24 ottobre

Limiti notevoli

1) $\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e$

<u>Dim</u> Ricordiamoci che $\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = e$

Se $\underset{\in\mathbb{Z}}{[x]}$ è la parte intera di $x\in\mathbb{R}$, abbiamo

$$[x] \le x < [x]+1 \qquad \text{e per } x>0$$

$$\frac{1}{[x]} \ge \frac{1}{x} > \frac{1}{[x]+1}$$

$$\frac{1}{[x]+1} \le \frac{1}{x} \le \frac{1}{[x]}$$

$$1+\frac{1}{[x]+1} \le 1+\frac{1}{x} \le 1+\frac{1}{[x]}$$

$$\left(1+\frac{1}{[x]+1}\right)^{[x]} \le \left(1+\frac{1}{x}\right)^{[x]} \le \left(1+\frac{1}{x}\right)^{x} \le \left(1+\frac{1}{[x]}\right)^{x} \le \left(1+\frac{1}{[x]}\right)^{[x]+1}$$

$$\underbrace{\left(1+\frac{1}{[x]+1}\right)^{[x]}}_{\xrightarrow[x\to+\infty]{} e} < \left(1+\frac{1}{x}\right)^{x} < \underbrace{\left(1+\frac{1}{[x]}\right)^{[x]+1}}_{\to e}$$

Ora dimostriamo che

$$\left.\begin{array}{l} \boxed{\lim_{x\to+\infty}\left(1+\frac{1}{[x]+1}\right)^{[x]} = e} \\ \lim_{x\to+\infty}\left(1+\frac{1}{[x]}\right)^{[x]+1} = e \end{array}\right\} \overset{\text{Carab}}{\Longrightarrow} \lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e$$

$$\lim_{x\to+\infty}\left(1+\frac{1}{[x]+1}\right)^{[x]} = e$$

Sappiamo che $\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = e$

Cioè $\boxed{\forall \varepsilon>0 \quad \exists\, N_\varepsilon>0 \text{ t.c. } n>N_\varepsilon \Rightarrow \left|e-\left(1+\frac{1}{n}\right)^n\right|<\varepsilon}$

$$\left(1+\frac{1}{[x]+1}\right)^{[x]} = \left(1+\frac{1}{[x]+1}\right)^{[x]+1} \underbrace{\frac{1}{1+\frac{1}{[x]+1}}}_{\to 1}$$

Poiché $\lim_{x\to+\infty}[x]=+\infty$

$$\Rightarrow \lim_{x\to+\infty}\frac{1}{1+\underbrace{\frac{1}{[x]+1}}_{\xrightarrow[x\to+\infty]{}0}} = 1$$

Resta da dimostrare

$$\lim_{x\to+\infty}\left(1+\frac{1}{[x]+1}\right)^{[x]+1} = e$$

Vogliamo dimostrare che

$$\forall \varepsilon>0 \quad \exists\, M_\varepsilon>0 \text{ t.c. } x>M_\varepsilon \Rightarrow$$
$$\left|e-\left(1+\frac{1}{[x]+1}\right)^{[x]+1}\right|<\varepsilon$$

Proviamo a scegliere $M_\varepsilon = N_\varepsilon$

$x > M_\varepsilon \Rightarrow [x]\in\mathbb{N}$ con $[x] > M_\varepsilon - 1 = N_\varepsilon - 1$

$$\Rightarrow \overbrace{\underbrace{[x]+1}_{\in\mathbb{N}}}^{n} > N_\varepsilon \Rightarrow \left|e-\left(1+\frac{1}{[x]+1}\right)^{[x]+1}\right| < \varepsilon$$

Abbiamo dimostrato $\lim_{x\to+\infty}\left(1+\frac{1}{[x]+1}\right)^{[x]+1} = e$

In modo analogo si dimostra $\lim_{x\to+\infty}\left(1+\frac{1}{[x]}\right)^{[x]+1} = e$

e per i Carabinieri resta dimostrata la tesi.

