? $f(X)(\omega) = \begin{cases} f(X(\omega)) & \text{SE } X(\omega) \in D \\ 0 & X(\omega) \notin D \end{cases}$

$P(X \in D) = 1$

$\frac{X}{Y}$ Y DISTRIBUZIONE NORMALE

$P(Y=0) = 0$

$(\Omega, \mathcal{F}, P)$

$\underline{X} : \Omega \longrightarrow \mathbb{R}^n \qquad \mathcal{B}^n$

$\underline{Y} : \Omega \longrightarrow \mathbb{R}^m \qquad \mathcal{B}^m$

$\sigma(\underline{X}, \underline{Y})$ = σ-ALGEBRA GENERATA DA $\underline{X}, \underline{Y}$

= PIÙ PICCOLA σ-ALGEBRA SU Ω TALE CHE RENDE SIA $\underline{X}$ CHE $\underline{Y}$ MISURABILI

$$\underline{X} : (\Omega, \sigma(\underline{X}, \underline{Y})) \longrightarrow (\mathbb{R}^n, \mathcal{B}^n)$$

$$\underline{Y} : \quad // \quad \longrightarrow (\mathbb{R}^m, \mathcal{B}^m)$$

MISURABILI

$$\sigma(\underline{X}, \underline{Y}) = \bigcap \left\{ \mathcal{G} \;\middle|\; \begin{array}{l} \underline{X} \quad \mathcal{G} \backslash \mathcal{B}^n \text{ MISURABILE} \\ \underline{Y} \quad \mathcal{G} \backslash \mathcal{B}^m \quad // \end{array} \right\}$$

INFORMAZIONE RISULTANTE DALL' OSSERVARE $\underline{X}, \underline{Y}$

$$= \sigma\Bigg(\overbrace{\left\{ \begin{array}{l} \underline{X}^{-1}(B),\ B \in \mathcal{B}^n \\ \underline{Y}^{-1}(D),\ D \in \mathcal{B}^m \end{array} \right\}}^{\text{NON È UNA } \sigma\text{-ALGEBRA}} \Bigg)$$

$$f : \Omega \longrightarrow \Omega'$$
$$\mathcal{F}'$$

$$\sigma(f) = \underbrace{\{ f^{-1}(A') \mid A' \in \mathcal{F}' \}}_{*}$$

$*$ È UNA σ-ALGEBRA

1) $\Omega \in *$ BASTA PRENDERE $A' = \Omega' \in \mathcal{F}'$

$$f^{-1}(\Omega') = \Omega \in *$$

2) $A \in * \Rightarrow \bar{A} \in *$

$$A = f^{-1}(A') \quad \text{con} \quad A' \in \mathcal{T}'$$

ALLORA

$$\overline{A} = \overline{f^{-1}(A')} = f^{-1}(\underbrace{\overline{A'}}_{\in \mathcal{T}'})$$

3) $(A_n) \subset \mathcal{A} \Rightarrow \bigcup_n A_n \in \mathcal{A}$

$$A_n = f^{-1}(A'_n) \quad \text{con} \quad A'_n \in \mathcal{T}', \quad \text{PER OGNI } n$$

