CRITERIO STANDARD DI MISURABILITÀ

$$\mathcal{G}' = \{ A' \subset \Omega' \mid f^{-1}(A') \in \mathcal{F} \}$$

VOGLIO FAR VEDERE CHE $\mathcal{F}' \subset \mathcal{G}'$

SO CHE $\mathcal{C}' \subset \mathcal{G}'$ PER IPOTESI

MOSTRIAMO CHE $\mathcal{G}'$ È UNA $\sigma$-ALGEBRA SU $\Omega'$

1) $\Omega' \in \mathcal{G}'$? OK, $f^{-1}(\Omega') = \Omega \in \mathcal{F}$

2) SE $A' \in \mathcal{G}' \Rightarrow \overline{A'} \in \mathcal{G}'$

$A' \in \mathcal{G}' \Rightarrow f^{-1}(A') \in \mathcal{F}$

$$f^{-1}(\overline{A'}) = \overline{f^{-1}(A')} \in \mathcal{F}$$

$\in \mathcal{F}$

3) $(A'_n) \subset \mathcal{G}' \Rightarrow \bigcup A'_n \in \mathcal{G}'$

$A'_n \in \mathcal{G}' \Rightarrow f^{-1}(A'_n) \in \mathcal{F}$

PER OGNI $n$

$$f^{-1}\left(\bigcup_n A'_n\right) = \underbrace{\bigcup_n \underbrace{f^{-1}(A'_n)}_{\in \mathcal{F}}}_{\in \mathcal{F}}$$

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$$\begin{cases} \mathcal{G}' \text{ È UNA } \sigma\text{-ALGEBRA} \\ \mathcal{C}' \subset \mathcal{G}' \\ \mathcal{F}' = \sigma(\mathcal{C}') \end{cases} \Rightarrow \boxed{\mathcal{F}' \subset \mathcal{G}'}$$

PER MINIMALITÀ DI $\mathcal{F}' = \sigma(\mathcal{C}')$

$f : \Omega \longrightarrow \mathbb{R}^n$

$f = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$ CON $f_i : \Omega \longrightarrow \mathbb{R}$

$f$ MISURABILE $\Longleftrightarrow$ $f_i$ MISURABILE PER OGNI $i$

<u>$n = 2$</u>

$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$

$\Longrightarrow$ $B \in \mathcal{B}$ $\quad f_1^{-1}(B) = f_1^{-1}(B) \cap \Omega =$

$$= f_1^{-1}(B) \cap f_2^{-1}(\mathbb{R}) = \underbrace{f^{-1}(\underbrace{B \times \mathbb{R}}_{\in \mathcal{B}^2})}_{} \in \mathcal{F}$$

$$\{\omega \in \Omega \mid f(\omega) \in B \times \mathbb{R}\}$$

$$= \left\{\omega \in \Omega \;\middle|\; \begin{bmatrix} f_1(\omega) \\ f_2(\omega) \end{bmatrix} \in B \times \mathbb{R}\right\}$$

$$= \{\omega \in \Omega \mid f_1(\omega) \in B,\ f_2(\omega) \in \mathbb{R}\}$$

$\Leftarrow$ $f_1, f_2$ È MISURABILE, ALLORA $f$ È MISURABILE

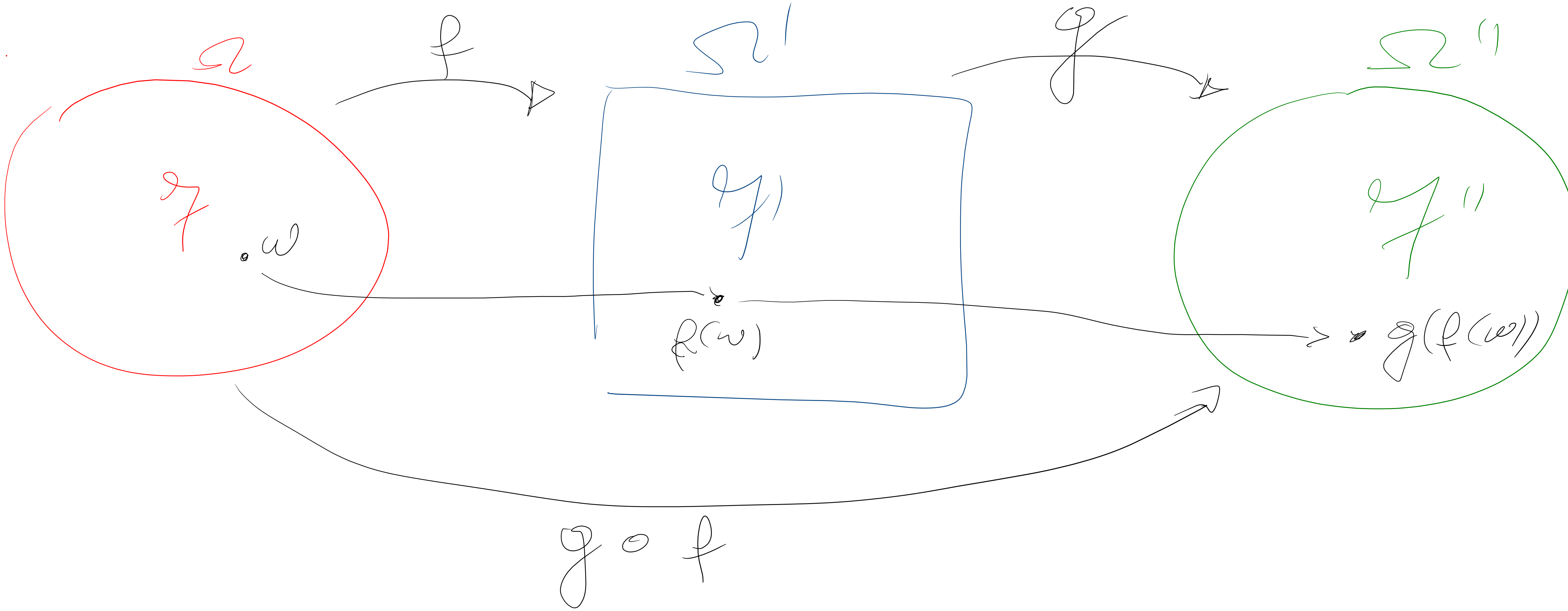
PER IL CRITERIO STANDARD DI MISURABILITÀ

