29 ottobre

Esempi di derivate

1) $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$.

$$\lim_{x \to x_0} \frac{\sin(x) - \sin(x_0)}{x - x_0} = \lim_{h \to 0} \frac{\sin(h + x_0) - \sin(x_0)}{h} =$$

$h = x - x_0$, $x = h + x_0$

$$= \lim_{h \to 0} \frac{\sin(h)\cos(x_0) + \cos(h)\sin(x_0) - \sin x_0}{h}$$

$$= \lim_{h \to 0} \left[\underbrace{\frac{\sin(h)}{h}}_{\xrightarrow{h \to 0} 1} \cos(x_0) + \sin(x_0) \underbrace{\frac{\cos(h) - 1}{h}}_{\xrightarrow{h \to 0} 0} \right]$$

$$= \cos(x_0) \qquad \forall x_0 \in \mathbb{R}$$

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x^2} = \frac{1}{2} \implies \lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

$$\frac{1 - \cos(x)}{x^2} \frac{1 + \cos x}{1 + \cos x} = \frac{1 - \cos^2 x}{x^2} \frac{1}{1 + \cos x} =$$

$$= \underbrace{\frac{\sin^2 x}{x^2}}_{\xrightarrow{x \to 0} 1^2} \underbrace{\frac{1}{1 + \cos x}}_{\xrightarrow{x \to 0} \frac{1}{2}} \xrightarrow{x \to 0} \frac{1}{2}$$

$$\lim_{x \to 0} \frac{1 - \cos x}{x} \frac{x}{x} = \lim_{x \to 0} \frac{1 - \cos x}{x^2} x = \frac{1}{2} \cdot 0 = 0$$

$$(\cos x)' = -\operatorname{sen} x \qquad h = x - x_0, \quad x = h + x_0$$

$$\lim_{x\to x_0} \frac{\cos x - \cos x_0}{x - x_0} = \lim_{h\to 0} \frac{\cos(h+x_0) - \cos(x_0)}{h} =$$

$$= \lim_{h\to 0} \frac{\cos(h)\cos(x_0) - \operatorname{sen}(h)\sin(x_0) - \cos x_0}{h}$$

$$= \lim_{h\to 0} \left[- \underbrace{\frac{\operatorname{sen}(h)}{h}}_{\to 1} \sin(x_0) + \cos(x_0) \underbrace{\frac{\cos(h) - 1}{h}}_{\to 0} \right]$$

