993SM - Laboratory of Computational Physics
IV week
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Random numbers and Monte Carlo(*) Techniques

(*) any procedure making use of random numbers

Exercise I: Linear Congruential Method: periodicity

$$I_{n+1} = (a\, I_n + c) \bmod m$$

(b) Test the program with $I_0 = 3$, $a=4$, $m=9$, $c=1$. Which is the interval over which the numbers are generated? Are ALL the numbers over the interval generated?

☆ Choice of multiplier, a

It was proven by M. Greenberger in 1961 that the sequence will have period m, if and only if:

c=1, ok	←	i) c is relatively prime to m;
a-1=3, 3 is a prime dividing 9: ok	←	ii) a-1 is a multiple of p, for every prime p dividing m;
this does not apply, ok	←	iii) a-1 is a multiple of 4, if m is a multiple of 4

ALL the numbers in [0,m-1] are generated, irrespectively on the choice of the seed I_0

Exercise I:
Linear Congruential Method: **periodicity**

$$I_{n+1} = (a\, I_n + c) \bmod m$$

(d) Test the program with $I_0 = 1$, a=3, m=32, c=4. Determine the period, that is, how many numbers are generated before the sequence repeats. Change and try with I_0. Does the period depends on I_0 ?

☆ Choice of multiplier, a

It was proven by M. Greenberger in 1961 that the sequence will have period m, if and only if:

c=4, m=32 ! NO! ←

- i) c is relatively prime to m;
- ii) a-1 is a multiple of p, for every prime p dividing m;
- iii) a-1 is a multiple of 4, if m is a multiple of 4

ONLY SOME numbers in [0,m-1] are generated, depending on the choice of the seed I_0

Exercise I:
Linear Congruential Method: **periodicity**

$$I_{n+1} = (a\, I_n + c) \bmod m$$

(e) Run the program with I_0=10, a=57, m=256, c=1. Determine the period. (how ? "by hands" ? Can the program itself do the job for you?)

```
integer :: i, number, old, seed, x, a,m,c, check

do i = 1, m !disgregard the first m numbers
   x = mod ((a*old+c), m)
   WRITE (unit=1,fmt=*) x
   old = x
end do
 check = old

 do i = 1, number
   x = mod ((a*old+c), m)
   WRITE (unit=1,fmt=*) x
   old = x
   if (old==check)then
   print*,'period is:',i;stop
   end if
end do
```

Added!

Exercise I: Linear Congruential Method: periodicity

$$I_{n+1} = (a\, I_n + c) \bmod m$$

(e) Run the program with I_0=10, a=57, m=256, c=1. Determine the period. (how ? "by hands" ? Can the program itself do the job for you?)

☆ Choice of multiplier, a

It was proven by M. Greenberger in 1961 that the sequence will have period m, if and only if:

i) c is relatively prime to m; → c=1, ok

ii) a-1 is a multiple of p, for every prime p dividing m; → a-1=56=2^3*7; m=56=2^8 => m multiple of 2, prime dividing m: ok

iii) a-1 is a multiple of 4, if m is a multiple of 4 → a-1=56=4*14; m=4*64 => ok

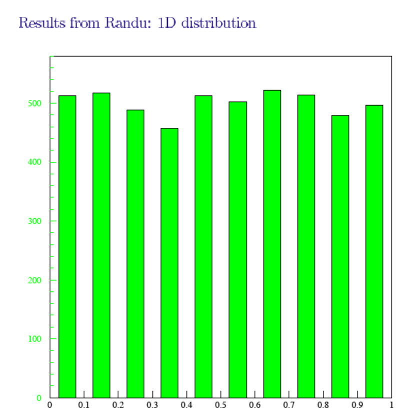
ALL the numbers in [0,m-1] are generated, irrespectively on the choice of the seed I_0

Exercise 2:
test of **uniformity** of the pseudorandom sequence

r(n), n=1, data is our random number sequence between 0 and 1

(b) Do a histogram with the sequence generated above and plot it using for instance **gnuplot** with the command `w[ith] boxes`. Is the distribution *uniform*?
Hint: to do the histogram, divide the range into a given number of channels of width Δr*, then calculate how many points fall in each channel,* $r/\Delta r$*:*