2) $\lim\limits_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = e$

Notare che per $x<-1$ si ha $\cdot\frac{1}{x}$

$\Downarrow$

$1 > -\frac{1}{x}$ $\quad\cdot(-1)$

$\boxed{-1<\frac{1}{x}}$ $\quad +1$

$0 < 1+\frac{1}{x}$ e $\left(1+\frac{1}{x}\right)^x$ è ben definita

Dom $\left(1+\frac{1}{x}\right)^x \supseteq (-\infty,-1)$

$\Rightarrow \inf D\left(1+\frac{1}{x}\right)^x = -\infty$

Ha senso quindi considerare $\lim\limits_{x\to-\infty}\left(1+\frac{1}{x}\right)^x$

$y=-x \qquad x=-y$

$$\left(1+\frac{1}{x}\right)^x = \left(1-\frac{1}{y}\right)^{-y} = \left(\frac{y-1}{y}\right)^{-y} =$$

$$= \left(\frac{y-1}{y-1+1}\right)^{-y} = \left(\frac{\frac{1}{y-1}}{\frac{1}{y-1}}\frac{(y-1)}{(y-1)+1}\right)^{-y}$$

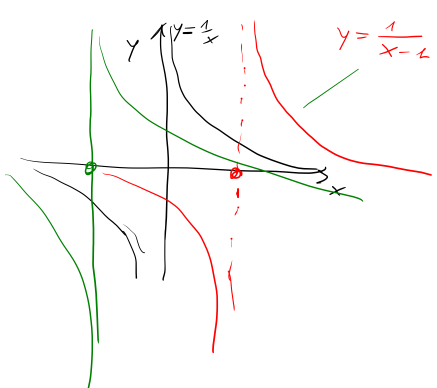
$$= \left(\frac{1}{1+\frac{1}{y-1}}\right)^{-y} = \frac{1}{\left(1+\frac{1}{y-1}\right)^{-y}}$$

$$= \left(1+\frac{1}{y-1}\right)^{y} = \underbrace{\left(1+\frac{1}{y-1}\right)^{y-1}}_{\xrightarrow{y\to+\infty}\ e}\ \underbrace{\left(1+\frac{1}{y-1}\right)}_{\xrightarrow{y\to+\infty}\ 1}$$

Esercizio Verificare che se $\lim\limits_{x\to+\infty} f(x) = L \in \overline{\mathbb{R}}$

allora $\lim\limits_{x\to+\infty} f(x-x_0) = L$, per ogni $x_0\in\mathbb{R}$

Es $f(x)=\frac{1}{x}$



$f(x-1) = \frac{1}{x-1}$

Il fatto che $\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e$ e che

$$\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = e$$ lo

esprimiamo scrivendo

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$$

3) $$\lim_{y \to 0} \frac{\lg(1+y)}{y} = 1$$

Dim $\quad x = \frac{1}{y}$

$$\frac{\lg(1+y)}{y} = \frac{\lg\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = x \lg\left(1+\frac{1}{x}\right)$$

$$= \lg\left(1+\frac{1}{x}\right)^x$$

$$\lim_{y \to 0} \frac{\lg(1+y)}{y} = \lim_{x \to \infty} \lg\left(1+\frac{1}{x}\right)^x =$$

$\xrightarrow[x \to \infty]{} e$

$$z = \left(1+\frac{1}{x}\right)^x \qquad = \lim_{z \to e} \lg z = \lg e = 1$$

4) $$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

Dim $\quad y = e^x - 1 \qquad e^x = y + 1 \qquad \cdot \lg$

$$x = \lg e^x = \lg(y+1)$$

$$x = \lg(y+1)$$

$$\frac{e^x - 1}{x} = \frac{y}{\lg(y+1)}$$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{y \to 0} \frac{y}{\lg(y+1)} = 1$$

5) $\forall\, \alpha \in \mathbb{R}$

$$\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha$$

<u>Dim</u>

$\alpha = 0$ si ottiene 0

$$\left(\lim_{x\to 0} \frac{0}{x} = \lim_{x\to 0} 0 = 0 \right.$$

$$\alpha = 1 \qquad \frac{\cancel{1} + x - \cancel{1}}{x} = \frac{x}{x} \qquad \lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}} = \lim_{x\to 0} 1 = 1 \Big)$$