$$= -\operatorname{sen}(x_0)$$

$$(e^x)' = e^x$$

2) $(e^{-x})' = -e^{-x}$ $\qquad h = x - x_0, \quad x = h + x_0$

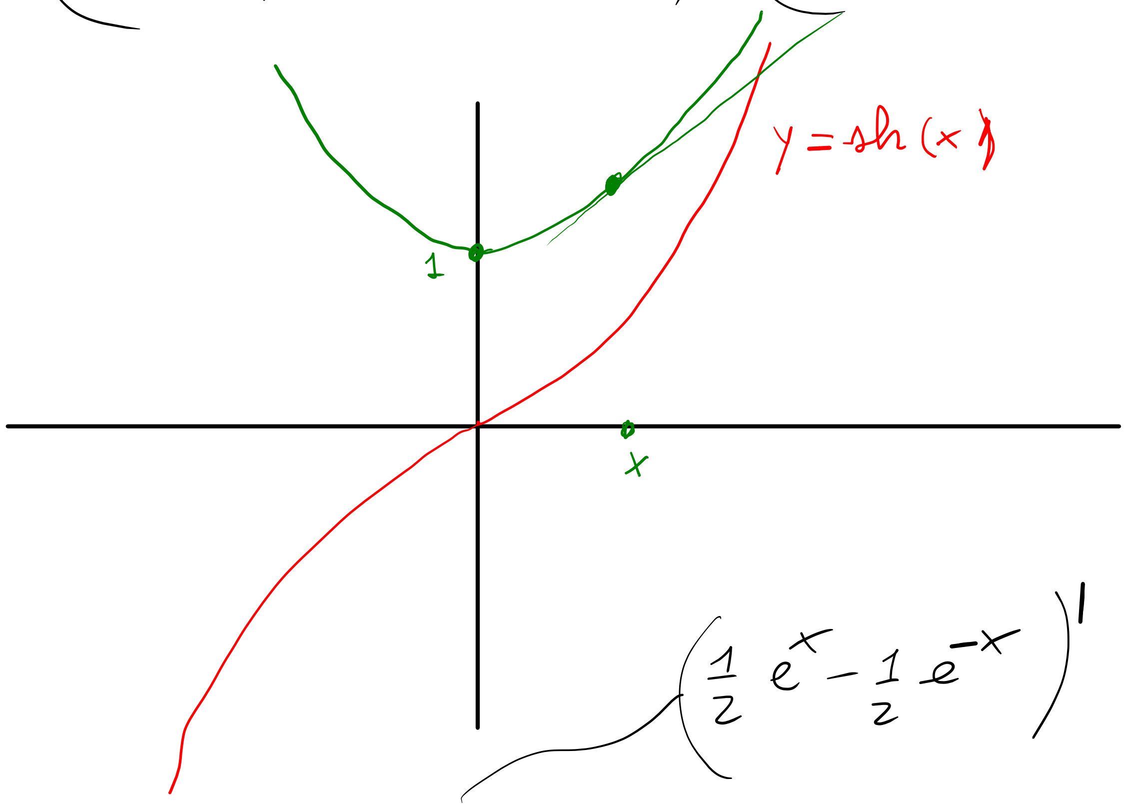
$$\lim_{x \to x_0} \frac{e^{-x} - e^{-x_0}}{x - x_0} = \lim_{h \to 0} \frac{e^{-h-x_0} - e^{-x_0}}{h} =$$

$$= \lim_{h \to 0} \frac{e^{-h} e^{-x_0} - e^{-x_0}}{h} =$$

$$= -e^{-x_0} \underbrace{\lim_{h \to 0} \frac{e^{-h} - 1}{-h}}_{=1} \qquad \left(\lim_{x \to 0} \frac{e^{x} - 1}{x} = 1 \right)$$

$$(e^{-x})'(x_0) = -e^{-x_0} \qquad \forall\, x_0 \in \mathbb{R}$$

3) $(\operatorname{sh} x)' = \operatorname{ch}(x)$, $(\operatorname{ch} x)' = \operatorname{sh}(x)$



$$\left(\frac{1}{2} e^{x} - \frac{1}{2} e^{-x}\right)'$$

$$\operatorname{sh}'(x) = \left(\frac{e^{x} - e^{-x}}{2}\right)' = \frac{1}{2}\left(e^{x}\right)' - \frac{1}{2}\left(e^{-x}\right)' =$$

$$= \frac{1}{2} e^{x} + \frac{1}{2} e^{-x} = \frac{e^{x} + e^{-x}}{2} =$$

$$= \operatorname{ch} x.$$

$$\operatorname{ch}'(x) = \left(\frac{e^{x} + e^{-x}}{2}\right)' = \left(\frac{1}{2} e^{x} + \frac{1}{2} e^{-x}\right)' =$$

$$= \frac{1}{2}\left(e^{x}\right)' + \frac{1}{2}\left(e^{-x}\right)' = \frac{1}{2} e^{x} - \frac{1}{2} e^{-x} = \frac{e^{x} - e^{-x}}{2}$$

$$= \operatorname{sh}(x)$$

5) $\quad th'(x) = \dfrac{1}{ch^2(x)} = 1 - th^2(x)$

$$\left(tg'(x) = \frac{1}{\cos^2(x)} = 1 + tg^2(x) \right)$$

4) $$tg'(x) = \left(\frac{\sin x}{\cos x}\right)' = \frac{\overbrace{(\sin x)'}^{\cos x}\cos x - \sin x\,\overbrace{(\cos x)'}^{-\sin x}}{\cos^2(x)} =$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$= \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = 1 + tg^2 x$$