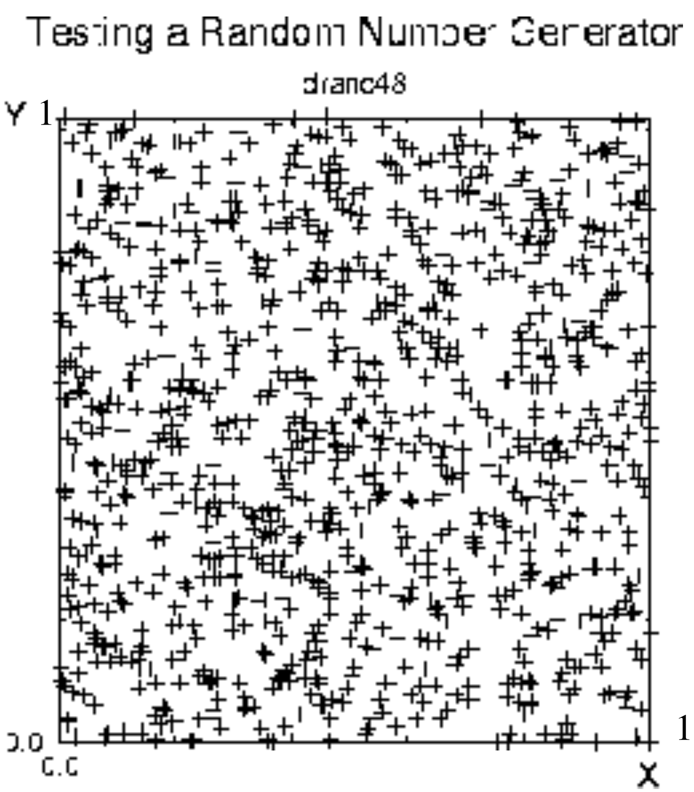
```
integer, dimension(20) :: histog
 :
histog = 0
do n = 1, ndata
   i = int(r(n)/delta_r) + 1
   histog(i) = histog(i) + 1
end do
```



<= counts the number of points falling between i*delta_r and (i+1)*delta_r and assign them to the “i+1” channel

Exercise 2:
intrinsic random number generator - test **correlations**

$$(x_i, y_i) = (r_{2i-1}, r_{2i}) \qquad i = 1, 2, 3....$$



write every 2

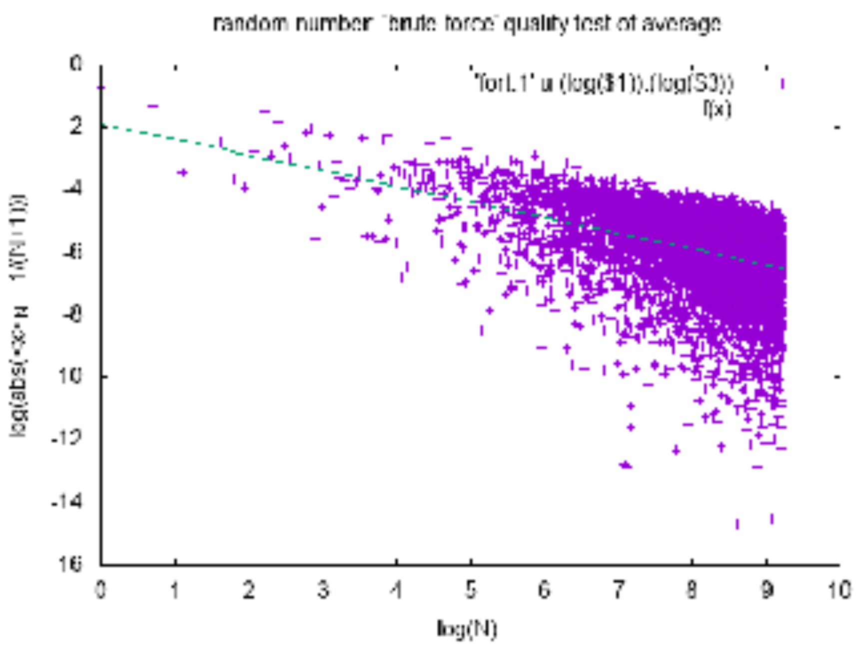
```
do i = 1, number/2, 2
   write(1,*)rnd(i),rnd(i+1)
   end do
```

