25th October

Dirichlet kernel

$$D_n(x) = \frac{1}{2} + \sum_{l=1}^{n} \cos(lx) = \frac{\sin((n+\frac{1}{2})x)}{2\sin\frac{x}{2}}$$

Lemma $f \in L^1(\pi)$

$$\boxed{S_n f(x) = \frac{a_0}{2} + \sum_{l=1}^{n} \left(a_l \cos(\cancel{kx}) + b_l \sin(lx)\right)}$$

$$S_n f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) D_n(x-t)\,dt$$

Pf

$$S_n f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\,dt +$$

$$+ \sum_{l=1}^{n} \left[\cos(lx) \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(lt)\,dt + \sin(lx) \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(lt)\,dt\right]$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} dt\, f(t) \underbrace{\left(\frac{1}{2} + \sum_{l=1}^{n} \underbrace{\left(\cos(lx)\cos(lt) + \sin(lx)\sin(lt)\right)}_{\cos(l(x-t))}\right)}_{D_n(x-t)}$$

$ev_0 \circ S_n$

G_δ subset of $C^0(\mathbb{T})$ such that for any $f \in C^0(\mathbb{T})$ in this subset we have

Then $\forall x \in \mathbb{T}$ ~~$\exists$~~ ~~s.t.~~

$$S_n f(x) \not\xrightarrow{n \to +\infty} f(x)$$

$$\boxed{\limsup_{n \to +\infty} |S_n f(x)| = +\infty}$$

$f \in L^2(-\pi, \pi)$ Carleson

Pf $x = 0$

$$|D_n(t)|_{L^1(\mathbb{T})} = \left| \frac{\sin((n+\frac{1}{2})t)}{2 \sin \frac{t}{2}} \right|_{L^1(\mathbb{T})}$$

$$= 2 \int_0^\pi \frac{|\sin((n+\frac{1}{2})t)|}{2 \sin \frac{t}{2}} \, dt \qquad 0 < \sin \frac{t}{2} < \frac{t}{2}$$

$$> 2 \int_0^\pi |\sin((n+\tfrac{1}{2})t)| \frac{dt}{t} \qquad s = (n+\tfrac{1}{2})t$$

$$= 2 \int_0^{(n+\frac{1}{2})\pi} |\sin(s)| \frac{ds}{s} > 2 \int_0^{n\pi} |\sin(t)| \frac{dt}{t}$$

$(j-1)\pi < t < j\pi \qquad \frac{1}{t} > \frac{1}{j\pi}$

$$> \frac{2}{\pi} \sum_{j=1}^{n} \frac{1}{j} \underbrace{\int_{(j-1)\pi}^{j\pi} |\sin(t)| \, dt}_{\underbrace{\int_0^\pi \sin t}_{=2}}$$

$$= \frac{4}{\pi} \sum_{j=1}^{n} \frac{1}{j} \xrightarrow{n \to +\infty} +\infty$$

$$g(t) = \operatorname{sign}(D_n(t)), \qquad \operatorname{sign} x = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

$$S_n g(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) D_n(t) \, dt = \frac{1}{2\pi} |D_n|_{L^1(\mathbb{T})}$$

By Lusin theorem $\forall j \in \mathbb{N}$ $\exists f_j \in C^0(\mathbb{T})$

$|f_j|_{L^\infty(\mathbb{T})} \leq |g|_{L^\infty(\mathbb{T})} = 1$

s.t. $|\{x : f_j(x) \neq g(x)\}| < \frac{1}{j}$.

