October 18

Corollary Y a vector subspace of a locally convex TVS X s.t. $\overline{Y} \subsetneq X$. Then $\exists\, f \in X'$ nontrivial s.t.

$$f(y) = 0 \quad \forall\, y \in Y.$$

Pf Let $x_0 \notin \overline{Y}$ $\quad x_0 \in X$

$A = \overline{Y}$ $\quad B = \{x_0\}$. Then $\exists\, f \in X'$ and an $\alpha \in \mathbb{R}$ s.t.

$$f(y) < \alpha < f(x_0) \qquad \forall\, y \in \overline{Y}$$

$f(y) < \alpha \quad \forall\, y \in \overline{Y} \implies f(y) = 0 \quad \forall\, y \in \overline{Y}$

Because if I had $f(y_0) \neq 0$ for some $y_0 \in \overline{Y}$ then

$$\sup_{t \in \mathbb{R}} f(t y_0) \leq \alpha$$

$$= \sup_{t \in \mathbb{R}} t\, f(y_0) = +\infty$$

Thm (Müntz-Szasz)

Let $I=[0,1]$ and consider a sequence

$0<\lambda_1<\lambda_2<\dots\quad \lambda_n \xrightarrow{n\to+\infty} +\infty$. In $C^0(I)$

Let $Y=\overline{sp\{1,t^{\lambda_1},t^{\lambda_2},\dots\}}$

1) If $\sum_{n=1}^{+\infty}\frac{1}{\lambda_n}=+\infty$ then $Y=C^0(I)$

2) If $\sum_{n=1}^{+\infty}\frac{1}{\lambda_n}<+\infty$ then for $\underset{\lambda>0}{\lambda\notin\{\lambda_1,\lambda_2,\dots\}}$

$t^{\lambda}\notin Y$.

Remark $\lambda_1=1,\ \lambda_2=2\ \dots$

Pf of 1) (Sketch)

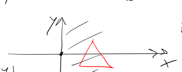
$(C^0(I))'$ is the space of Borel measures in I.

Under the hypothesis (1) we will sketch the fact that $Y=\overline{sp\{1,t^{\lambda_1},t^{\lambda_2}\dots\}}$

$$\int_I t^{\lambda_n}\,d\mu(t)=\int_I 1\,d\mu(t)=0 \implies \mu\equiv 0$$

$$f(z)\doteq\int_I t^z\,d\mu(t) \qquad \operatorname{Re} z>0$$

$z=x+iy$



$$\left|\int_I t^z\,d\mu(t)\right|\le\int_0^1\left|t^{x+iy}\right|\,d|\mu|(t) \qquad |t^{iy}|=1$$

$$=\int_0^1 t^x\,d|\mu|(t)\le\int_0^1 d|\mu|<\infty$$

Using Morera Theorem

$$\int_{\partial\Delta}f(z)\,dz=\int_{\partial\Delta}\left(\int_I t^z\,d\mu(t)\right)dz=$$

$$=\int_I\left(\int_{\partial\Delta}t^z\,dz\right)d\mu=0$$

$\implies$ f is holomorphic.

$U=D_{\mathbb{C}}(0,1)\longrightarrow\{w:\operatorname{Re}w>0\}\xrightarrow{f}\mathbb{C}$

$z\longrightarrow\frac{1+z}{1-z}$

$f(\lambda_n)=0\ \forall n$

$g(z)=f\left(\frac{1+z}{1-z}\right)$

$U\ni\alpha_n\longrightarrow\lambda_n=\frac{1+\alpha_n}{1-\alpha_n}$

$g\in H^\infty(U)$

$g(\alpha_n)=0$

$\alpha_n=\frac{\lambda_n-1}{1+\lambda_n}$

$\{\lambda_n\}$ is increasing $\implies$ α_n is increasing

because $t\to\left(\frac{t-1}{1+t}\right)'=\frac{2}{(t+1)^2}>0$

$$+\infty=\sum_{n=1}^{\infty}\frac{1}{\lambda_n}=\sum_{n=1}^{\infty}\frac{1-\alpha_n}{1+\alpha_n}\le\frac{1}{1+\alpha_1}\sum_{n=1}^{\infty}(1-\alpha_n)$$

$\frac{1-\alpha_n}{1+\alpha_n}\le\frac{1-\alpha_n}{1+\alpha_1}$ $\qquad =+\infty$

So $g\in H^\infty(U)$ $\quad g(\alpha_n)=0$ $\quad \sum_{n=1}^{\infty}(1-|\alpha_n|)=+\infty$

$\implies g\equiv 0$

$\Downarrow$

$f\equiv 0$

$$\implies f(z)=\int_I t^z\,d\mu(t)=0$$
$\forall$ z with $\operatorname{Re} z>0$

$\int_I t^n\,d\mu(t)=0 \qquad n\in\mathbb{N}=\{1,2,\dots\}$

$\int_I 1\,d\mu(t)=0 \implies$ The functional associated to $d\mu$ is zero in $\overline{span\{1,t,t^2,t^3\dots\}}$