$$sec(x) = \frac{1}{\cos x} \qquad \boxed{tg'(x) = sec^2(x)}$$

$$cosec(x) = \frac{1}{sen\, x}$$

Dim di 5) $$(th\, x)' = \left(\frac{sh\, x}{ch\, x}\right)' =$$

$$= \frac{\overbrace{(sh\, x)'}^{ch(x)}\, ch(x) - sh(x)\,\overbrace{(ch(x))'}^{sh(x)}}{ch^2(x)} =$$

$$= \frac{ch^2(x) - sh^2(x)}{ch^2(x)} = \frac{1}{ch^2(x)}$$

$$= 1 - th^2(x)$$

Teorema Sia $f : I \to J$ con I e J due intervalli.

Siano $x_0 \in I$, $y_0 = f(x_0) \in J$, sia f biettiva

e supponiamo esista $f'(x_0) \neq 0$. Allora se

$g : J \to I$ è la funzione inversa di f

allora esiste $g'(y_0)$ e si ha $g'(y_0) = \dfrac{1}{f'(x_0)}$.

Dim

$$\lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} =$$

Poniamo $y = f(x)$, $y_0 = f(x_0)$

Ricordando che $g(y) = x$ e $g(y_0) = x_0$

$$= \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0)} =: g'(y_0)$$

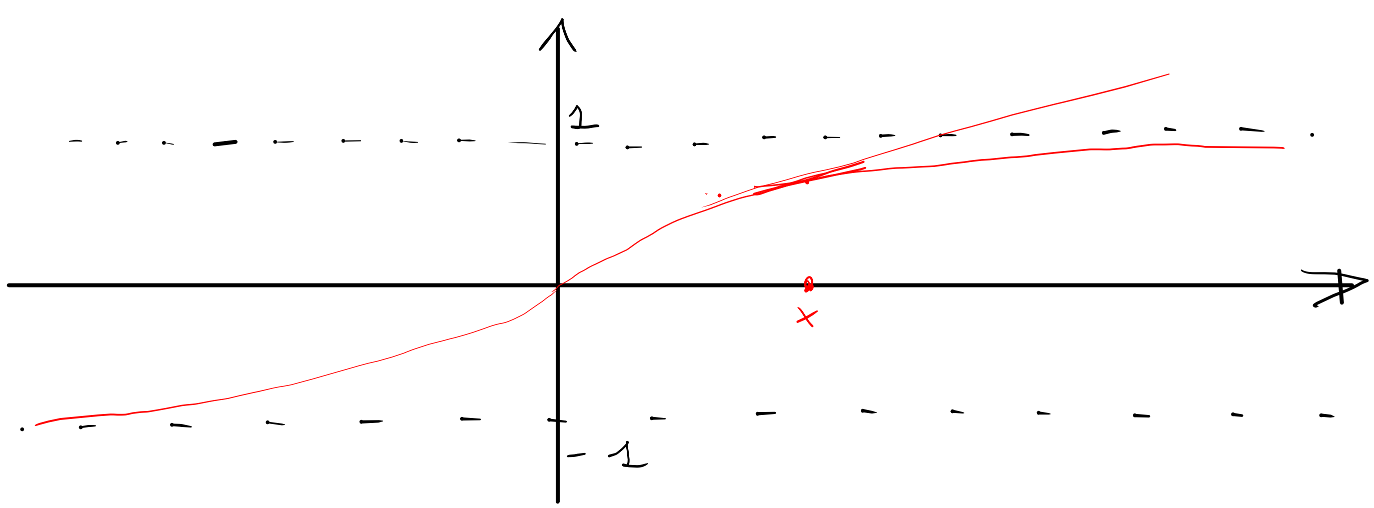
Esempi

1) $\left(\arctan y\right)' = \dfrac{1}{1+y^2}$

$x = \arctan y \implies \boxed{y = \tan x}$

$$\left(\arctan y\right)' = \frac{1}{\tan' x} = \frac{1}{1+\tan^2 x} = \frac{1}{1+y^2}$$

$$\left(\arctan x\right)' = \frac{1}{1+x^2}$$



Esercizio: se f è definita in $\mathbb{R}$ è pari allora f' è dispari

(dispari) (pari)