Exercise 3:
intrinsic random number generator - test **uniformity**

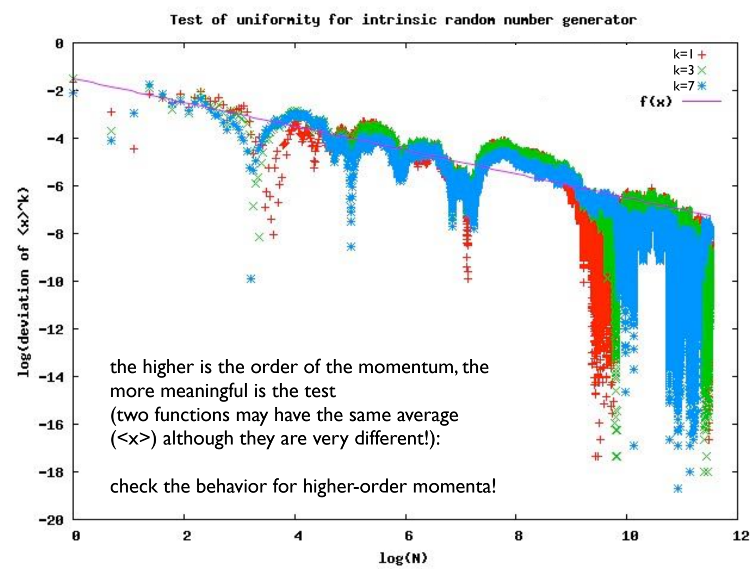
$$\langle x^k \rangle^{calc} = \frac{1}{N}\sum_{i=1}^{N} x_i^k, \qquad \langle x^k \rangle^{th} = \int_0^1 dx\; x^k\; p_u(x) = \frac{1}{k+1}$$

$$\Delta_N(k) = \left| \langle x^k \rangle^{calc} - \langle x^k \rangle^{th} \right| \sim 1/\sqrt{N}.$$

$$\text{If } f(x) \sim 1/\sqrt{N} \Longrightarrow log(f(x)) \sim -\frac{1}{2}\, log(N)$$



Test on one sequence, several momenta



Exercise 3:
intrinsic random number generator - test **correlation**

$$C(k)^{calc} = \frac{1}{N} \sum_{i=1}^{N} x_i x_{i+k}$$

if N is the length of the sequence:

$$C(k)^{calc} = \frac{1}{N-k} \sum_{i=1}^{N-k} x_i x_{i+k}$$

this is correct!

$$\Delta_N(k) = \left|C(k)^{calc} - 1/4\right| \sim 1/\sqrt{N}.$$

fit raw data or their log?

fit raw data :

```
gnuplot> f(x)=a*x**(-b)+c
gnuplot> b=0.5
gnuplot> c=0
gnuplot> fit f(x) ‘[file]’ u … via a,b
```

A power low fit!

do you want to fit with gnuplot?

Suppose you have the data in two columns, x and y, and you suspect a power low $y = x^a + const$

Consider that: $\log(y) = a * \log(x) + b$

```
gnuplot> f(x) = a * x + b

gnuplot> fit f(x) 'data.dat' u (log($1)):(log($2)) via a,b

gnuplot> plot f(x), 'data.dat' u (log($1)):(log($2))
```

a linear fit is better!

More exercises:

suggestions of “good” LCM generators

G1. LCG with $m = 2^{31} - 1$ and $a = 16807$. and c = 0

G2. LCG with $m = 2^{31} - 1$ and $a = 630360016$. and c = 0

G3. LCG with $m = 2^{31} - 1$ and $a = 742938285$. and c = 0

from: 10.1145/167293.167354