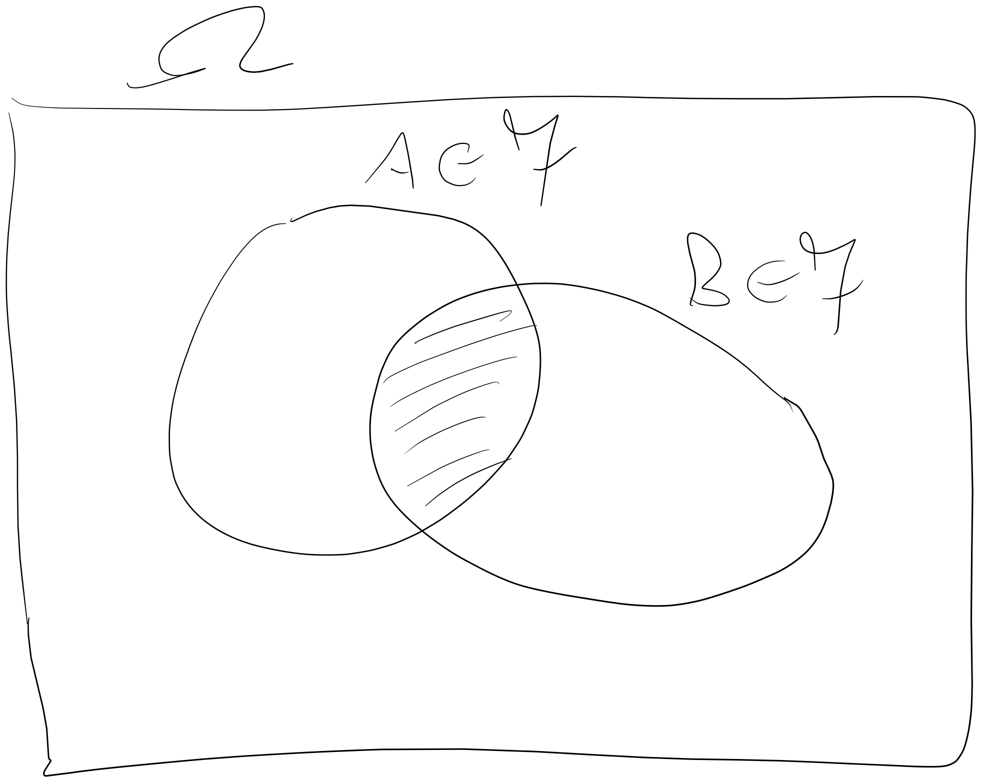
$$\bigcup_n A_n = \bigcup_n f^{-1}(A'_n) = f^{-1}(\underbrace{\bigcup_n A'_n}_{\in \mathcal{T}'})$$

X, Y V.A.

Y È $\sigma(X)$ MISURABILE

SE CONOSCO X, POSSO "RICAVARE" Y

CIOÈ $\quad Y = g(X)$



$B \cap A \subset A$

$f : \Omega \to \mathbb{R}$ $\quad \mathcal{F}$ MISURABILE

A ATOMO DI $\mathcal{F} \Rightarrow f$ COSTANTE SU A

$A \neq \emptyset \quad \omega^* \in A \quad f(\omega^*) = x^* \in \mathbb{R}$

$$B = f^{-1}(\underbrace{\{x^*\}}_{\text{BORELIANO}}) \in \mathcal{F}$$

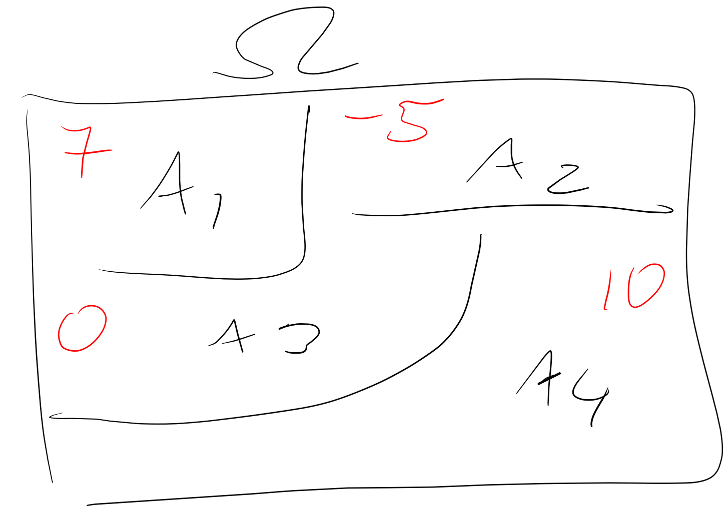
$\omega^* \in A \cap B$ CIOÈ $A \cap B \neq \emptyset$

INOLTRE $A \cap B \subset A$

QUINDI (A È UN ATOMO)