$\qquad = C^0(I)$

The functional is trivial

Bidual and orthogonality

X B space

X' B space

$X'' = (X')'$ B space

There exists canonical map $X \xrightarrow{J} X''$

$x \in X$ $x' \in X'$ recall

$|\langle x, x' \rangle_{X \times X'}| \leq |x'|_{X'} \, |x|_X$

$x' \longrightarrow \langle x, x' \rangle_{X \times X'}$ is a functional on X'

We define $(Jx)(x') = \langle x, x' \rangle_{X \times X'} = \langle Jx, x' \rangle_{X'' \times X'}$

<u>Lemma</u> J is an isometric immersion $J : X \hookrightarrow X''$

$$X \xrightarrow{J} JX \subseteq X''$$

<u>Pf</u> We need to show that $\|Jx\|_{X''} = |x|_X$

$|\langle Jx, x' \rangle_{X'' \times X'}| = |\langle x, x' \rangle_{X \times X'}| \leq |x|_X \, |x'|_{X'}$ $\forall x' \in X'$

$\Rightarrow$ $\boxed{\|Jx\|_{X''} \leq |x|_X \quad \forall x \in X}$

Now we will show $\boxed{|Jx|_{X''} \geq |x|_X \quad \forall x \in X}$

Let $x \in X$ $\exists$ $x' \in X'$ with $|x'|_{X'} = 1$ st

$\|x\|_X = \langle x, x' \rangle_{X \times X'} = |\langle Jx, x' \rangle_{X'' \times X'}|$

Hahn-Banach

$\leq |Jx|_{X''} \underbrace{|x'|_{X'}}_{1}$

$|x|_X \leq |Jx|_{X''}$.

Def X normed space and $M \subseteq X$ a subset

$$M^{\perp} = \{ f \in X' : \langle f, x \rangle_{X' \times X} = 0 \quad \forall x \in M \}$$
$$M^{\perp} \subseteq X'$$

Let N be a subset of X'

$$N^{\perp} = \{ x \in X : \langle f, x \rangle_{X' \times X} = 0 \quad \forall f \in N \}$$
$$N^{\perp} \subseteq X$$

$N^{\perp} \hookrightarrow N_2^{\perp} \subseteq X''$

Lemma 1 $M \subseteq X$ a linear space

$$(M^{\perp})^{\perp} = \overline{M}!$$

$N \subseteq X'$ a linear space

$$(N^{\perp})^{\perp} \supseteq \overline{N}$$

Remark $\ell^1(\mathbb{N}) = \{ f : \mathbb{N} \to \mathbb{R} \text{ s.t. } \sum_{n=1}^{\infty} |f(n)| < +\infty \}$

$\ell^{\infty}(\mathbb{N}) = \{ f \quad \ldots \quad \sup_n |f(n)| < +\infty \}$

$\cup$

$c_0(\mathbb{N}) = \{ \ldots \quad , \quad \lim_{n \to +\infty} f(n) = 0 \}$

$(\ell^1(\mathbb{N}))' = \ell^{\infty}(\mathbb{N}) \supseteq c_0(\mathbb{N})$

$(c_0(\mathbb{N}))^{\perp} = \{ f \in \ell^1(\mathbb{N}) : \sum_{n=1}^{\infty} f(n) g(n) = 0 \quad \forall g \in c_0(\mathbb{N}) \} = 0$

$(c_0(\mathbb{N}))' = \ell^1(\mathbb{N})$

$\langle g, f \rangle_{c_0(\mathbb{N}) \times \ell^1(\mathbb{N})} =$

$\neq (c_0(\mathbb{N}))'$

$\ell^1(\mathbb{N})$

$((c_0(\mathbb{N}))^{\perp})^{\perp} = (\{0\})^{\perp} = \ell^{\infty}(\mathbb{N}) \neq c_0(\mathbb{N})$

$\ell^{\infty}(\mathbb{N})$

$c_0 \subsetneq (\ell^1)'$ $\quad (c_0^{\perp})^{\perp} \supsetneq c_0$

Lemm $T: X \to Y$ B-spaces and T bounded

$T': Y' \to X'$

$R(T) = TX \subseteq Y$

$R(T') = T Y' \subseteq X'$

$\to \boxed{\ker T = R(T')^{\perp}}$

$\ker T' = R(T)^{\perp}$

$(\ker T)^{\perp} \supseteq \overline{R(T')}$ $Tx = y_0$

$(\ker T')^{\perp} = \overline{R(T)}$

$y_0 \in (\ker T')^{\perp}$

$\langle Tx, y' \rangle_{Y \times Y'} = \langle x, T'y' \rangle_{X \times X'}$

$x \in \ker T \Rightarrow \langle x, T'y' \rangle_{X \times X'} = 0 \ \forall y' \Rightarrow x \in R(T')^{\perp}$

If $x \in R(T')^{\perp} \Rightarrow \langle x, T'y' \rangle_{X \times X'} = 0 \ \forall y' \in Y'$

$\Rightarrow \langle Tx, y' \rangle_{Y \times Y'} = 0 \ \forall y' \in Y'$

$\Rightarrow Tx \Rightarrow x \in \ker T$