$\alpha \in \mathbb{R}$

$$\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha$$

$$y = (1+x)^\alpha - 1$$

$$\frac{\lg(1+y)}{y} = \frac{\lg(\cancel{1} + (1+x)^\alpha - \cancel{1})}{(1+x)^\alpha - 1} = \frac{\alpha \lg(1+x)}{(1+x)^\alpha - 1}$$

$$= \alpha \frac{x}{(1+x)^\alpha - 1} \frac{\lg(1+x)}{x}$$

$$\boxed{\frac{\frac{\lg(1+y)}{y}}{\frac{\lg(1+x)}{x}} = \alpha \frac{x}{(1+x)^\alpha - 1}} \qquad x \to 0$$

dove $\boxed{y = (1+x)^\alpha - 1}$

$$\alpha \lim_{x\to 0} \frac{x}{(1+x)^\alpha - 1} = \lim_{x\to 0} \frac{\frac{\lg(1+y)}{y}}{\frac{\lg(1+x)}{x}}$$

$$= \frac{\lim_{x\to 0} \frac{\lg(1+y)}{y}}{\lim_{x\to 0} \frac{\lg(1+x)}{x}} = \frac{\lim_{y\to 0} \frac{\lg(1+y)}{y}}{\lim_{x\to 0} \frac{\lg(1+x)}{x}} = 1$$

Cioè, se $\alpha \neq 0$ concludo che

$$\lim_{x\to 0} \frac{x}{(1+x)^\alpha - 1} = \frac{1}{\alpha} \Rightarrow \text{tesi}$$

per $n \in \mathbb{N}$

$$\lim_{x \to 0} \frac{(1+x)^n - 1}{x} = n$$

$$\frac{(1+x)^n - 1}{x} = \frac{\sum_{k=0}^{n} \binom{n}{k} x^k - 1}{x} =$$

$$= \frac{\cancel{\underbrace{\binom{n}{0}}_{=1} \underbrace{x^0}_{=1}} + \sum_{k=1}^{n} \binom{n}{k} x^k - \cancel{1}}{x}$$

$$= \frac{\sum_{k=1}^{n} \binom{n}{k} x^k}{x} = \sum_{k=1}^{n} \binom{n}{k} x^k \frac{1}{x}$$

$$= \sum_{k=1}^{n} \binom{n}{k} x^{k-1} = \binom{n}{1} \underbrace{x^0}_{=1} + \sum_{k=2}^{n} \binom{n}{k} x^{k-1}$$

$$= n + \sum_{k=2}^{n} \binom{n}{k} \underbrace{x^{k-1}}_{\xrightarrow{x \to 0} 0} \xrightarrow{x \to 0} n$$

Derivata

<u>Terminologia</u> Sia $f : X \to \mathbb{R}$ $\quad X \subseteq \mathbb{R}$

e siano x e x_0 due punti di X.

Allora
$$\frac{f(x)-f(x_0)}{x-x_0} = \frac{f(x_0)-f(x)}{x_0-x}$$

è un rapporto incrementale

$$\Delta f = f(x)-f(x_0) \qquad \frac{f(x)-f(x_0)}{x-x_0} = \frac{\Delta f}{\Delta x}$$
$$\Delta x = x - x_0$$