SI HA CHE $A \cap B = A$ CIOÈ $A \subset B = f^{-1}(\{x^*\})$

DA CUI PER OGNI $\omega \in A$, $f(\omega) = x^*$



$\mathbb{P} = \{A_1, A_2, A_3, A_4\}$

$\sigma(\mathbb{P}) = \{\emptyset, A_1, \ldots, A_4,$

$A_1 \cup A_2, A_1 \cup A_3, \ldots$

$A_1 \cup A_2 \cup A_3, \ldots, \Omega\}$

$A_1, A_2, \ldots, A_4$ SONO ATOMI DI $\sigma(\mathbb{P})$

f È $\sigma(\mathbb{P})$ MISURABILE $\Longleftrightarrow$ f È COSTANTE SU OGNUNO DEGLI A_i

$$\sigma(1_A) = \sigma(\underbrace{\{A, \bar{A}\}}_{\text{PARTIZIONE}})$$

f è $\sigma(1_A)$ MISURABILE

$$f(\omega) = \begin{cases} x_1 & \omega \in A \\ x_2 & \omega \notin A \end{cases}$$

$$f = x_1 \cdot 1_A + x_2 \cdot 1_{\bar{A}} = (x_1 - x_2) 1_A + x_2$$

$$X = \begin{cases} x_1 \\ x_2 \\ x_3 \end{cases}$$

$$\sigma(X) = \left\{ \{X \in B\} \ , \ B \in \mathcal{B} \right\}$$

$$= \left\{ \emptyset, \{X = x_1\}, \{X = x_2\}, \{X = x_3\}, \right.$$

$$\{X = x_1 \text{ o } x_2\}, \{X = x_1 \text{ o } x_3\}, \{X = x_2 \text{ o } x_3\}$$

$$\left. \Omega \right\} = \sigma\Big(\underbrace{\{ \{X = x_1\}, \{X = x_2\}, \{X = x_3\} \}}_{\text{PARTIZIONE}} \Big)$$

Y È $\sigma(X)$ - MISURABILE

$\Leftrightarrow$ Y È COSTANTE SU $\{X = x_i\}$ $\Leftrightarrow$ Y È UNA FUNZIONE DI X

$$Y = \begin{cases} y_1 & X = x_1 \\ y_2 & X = x_2 \\ y_3 & X = x_3 \end{cases} = y_1 \mathbb{1}_{\{X = x_1\}} + y_2 \mathbb{1}_{\{X = x_2\}} + y_3 \mathbb{1}_{\{X = x_3\}}$$

$$Y = \begin{cases} 10 & X = x_1 \\ 10 & X = x_2 \\ 5 & X = x_3 \end{cases}$$

$$\sigma(Y) \subset \sigma(X)$$

$$\begin{tikzcd}
\Omega \ \ \boxed{\sigma(\underline{X})} \arrow[r, "\underline{X}"] \arrow[d, "\underline{Y}"'] & \mathbb{R}^n \arrow[dl, "g"] \\
\mathbb{R}^m &
\end{tikzcd}$$

