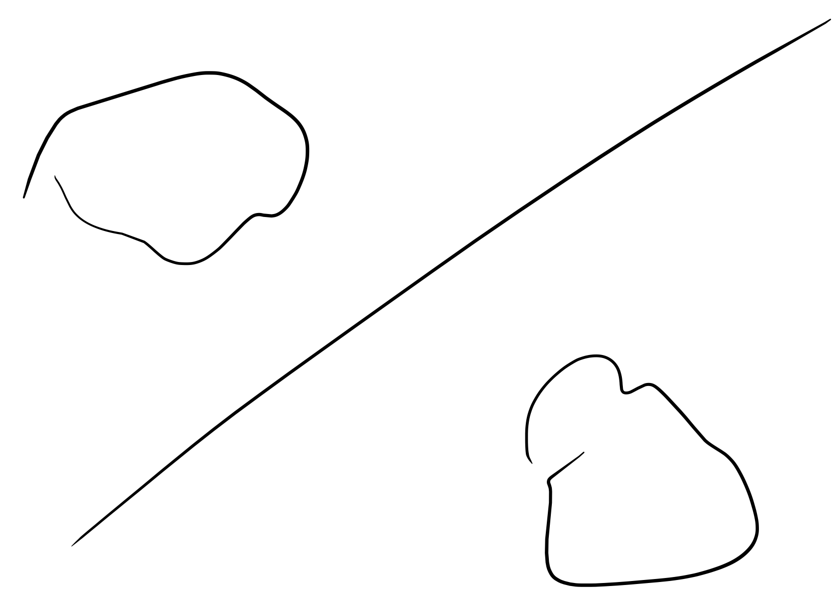
Def Let A, B subsets of a TVS X on $\mathbb{R}$

Let $H = f^{-1}(a)$ $f: X \to \mathbb{R}$ linear

be a Hyperplane.

1) H separates A and B if we can take f s.t. that

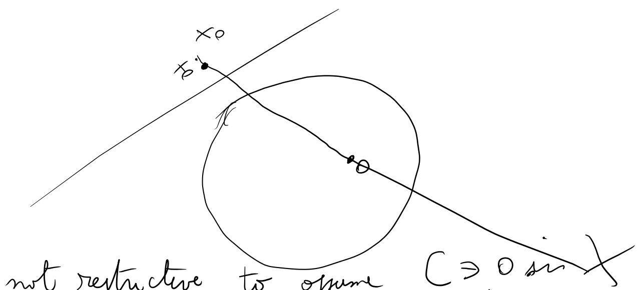
$$f(A) \subseteq (-\infty, a] \text{ and } f(B) \subseteq [a, +\infty)$$



2) H separates strictly A and B if $\exists\, \varepsilon > 0$ s.t.

$$f(A) \subseteq (-\infty, a-\varepsilon] \text{ and } f(B) \subseteq [a+\varepsilon, +\infty)$$

Lemma Let X be a TVS on $\mathbb{R}$, C open convex subspace and $x_0 \notin C$. Then $\exists$ a continuous $f : X \longrightarrow \mathbb{R}$ s.t. $f(x) < f(x_0)$ $\forall x \in C$



Pf It is not restrictive to assume $C \ni 0$ since X

Let p be the Minkowski seminorm associated to C.

$x \in C \iff p(x) < 1$

$x_0 \notin C \implies p(x_0) \geq 1$.

Let $Y = \text{span}\{x_0\} = \mathbb{R} x_0$

$g : Y \to \mathbb{R}$ $\quad g(t x_0) = t$ $\quad \boxed{g(x_0) = 1 \leq p(x_0)}$

$\implies g(t x_0) \leq p(t x_0)$ $\quad \forall\, t \in \mathbb{R}$

$t > 0$ $\quad t\, g(x_0) \leq t\, p(x_0)$

$t < 0$ $\quad g(t x_0) = t < 0 \leq p(t x_0)$

$g(y) \leq p(y)$ $\quad \forall y \in Y$

$\implies f : X \longrightarrow \mathbb{R}$ $\quad f|_Y = g$

s.t. $f(x) \leq p(x)$ $\quad \forall\, x \in X$

$\forall\, x \in C$ $\quad f(x) \leq p(x) < 1 \leq f(x_0) = g(x_0)$

$f^{-1}(1)$ is a hyperplane separating, not strictly $\{x_0\}$ and C

$\underset{H}{f^{-1}(1)} \cap C = \emptyset$ $\quad H$ is not dense

$\implies H$ is closed.

$\implies f$ is continuous

If $x \in H$ $\quad \ker f = H - x$

Theorem (H-B, geometric form)

X T.V.S, A and B convex disjoint subsets, A open.

Then $\exists$ H, closed hyperplane, which separates them

Pf $C \doteq A - B = \{ a - b : a \in A, b \in B \}$

C is convex

$a_0 - b_0$ $\qquad$ $\underset{A}{a_t} = (1-t) a_0 + t a_1 \quad 0 \le t \le 1$

$a_1 - b_1$ $\qquad$ $\underset{B}{b_t} = (1-t) b_0 + t b_1 \quad$ ''

$C \ni (1-t)(a_0 - b_0) + t(a_1 - b_1) = a_t - b_t \quad \forall \ t \in [0,1]$

By $C = \bigcup_{b \in B} (A - b)$, C is open

$A \cap B = \phi \Rightarrow 0 \notin C$.

By the lemma $\exists$ $f \in X'$ s.t

$f(c) \le f(0) = 0 \qquad \forall c \in C$

$f(a-b) \le 0 \qquad \forall a \in A, b \in B$

$f(a) \le f(b) \Rightarrow \exists \ \alpha$

s.t $f(a) \le \alpha \le f(b) \quad \forall a \in A, \forall b \in B$

$H = f^{-1}(\alpha)$

Theorem (H-B, Geom form, 2° version)

X locally convex TVS. A, B convex disjoint A closed B compact. $\exists$ a closed hyperplane H separating them strictly.

Proof (only for X normed)

If X is normed we claim

$\exists\ \varepsilon > 0$ s.t. $\left(A + D_X(0,\varepsilon)\right) \cap \left(B + D_X(0,\varepsilon)\right) = \phi$

If not for any sequence $\varepsilon_n \to 0$ $\exists$ z_n belonging to the intersection and there are sequences $\{a_n\}$ in A and $\{b_n\}$ in B s.t. $\| z_n - a_n \|_X < \varepsilon$

$\| z_n - b_n \|_X < \varepsilon$

$\| a_n - b_n \|_X < 2\varepsilon$

Since B is compact $\{b_n\}$ has a convergent subsequence. It is not restrictive to assume $b_n \to b \in B$

$a_n \nearrow$

Since A is closed $b \in A$

$\phi = A \cap B \ni \{b\}$ contradiction

so $\exists\ \varepsilon > 0$ s.t.

$\left(A + D_X(0,\varepsilon)\right) \cap \left(B + D_X(0,\varepsilon)\right) = \phi$

Let $H = f^{-1}(\alpha)$ for $f \in X'$ separate

$A + D_X(0,\varepsilon)$ and $B + D_X(0,\varepsilon)$

Show that H separates strictly A and B.

$\mathbb{R}^2$ $A = \{(x,y) : y \geq \frac{1}{x}, x > 0\}$

$B = \{(x,y) : y \leq 0\}$