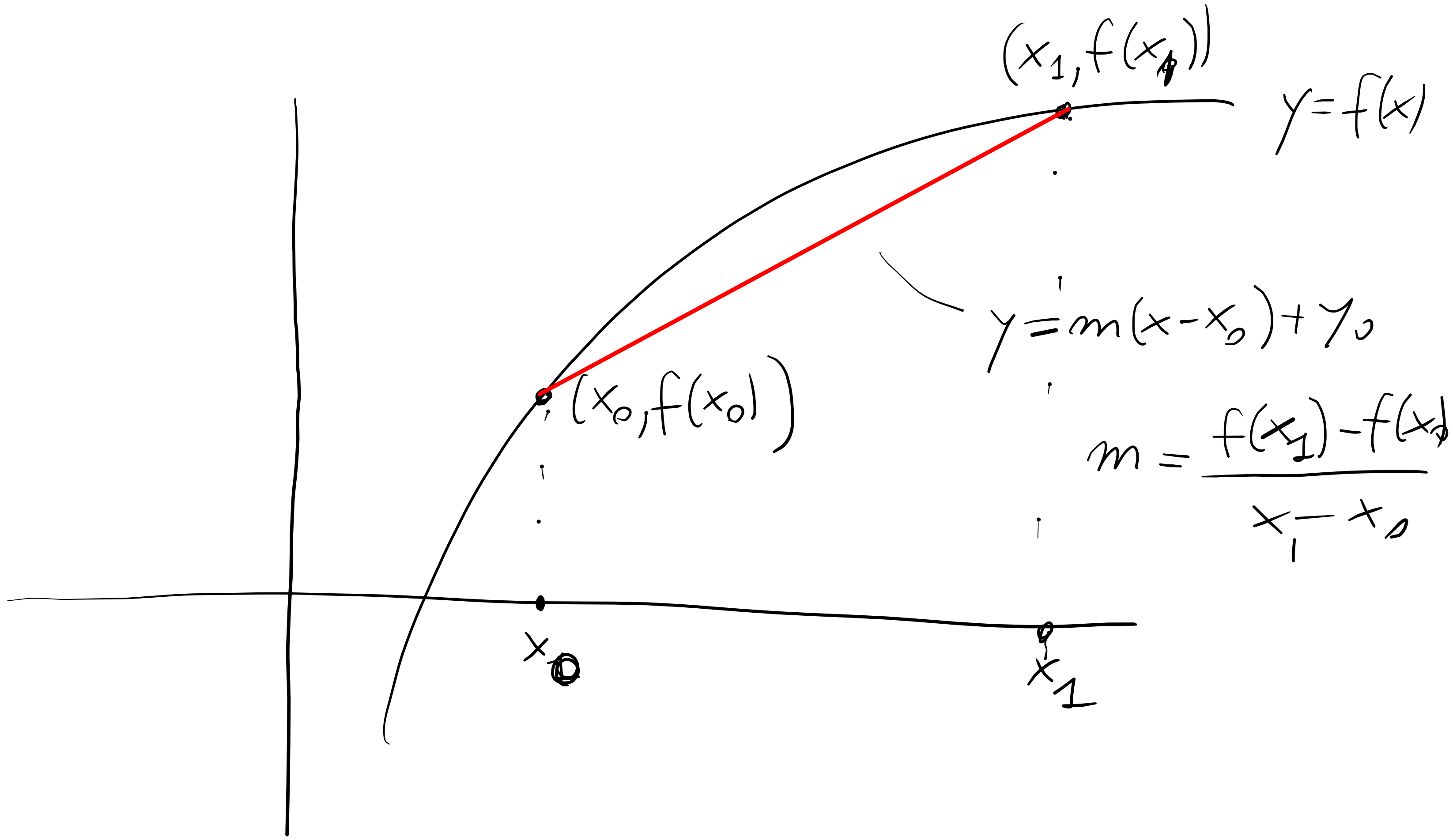
Esempio Sia $[t_0, t_1] \longrightarrow x(t) \in \mathbb{R}$

una legge di moto. La velocità media

$$v_{medio} = \frac{x(t_1) - x(t_0)}{t_1 - t_0}$$

Esempio $[t_0, t_1] \longrightarrow n(t) \in \mathbb{N}$

$$\frac{n(t_1) - n(t_0)}{t_1 - t_0}$$



$y - f(x_0) = m\,(x - x_0)$ è la generica retta per $(x_0, f(x_0))$

Voglio m in modo tale che sia vero anche

$$f(x_1) - f(x_0) = m\,(x_1 - x_0) \iff m = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$