$$\underline{Y} = g(\underline{X})$$

$$Y_1 = X_1$$

$$Y_2 = X_1 + X_2$$

$$\vdots$$

$$Y_n = X_1 + \ldots + X_n = Y_{n-1} + X_n$$

$$\Rightarrow X_n = Y_n - Y_{n-1} \quad \text{PER } n > 1$$

$$X_1 = Y_1$$

$\{\phi, \Omega\}$ PIÙ PICCOLA

$\mathcal{F}$ // GRANDE

$$\sigma(X) = \sigma(X + c)$$

$$\cup$$

$$\sigma(|X|) = \sigma(X^2)$$

$$X^2 = |X|^2$$

$$|X| = \sqrt{X^2}$$

$$\sigma(X, Y) = \sigma(X, X+Y) = \sigma(X+Y, X-Y)$$

$$\cup$$

$$\sigma(X) \subset \sigma(X, X^2 - Y^2) = \sigma(X, Y^2)$$

$x_1 \_\_\_ x_n$

$Y_1 = X_1$

$Y_2 = X_1 \cdot X_2$

$\vdots$

$Y_n = X_1 \cdot \_\_\_ \cdot X_n$

$\sigma(X_1 \_\_\_ X_n) \supset \sigma(Y_1 \_\_\_ Y_n)$

SE $X_2 = 0$

$\Rightarrow Y_2 = 0$

DA (Y_1, Y_2, Y_3) NON RIESCO A CALCOLARE (X_1, X_2, X_3)

$(Y_1 = 5, Y_2 = 0, Y_3 = 0)$

$\rightarrow (X_1 = 5, X_2 = 0, X_3 = ?)$

1) F 2) F

3) V OSSERVIAMO CHE

$$Z = X_1 + X_2 = U + V$$

$$\sigma(Z, U) = \sigma(Z, V)$$

$$V = Z - U$$

4) $V - U = |X_1 - X_2|$

$$1_{X_1 > X_2} = 1_{X_1 - X_2 > 0}$$

$$\left\{ \begin{array}{l} \sigma(X_1 - X_2) \supset \\ \supset \sigma(V - U, 1_{X_1 > X_2}) \end{array} \right.$$

$$X_1 - X_2 = (V - U)\,1_{X_1 > X_2}$$

$$- (V - U)(1 - 1_{X_1 > X_2})$$

$$\sigma(X_1 - X_2) \subset \sigma(V - U,\ 1_{X_1 > X_2})$$

5) F

$\sigma(X_1, U) \cap \sigma(X_2, V) = \{\emptyset, \Omega\}$? <u>no</u>

$$\overbrace{\{X_1 = \overset{5}{i}, U = \overset{2}{j}\}}^{\in} = \overbrace{\{X_2 = 2, V = 5\}}^{\in}$$

con $j < i$

6) F

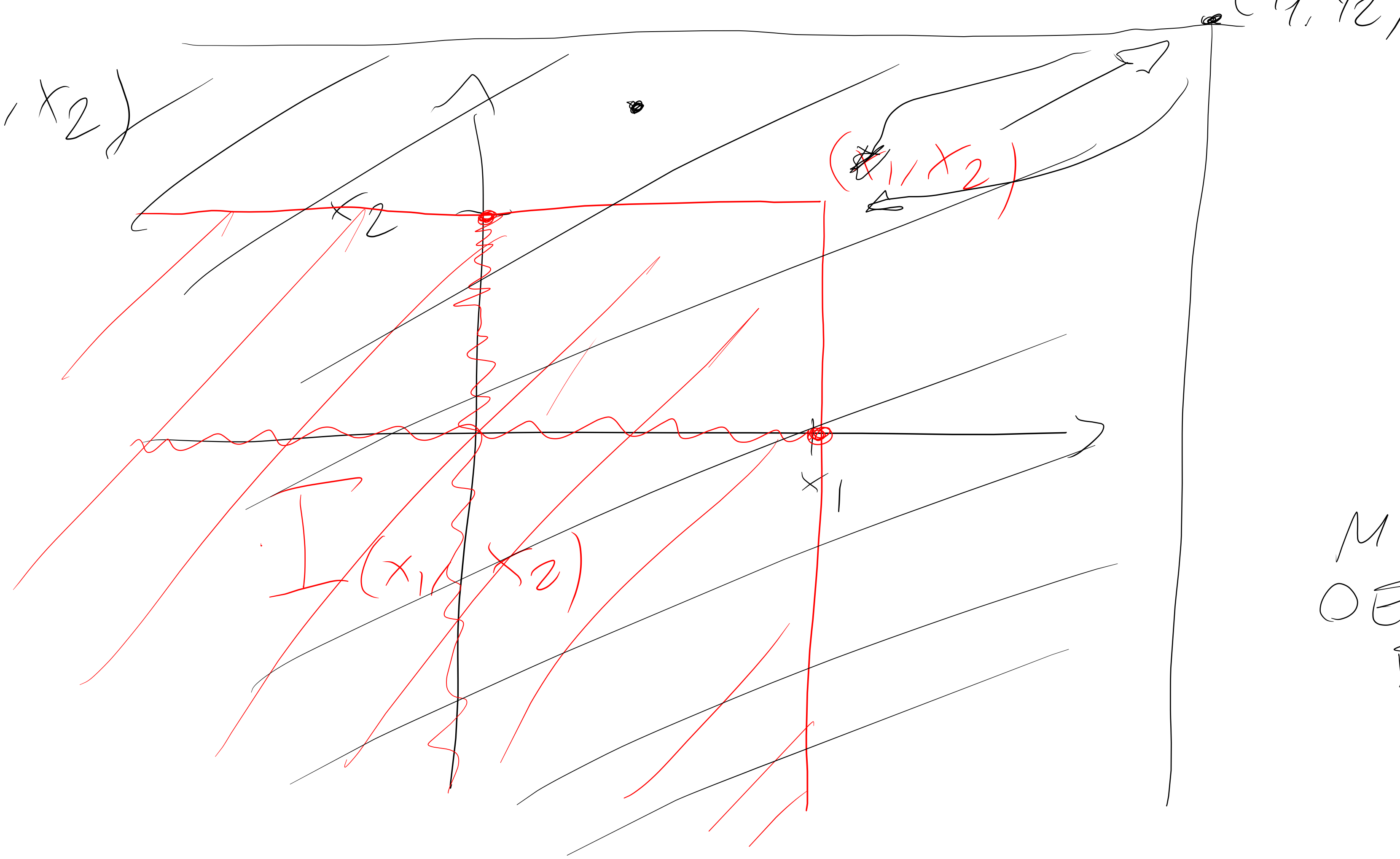
$\{Z = 2\} = \{W = 1\} \in \sigma(Z) \cap \sigma(W)$