Then $f_j \xrightarrow{j \to +\infty} g$ in $L^1(\mathbb{T})$ because

$$|f_j - g|_{L^1(\mathbb{T})} = \int_{\{f_j \neq g\}} |f_j - g| \, dx \leq 2 \frac{1}{j} \xrightarrow{j \to +\infty} 0$$

$$S_n f_j(0) \longrightarrow S_n g(0) = \frac{1}{2\pi} |D_n|_{L^1(\mathbb{T})}$$

$f \dashrightarrow S_n f \longrightarrow S_n f(0)$

$ev_0 \circ S_n$

If for any $f \in C^0(\mathbb{T})$ we had $S_n f(0) \xrightarrow{n \to +\infty} f(0)$

then $\sup_{n \in \mathbb{N}} |S_n f(0)| < +\infty$

$f \in C^0(\mathbb{T}) \xrightarrow[ev_0 \circ S_n]{} S_n f(0)$ is a functional on $C^0(\mathbb{T})$

$f(0) = ev_0 f$

$\{ev_0 \circ S_n\}_{n \in \mathbb{N}}$ $\qquad \sup_n |\overbrace{ev_0 \circ S_n f}^{S_n f(0)}| < +\infty$

By Banach Steinhaus $\exists$ $C_0 > 0$ st

$$|ev_0 \circ S_n|_{(C^0(\mathbb{T}))'} < C_0 \quad \forall n$$

$$|S_n f(0)| = |ev_0 \circ S_n f| \leq |ev_0 \circ S_n|_{(C^0(\mathbb{T}))'} |f|_{L^\infty(\mathbb{T})}$$

$$\leq C_0 |f|_{L^\infty(\mathbb{T})} \quad \forall n$$

But earlier we found a sequence $\{f_j\}_{j \in \mathbb{N}}$ in $C^0(\mathbb{T})$

$|f_j|_{L^\infty(\mathbb{T})} \leq 1$ s.t.

$$|S_n f_j(0)| \longrightarrow \frac{1}{2\pi} |D_n|_{L^1(\mathbb{T})}$$

$$|S_n f_j(0)| \leq C_0 \implies$$

$$\frac{1}{2\pi} |D_n|_{L^1(\mathbb{T})} \leq C_0 \quad \forall n$$

$\downarrow n \to +\infty$

$+\infty$

Theorem (Open Mapping)

Let E and F be B-spaces and let $T: E \to F$ onto $R(T) = F$. Then $\exists$ $c>0$ st

$$T D_E(0,1) \supset D_F(0,c).$$

Corollary If furthermore T is 1-1 then T is an isomorphism, that is $T^{-1} \in \mathcal{L}(F,E)$

Example $x \in \mathbb{R}^d$ $\langle x \rangle = \sqrt{1+|x|^2}$

$1 \le p < +\infty$ $L^p(\mathbb{R}^d) \xrightarrow{T} L^p(\mathbb{R}^d)$

$Tf(x) = \langle x \rangle^{-1} f(x)$

$\mathbb{R}^d \to \mathbb{R}$

$\frac{1}{\langle x \rangle}$ has image $\overline{(0,1]} = [0,1]$

$\sigma(T) = [0,1]$

In particular T^{-1} is not a bounded operator

$(T-0)^{-1}$

T is $L^p(\mathbb{R}^d) \xrightarrow{T} R(T)$, $\xleftarrow{T^{-1}}$

$R(T)$ is dense in $L^p(\mathbb{R}^d)$

$T^{-1} \notin \mathcal{L}(R(T), L^p(\mathbb{R}^d))$

Pf We first show $\exists$ $c>0$ s.t.

$$\overline{T D_E(0,1)} \supseteq D_F(0,2c)$$

$$X_n = n\overline{T D_E(0,1)} = \overline{T D_E(0,n)}$$

$$\bigcup_{n=1}^{\infty} X_n = \bigcup_{n=1}^{\infty} \overline{T D_E(0,n)} \supseteq \bigcup_{n=1}^{\infty} T D_E(0,n) =$$

$$= R(T) = F$$

$\exists$ $\overset{\circ}{X}_n \neq \phi$

$$X_n = n X_1 \Rightarrow \overset{\circ}{X}_1 \neq \phi$$

$\exists$ $D_F(y_0, 4c) \subseteq \overline{T D_E(0,1)} = X_1$

$y_0, -y_0$

$$D_F(y_0, 4c) + (-y_0) = D_F(0,4c)$$

$$\subseteq \frac{1}{2}\overline{T D_E(0,1)} + \frac{1}{2}\overline{T D_E(0,1)} \subseteq \overline{T D_E(0,1)}$$