2) $\left(\arcsin y\right)' = \dfrac{1}{\sqrt{1-y^2}}$

$y \in [-1,1]$, $x = \arcsin y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

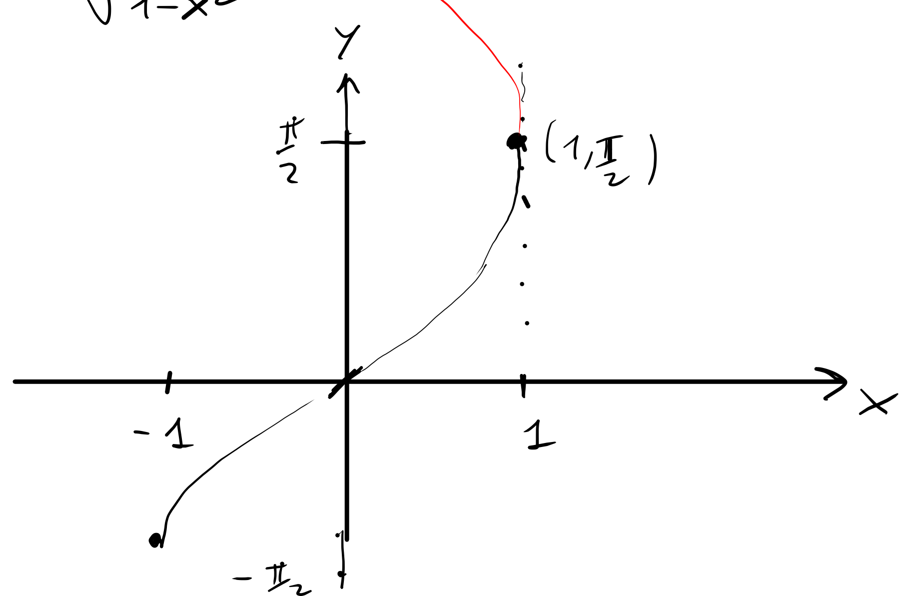
$y = \sin x$

$$\left(\arcsin y\right)' = \frac{1}{\sin' x} = \frac{1}{\cos x} = \frac{1}{\sqrt{1-\sin^2 x}} = \frac{1}{\sqrt{1-y^2}}$$

$\cos x = \pm\sqrt{1-\sin^2 x}$ ma qui $\cos x = \sqrt{1-\sin^2 x}$

perché $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\left(\arcsin x\right)' = \frac{1}{\sqrt{1-x^2}}$$



Esercizio. Sia $f : I \to \mathbb{R}$, $x_0 \in \mathring{I}$. Sono equivalenti le seguenti due proposizioni

1) $\exists \; f'(x_0)$

2) $\exists \; c \in \mathbb{R}$ t.c.

è l'approssimante lineare di $f(x)$ in x_0

$$f(x) = \boxed{f(x_0) + c\,(x-x_0)} + o(x-x_0)$$

(Qui ricordo che $\displaystyle\lim_{x \to x_0} \frac{o(x-x_0)}{x-x_0} = 0$)

Quando 1) e 2) sono vere, si ha $c = f'(x_0)$.

Teor (Regola della catena)

Siano $f: I \longrightarrow J$

$g: J \longrightarrow \mathbb{R}$

$x_0 \in I$, $y_0 \in J$ con $y_0 = f(x_0)$

$$\begin{tikzcd} I \arrow[r, "f"] \arrow[rr, bend right, "g(f(x))"'] & J \arrow[r, "g"] & \mathbb{R} \end{tikzcd}$$

Allora, se esistono $g'(y_0)$ e $f'(x_0)$

si ha $\big(g(f(x))\big)'(x_0) = g'(y_0)\, f'(x_0)$

Osservazione Poniamo $f(x) = y(x)$

$$\begin{tikzcd} I \arrow[r, "y(x)"] & J \arrow[r, "g(y)"] & \, \end{tikzcd}$$

$$\frac{d}{dx} g(y(x)) = \frac{d}{dx} g(y) = \frac{d}{dy} g(y) \, \frac{dy}{dx}$$