$$f^{-1}((-\infty, a_1] \times (-\infty, a_2]) = \qquad a_1, a_2 \in \mathbb{R}$$

$$= \{\omega \in \Omega \mid f(\omega) = \begin{bmatrix} f_1(\omega) \\ f_2(\omega) \end{bmatrix} \in (-\infty, a_1] \times (-\infty, a_2]\}$$

$$= \{\omega \in \Omega \mid f_1(\omega) \in (-\infty, a_1],\ f_2(\omega) \in (-\infty, a_2]\}$$

$$= \underbrace{f_1^{-1}((-\infty, a_1])}_{\in \mathcal{F}} \cap \underbrace{f_2^{-1}((-\infty, a_2])}_{\in \mathcal{F}} \qquad \in \mathcal{F}$$




$$(g \circ f)(\omega) = g(f(\omega))$$

$A'' \in \mathcal{F}''$ $\qquad (g \circ f)^{-1}(A'') \in \mathcal{F}$

$$(g \circ f)^{-1}(A'') = f^{-1}\Big(\underbrace{g^{-1}(A'')}_{\in \mathcal{F}'}\Big) \in \mathcal{F}$$

PERCHÉ?

$h = \text{MAX}\{f, g\}$ È MISURABILE

$$\underbrace{h^{-1}((-\infty, a])}_{\{\omega \in \Omega \mid h(\omega) \in (-\infty, a]\}} \in \mathcal{F} \qquad \text{PER OGNI } a \in \mathbb{R}$$

$$= \{\omega \in \Omega \mid h(\omega) \leq a\}$$

$$= \{\omega \in \Omega \mid \text{MAX}\{f(\omega), g(\omega)\} \leq a\}$$

$$= \{\omega \in \Omega \mid f(\omega) \leq a, \ g(\omega) \leq a\} =$$

$$= \underbrace{f^{-1}((-\infty, a])}_{\in \mathcal{T}} \cap \underbrace{g^{-1}((-\infty, a])}_{\in \mathcal{T}} \in \mathcal{T}$$

$$\Omega = \overline{\mathbb{R}}$$

$$\mathbb{R} \in \overline{\mathcal{B}}$$

$$\overline{\mathbb{R}} = \{-\infty, +\infty\} \in \overline{\mathcal{B}}$$

$$A_n = (-\infty, -n) \qquad \bigcap_n A_n = \emptyset$$

$$B_n = [-\infty, -n) \qquad \bigcap_n B_n = \{-\infty\}$$

$E_1 = \{ \omega \in \Omega \mid \lim f_n(\omega)$ ESISTE IN $\mathbb{R} \} \in \mathcal{F}$

$P(E_1) = 1$

$\lim f_n$ MISURABILE $\mathcal{F}/\mathcal{B}$

$\left(\lim_n f_n\right)(\omega) = \lim_n f_n(\omega)$ ESISTE IN $\mathbb{R}$ PER OGNI $\omega \in E_1$

AL DI FUORI DI $E_1$, SI DEFINISCE LA VARIABILE IN OGGETTO A PIACERE

$\left(\lim_n f_n\right)(\omega) = 0$ PER $\omega \notin E_1$

$\Omega = [-1, 1]$

$f_n(x) = x^n$

$$\lim_{n \to +\infty} f_n(x) = \begin{cases} \text{NON ESISTE} & x = -1 \\ 0 & -1 < x < 1 \\ 1 & x = 1 \end{cases}$$

$E_1 = (-1, 1]$

$f(x) = e^{-x^2} : \mathbb{R} \to \mathbb{R}$

$\ast\ \underbrace{f^{-1}((-\infty, a])}_{} \in \mathcal{B}$ per ogni $a \in \mathbb{R}$

$\{ x \in \mathbb{R} \mid f(x) \leq a \}$

$e^{-x^2} \leq a$

<u>SE $a < 0$</u> $\quad \ast = \emptyset \in \mathcal{B}$

$\underline{a > 0}$

$$e^{-x^2} \leq a \iff -x^2 \leq \log a$$
$$\iff x^2 \geq -\log a$$

SE $-\log a \leq 0 \iff \log a \geq 0 \iff \underline{a \geq 1}$

ALLORA $* = \mathbb{R}$ c.B

$\underline{0 < a < 1}$ $\quad x^2 \geq -\log a$

$$|x| \geq \sqrt{-\log a} \iff x \geq \sqrt{-\log a}$$
$$\vee \; x \leq -\sqrt{-\log a}$$

$$* = \left(-\infty, -\sqrt{-\log a}\,\right] \cup \left[\sqrt{-\log a}, +\infty\right) \in \mathcal{B}$$