24th of October

Theorem of Banach and Steinhaus

Def X topological space is a Baire space if one of the following two equivalent statements hold

1) For any sequence $\{A_n\}$ of dense open subspaces of X then $\bigcap_{n=1}^{\infty} A_n$ is dense

2) For any sequence $\{C_n\}$ of closed subspaces s.t. $\mathring{C}_n = \emptyset$ $\forall n$ also the interior of $\bigcup_{n=1}^{\infty} C_n$ is empty

A subspace of a topological space X which contains the intersection of a sequence of open dense sets is called a G_δ subspace X.

Theorem

1) Every locally compact Hausdorff space X is a Baire space

2) Every complete metric space is a Baire space.

Theorem Let X and Y be Banach spaces and let $\{T_j ;\ j\in J\}$ be a family of operators in $\mathcal{L}(X,Y)$.

Let $\sup_{j\in J} |T_j x|_Y < +\infty \quad \forall\ x\in X$.

Then $\exists\ M>0$ s.t. $|T_j|_{\mathcal{L}(X,Y)} \leq M \quad \forall\ j\in J$.

If it is not true that $\sup_{j\in J} |T_j|_{\mathcal{L}(X,Y)} < +\infty$

then $\sup_{j\in J} |T_j x|_Y = +\infty$ for all the x's of a

G_δ subspace of X.

Pf Suppose that $\sup_{j\in J} |T_j x|_Y < +\infty \quad \forall\ x\in X$

Let $X_n = \{ x\in X :\ |T_j x|_Y \leq n \quad \forall\ j\in J\}$

X_n is closed $\forall n$ since

$$X_n = \bigcap_{j\in J} T_j^{-1} \overline{D_Y(0,n)} .$$

$\bigcup_{n\geq 1}^{\infty} X_n = X$. Indeed $\forall\ x\in X$

$\sup_{j\in J} |T_j x|_Y < \infty \Rightarrow \exists\ n\in\mathbb{N}$ s.t.

$|T_j x|_Y \leq n \quad \forall\ j\in J \Rightarrow x\in X_n$

Since X is Baire space, it follows that one of the X_n has $\mathring{X}_n \neq \emptyset$

$\Rightarrow \exists\ x_0\in X_n$ and $r>0$ st.

$x_0 + D_X(0,r) = D_X(x_0,r) \subseteq X_n \Rightarrow \forall\ z\in D_X(0,1)$

$D_X(0,r) = r D_X(0,1)$

$|T_j(x_0+rz)|_Y \leq n \quad \forall\ j\in J.$

$\geq |T_j rz|_Y - |T_j x_0|_Y$

$|T_j rz|_Y \leq n + \overset{\leq n}{|T_j x_0|_Y} \quad \forall\ j\in J \quad \forall z\in D_X(0,1)$

$|T_j z|_Y \leq \frac{2n}{r} \Rightarrow \sup_{z\in D_X(0,1)} |T_j z|_Y \leq \frac{2n}{r} \quad \forall\ j\in J \quad \forall z\in D_X(0,1)$

$\Rightarrow |T_j|_{\mathcal{L}(X,Y)} \leq \frac{2n}{r} \quad \forall\ j\in J$

Suppose $\sup_{j \in J} |T_j|_{\mathcal{L}(X,Y)} = +\infty$ ($\Longrightarrow X_n$ does not contain open sets)

It follows that $\overset{\circ}{X}_n = \phi \quad \forall \; n \geq 1$

X_n closed $\Longrightarrow \quad X \setminus X_n$ is an open dense subset of X.

$$\Longrightarrow \quad \bigcap_{n=1}^{\infty} (X \setminus X_n) = X \setminus \left(\bigcup_{n=1}^{\infty} X_n \right)$$

is dense and is a G_δ subspace of X.