Dim So che $\exists\, g'(y_0)$ e $f'(x_0)$ e applicando l'espansione

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + o(x-x_0)$$

$$g(y) = g(y_0) + g'(y_0)(y-y_0) + o(y-y_0) \qquad y_0 = f(x_0)$$

$$g(f(x)) = g(f(x_0)) + g'(f(x_0))\big(f(x)-f(x_0)\big) + o\big(f(x)-f(x_0)\big)$$

$$= g(f(x_0)) + g'(f(x_0))\big(f'(x_0)(x-x_0) + o(x-x_0)\big)$$

$$+\, o\big(f'(x_0)(x-x_0) + o(x-x_0)\big) \quad +$$

$$= \boxed{g(f(x_0)) + g'(f(x_0))\, f'(x_0)\,(x-x_0)} \; +$$

$$+ \underbrace{\Big[g'(f(x_0))\, o(x-x_0) + o\big(f'(x_0)(x-x_0) + o(x-x_0)\big) \Big]}_{\text{verifichiamo } o(x-x_0)}$$

Si tratta di dimostrare che

$$\lim_{x \to x_0} \frac{\big(g'(f(x_0))\, o(x-x_0)\big) + o\big(f'(x_0)(x-x_0) + o(x-x_0)\big)}{x - x_0} = 0$$

$$\lim_{x \to x_0} \Big[g'(f(x_0)) \underbrace{\frac{o(x-x_0)}{x-x_0}}_{\to 0} + \frac{o\big(f'(x_0)(x-x_0) + o(x-x_0)\big)}{x-x_0} \Big] = 0$$

$$= \lim_{x \to x_0} \frac{o\big(f'(x_0)(x-x_0) + o(x-x_0)\big)}{x-x_0}$$

supponiamo che $f'(x_0) \neq 0$

Se $f'(x_0) \neq 0$

$$\lim_{x \to x_0} \frac{o\big(f'(x_0)(x-x_0) + o(x-x_0)\big)}{x-x_0} \quad \frac{f'(x_0)(x-x_0) + o(x-x_0)}{f'(x_0)(x-x_0) + o(x-x_0)}$$

$$= \lim_{x \to x_0} \frac{o\big(f'(x_0)(x-x_0) + o(x-x_0)\big)}{f'(x_0)(x-x_0) + o(x-x_0)} \quad \frac{f'(x_0)(x-x_0) + o(x-x_0)}{x-x_0}$$

$$= \lim_{x \to x_0} \underbrace{\frac{o\big(f'(x_0)(x-x_0) + o(x-x_0)\big)}{f'(x_0)(x-x_0) + o(x-x_0)}}_{\to 0} \underbrace{\Big[f'(x_0) + \frac{o(x-x_0)}{x-x_0} \Big]}_{\to f'(x_0)} = 0$$

Ora supponiamo $f'(x_0) = 0$

$$\lim_{x \to x_0} \frac{o(o(x-x_0))}{x-x_0} = 0$$

Il teorema è dimostrato.

Esempi

1) $\left(x^{x}\right)' = ?$ $\qquad x > 0$

$x^{x} = e^{\lg(x^{x})} =$ $\qquad y = e^{\lg y} \quad y > 0$

$= e^{x \lg x}$ $\qquad g(y) = e^{y} \qquad f(x) = x \lg x$

$\left(x^{x}\right)' = \left(e^{x \lg x}\right)' = e^{x \lg x} \left(x \lg x\right)' =$

$= x^{x} \left(\lg x + x \frac{1}{x}\right) = x^{x}\left(\lg x + 1\right)$

Esercizio $\quad [x] : \mathbb{R} \longrightarrow \mathbb{Z} \quad$ t.c.

$[x] \leq x < [x] + 1$

Dimostrate che $f(x) = \left([x]^{3} + [x]\right) e^{-[x]^{2}}$

ha punti di minimo e punti di massimo assoluto in $\mathbb{R}$.