$$D_F(0,4c) \subseteq \overline{T D_E(0,1)} + \overline{T D_E(0,1)} \subseteq 2\overline{T D_E(0,1)}$$

$$D_F(0,4c) \subseteq 2\overline{T D_E(0,1)}$$

$$\frac{1}{2} D_F(0,4c) = D_F(0,2c) \subseteq \overline{T D_E(0,1)}$$

Now we want to prove $D_F(0,c) \subset T D_E(0,1)$

$$D_F(0,2c) \subseteq \overline{T D_E(0,1)} \qquad \cdot \frac{1}{2}$$

$$D_F(0,c) \subseteq \overline{T D_E(0,\tfrac{1}{2})}$$

This means that for any $y \in D(0,c)$ $\exists$ $z_1 \in D_E(0,\frac{1}{2})$

$$|y - Tz_1|_F < \varepsilon \qquad \forall \varepsilon>0 \text{ preassigned}$$

$$|y - Tz_1|_F < \frac{c}{2}$$

$$y - Tz_1 \in D_F(0, \tfrac{c}{2})$$

$$D_F(0,\tfrac{c}{2}) \subseteq \overline{T D_E(0,\tfrac{1}{4})}$$

$\Rightarrow$ $\exists$ $z_2 \in D_E(0,\frac{1}{4})$ s.t.

$$|y - Tz_1 - Tz_2|_F < \frac{c}{2^2}$$

By induction is possible to prove the existence of a sequence $\{z_n\}$ in E $\quad z_n \in D_E(0,\frac{1}{2^n})$

s.t. $\left|y - \sum_{j=1}^{n} Tz_j\right|_F < \frac{c}{2^n}$ $\forall n$

$$x = \sum_{n=1}^{\infty} z_n \qquad |y - Tx|_F = 0 \qquad y = Tx$$

$$\|x\|_E \leq \sum_{n=1}^{\infty} \|z_n\|_E < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

We have found a $c>0$ s.t.

$\forall$ ~~$y \in$~~ $\|y\|_y < c$ $\exists$ x with $\|x\|<1$

and with $y = Tx$

$$T D_E(0,1) \supseteq D_F(0,c)$$

Theorem The map $f \in L^1(\mathbb{T}) \xrightarrow{\mathcal{F}} \hat{f}(n) \in c_0(\mathbb{Z})$ is not onto

Pf $\mathcal{F}$ is 1-1 map. If we assume that $\mathcal{F}$ is onto then $\mathcal{F}$ is an isomorphism.

$$\mathcal{F}^{-1} : c_0(\mathbb{Z}) \to L^1(\mathbb{T})$$

$$\|\mathcal{F}^{-1}\| \leq C_0$$

$$|\mathcal{F}^{-1}\hat{f}|_{L^1(\mathbb{T})} \leq C_0 \|\hat{f}\|_{\ell^\infty(\mathbb{Z})}$$

$$\|f\|_{L^1(\mathbb{T})} \leq C_0 \|\hat{f}\|_{\ell^\infty(\mathbb{Z})} \quad \forall f \in L^1(\mathbb{T})$$

$$D_n(t) = \frac{1}{2} + \sum_{j=1}^{n} \cos(jt) = \frac{1}{2} + \sum_{j=1}^{n} \frac{e^{ijt} + e^{-ijt}}{2}$$

$$+\infty \xleftarrow{n \to +\infty} \|D_n\|_{L^1(\mathbb{T})} \leq C_0 \|\hat{D}_n\|_{\ell^\infty(\mathbb{Z})} = C_0 \frac{1}{2}$$

$$\hat{D}_n(k) = \begin{cases} \frac{1}{2} & |k| \leq n \\ 0 & |k| > n \end{cases}$$

Remark $\sum_{n=2}^{\infty} \frac{\sin(nx)}{\log n}$ is not the Fourier series of an $L^1(\mathbb{T})$ function

$\sum_{n=2}^{\infty} \frac{\cos(nx)}{\log n}$ is Fourier series of an $L^1(\mathbb{T})$ function