$\forall \; x \in \underline{X \setminus \left(\bigcup_{n=1}^{\infty} X_n \right)}$ we have $x \notin X_n \quad \forall n$

$$\Longrightarrow \quad \sup_{j \in J} |T_j x|_Y > n \qquad \forall \; n \in \mathbb{N}$$

$$\Longrightarrow \quad \sup_{j \in J} |T_j x|_Y = +\infty \qquad \forall \; x \in X \setminus \left(\bigcup_{n=1}^{\infty} X_n \right).$$

Fourier Series

$P(z_1, z_2)$ a polynomial

$f(x) = P(\cos x, \sin x)$ is a trigonometric polynomial

Prostaferese formulas imply

$$\sin(nx)\sin(mx) = \frac{\cos((n-m)x) - \cos((n+m)x)}{2}$$

$$\cos(nx)\sin(mx) = \frac{\sin((n+m)x) - \sin((n-m)x)}{2}$$

$$\cos(nx)\cos(m) = \frac{\cos((n-m)x) + \cos((n+m)x)}{2}$$

It is easy to show that any trigonometric poly is of the form

$$f(x) = \frac{a_0}{2} + \sum_{\ell=1}^{n} \left(a_\ell \cos(\ell x) + b_\ell \sin(\ell x) \right)$$

Lemma

$$a_\ell = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(\ell x)\, dx = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(\ell x)\, dx$$

$\ell = 0, 1, \dots$

$$b_\ell = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(\ell x)\, dx \qquad \ell = .$$

Pf

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \sin(nx)\sin(mx) = \delta_{nm}$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nx)\cos(mx) = \delta_{nm}$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nx)\sin(mx)\, dx = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{\ell=1}^{n} \left(a_\ell \cos(\ell x) + b_\ell \sin(\ell x) \right)$$

$a_m \qquad \ell \geq m \geq 1$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(mx) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{a_0 \cos(mx)}{2} + \sum_{\ell=1}^{n} \frac{1}{\pi} \int_{-\pi}^{\pi} \left(a_\ell \cos(\ell x)\cos(mx) + b_\ell \sin(\ell x)\cos(mx) \right)$$

$$= \frac{1}{\pi} a_m \underbrace{\int_{-\pi}^{\pi} \cos^2(mx)\, dx}_{1}$$

Def $\forall f \in L^1(-\pi, \pi)$ its trigonometric series is

$$\frac{a_0}{2} + \sum_{\ell=1}^{\infty} \left(a_\ell \cos(\ell x) + b_\ell \sin(\ell x) \right)$$

where
$$a_\ell = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(\ell x)\, dx \qquad \ell = 0, \ldots$$

$$b_\ell = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(\ell x)\, dx \qquad \ell = 1, \ldots$$

Alternatively we will write the Fourier series in the form

$$\sum_{\ell \in \mathbb{Z}} \hat{f}(\ell)\, e^{i\ell x} \qquad \hat{f}(\ell) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\ell x} f(x)\, dx$$

$$\frac{\mathbb{R}^d}{2\pi\mathbb{Z}^d} =: \mathbb{T}^d \leftarrow \mathbb{R}^d$$

There is an identification of $L^1(\mathbb{T}^d)$ and $L^1((-\pi,\pi)^d)$

$$\sum_{n\in\mathbb{Z}^d} \hat{f}(n)\, e^{in\cdot x} \qquad \hat{f}(n) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} f(x)\, e^{-in\cdot x}\,dx$$

$$\sum_{|n|\le N} \hat{f}(n)\, e^{in\cdot x}$$

$$\alpha = (\alpha_1, \dots, \alpha_d)$$
$$|\alpha| = |\alpha_1| + \dots + |\alpha_d|$$

α

$$\widehat{\partial_x^\alpha f}(n) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} e^{-inx}\,\partial_x^\alpha f(x)\,dx =$$

$$= (-1)^{|\alpha|}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \partial_x^\alpha\left(e^{-inx}\right) f(x)\,dx$$

$$= \cancel{(-1)^{|\alpha|}}\,\frac{1}{(2\pi)^d}\,(\cancel{-}in)^\alpha \int_{\mathbb{T}^d} e^{-inx} f(x)\,dx$$

$$\widehat{\partial_x^\alpha f}(n) = i^{|\alpha|}\, n^\alpha\, \hat{f}(n)$$

$$\partial_x^\alpha f \qquad n^\alpha = n_1^{\alpha_1}\cdots n_d^{\alpha_d}$$
$$\downarrow$$
$$m(n)\,\hat{f}(n)$$

$$\hat{f}: \mathbb{Z}^d \to \mathbb{C}$$

$$\Omega \subset \mathbb{Z}^d$$
$$|\Omega| = \operatorname{card}\Omega$$

$$m(n) = i^{|\alpha|}\, n^\alpha$$

$$\mathbb{T}^d = (-\pi, \pi)^d$$

<u>Lemma</u> $\forall \; f \in L^1(\mathbb{T}^d) \implies |\hat{f}(n)| \leq \frac{1}{(2\pi)^d} |f|_{L^1(\mathbb{T}^d)}$

Pf

$$\hat{f}(n) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) e^{-in\cdot x} \, dx \qquad \forall \, n \in \mathbb{Z}^d$$

$$|\hat{f}(n)| \leq \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} |f(x)| \, dx = \frac{1}{(2\pi)^d} |f|_{L^1(\mathbb{T}^d)}$$

$$f \xrightarrow{\mathcal{F}} \hat{f}$$

$$L^1(\mathbb{T}^d) \longrightarrow L^\infty(\mathbb{Z}^d)$$

$$\| \mathcal{F} \|_{L^1(\mathbb{T}^d) \to L^\infty(\mathbb{Z}^d)} \leq \frac{1}{(2\pi)^d}$$

(Riemann Lebesgue lemma)

Lemma $\forall\ f \in L^1(\mathbb{T}^d)$ $\qquad [-\pi,\pi]^d$

$$\lim_{n\to\infty} \hat{f}(n) = 0$$

Pf In $L^1(\mathbb{T}^d)$ the linear combination of ~~charact~~ function

$$\boxed{\hat{\chi}_{[a_1,b_1]\times\cdots\times[a_d,b_d]}(n)} \xrightarrow[n\to\infty]{} 0$$

$$= \int_{[a_1,b_1]\times\cdots\times[a_d,b_d]} e^{-in_1x_1} e^{-in_2x_2} \cdots e^{-in_dx_d}\, dx_1 \cdots dx_d =$$

$$= \int_{a_1}^{b_1} e^{-in_1x_1} dx_1 \cdots \int_{a_d}^{b_d} e^{-in_dx_d}\, dx_d$$

$$= \frac{e^{-in_1b_1} - e^{-in_1a_1}}{-i n_1} \cdots \frac{e^{-in_db_d} - e^{-in_da_d}}{-i n_d} \xrightarrow{n\to\infty} 0$$

$\hat{f}(n) \xrightarrow{n\to\infty} 0 \qquad \forall\ f \in L^1$

We need to show that

$\forall\ \varepsilon > 0 \quad \exists\ N_\varepsilon \implies |n| > N_\varepsilon \implies |\hat{f}(n)| < \varepsilon$

Take $g = \sum_{j=1}^{m} \lambda_j \chi_{R_j}$ $\qquad \exists$ such g s.t.

$$|f-g|_{L^1(\mathbb{T}^d)} < \varepsilon$$

$$|\hat{f}(n)| = |\widehat{f-g}(n) + \hat{g}(n)| \le |\widehat{f-g}(n)| + |\hat{g}(n)|$$

$$\le \frac{1}{(2\pi)^d}\varepsilon + |\hat{g}(n)| \qquad \hat{g}(n) \xrightarrow{n\to+\infty} 0$$

$\forall\ \varepsilon > 0 \quad \exists\ M_\varepsilon \implies |n| > M_\varepsilon \implies |\hat{g}(n)| < \varepsilon$

Let $N_\varepsilon = M_{\frac{\varepsilon}{2}}$

$\implies |n| > N_\varepsilon = M_{\frac{\varepsilon}{2}}$

$$|\hat{f}(n)| \le \frac{\varepsilon}{(2\pi)^d} + \frac{\varepsilon}{2} < \varepsilon$$

__Def__ Dirichlet kernel $n \geq 1$

$$D_n(x) = \frac{1}{2} + \sum_{\ell=1}^{n} \cos(\ell x) = \frac{\sin\left(\left(n+\frac{1}{2}\right)x\right)}{2 \sin \frac{x}{2}}$$

$$\sin\left(\left(n+\frac{1}{2}\right)x\right) = \sin\left(\frac{x}{2}\right) + \sum_{\ell=1}^{n} \left(\sin\left(\left(\ell+\frac{1}{2}\right)x\right) - \sin\left(\left(\ell-\frac{1}{2}\right)x\right) \right)$$

$$= \sin\left(\frac{x}{2}\right) + \sum_{\ell=1}^{n} 2 \sin\left(\frac{x}{2}\right) \cos(\ell x)$$