7)

$\{U = 6\} \not\subset \{V = 6\}$

V $\quad \sigma(U) \cap \sigma(V) = \{\emptyset, \Omega\}$

$I_{(x_1,x_2)}$



(X_1, X_2)

MONOTONIA
DELLA
PROBABILITÀ

$$\{(X_1,X_2)\in I_{(x_1,x_2)}\} \qquad \{(X_1,X_2)\in I_{(y_1,y_2)}\}$$

$$\{X_1\le x_1,\ X_2\le x_2\} \subset \{X_1\le y_1,\ X_2\le y_2\}$$

$$P_{\underline{X}}(B) = P(\underline{X}^{-1}(B)) \qquad B \in \mathcal{B}^n$$

$$P_{\underline{X}} : \mathcal{B}^n \longrightarrow [0, 1]$$

↳ È UNA PROBABILITÀ SU $(\mathbb{R}^n, \mathcal{B}^n)$

1) $P_{\underline{X}}(\emptyset) = 0 \qquad P_{\underline{X}}(\mathbb{R}^n) = 1$

$$P_{\underline{X}}(\emptyset) = P(\underline{X} \in \emptyset) = P(\emptyset) = 0$$

$$P_{\underline{X}}(\mathbb{R}^n) = P(\underline{X} \in \mathbb{R}^n) = P(\Omega) = 1$$

2) σ- ADDITIVITÀ

$B_1, \ldots, B_n, \ldots \in \mathcal{B}^n$

A DUE A DUE DISGIUNTI

$$P_{\underline{X}}\left(\bigcup_n B_n\right) = \sum_n P_{\underline{X}}(B_n)$$

σ- ADDITIVITÀ DI P

$$P_{\underline{X}}\left(\bigcup B_n\right) = P\left(\underline{X}^{-1}\left(\bigcup_n B_n\right)\right) =$$

$$= P\left(\bigcup_n \underline{X}^{-1}(B_n)\right) = \sum_n P(\underline{X}^{-1}(B_n)) =$$

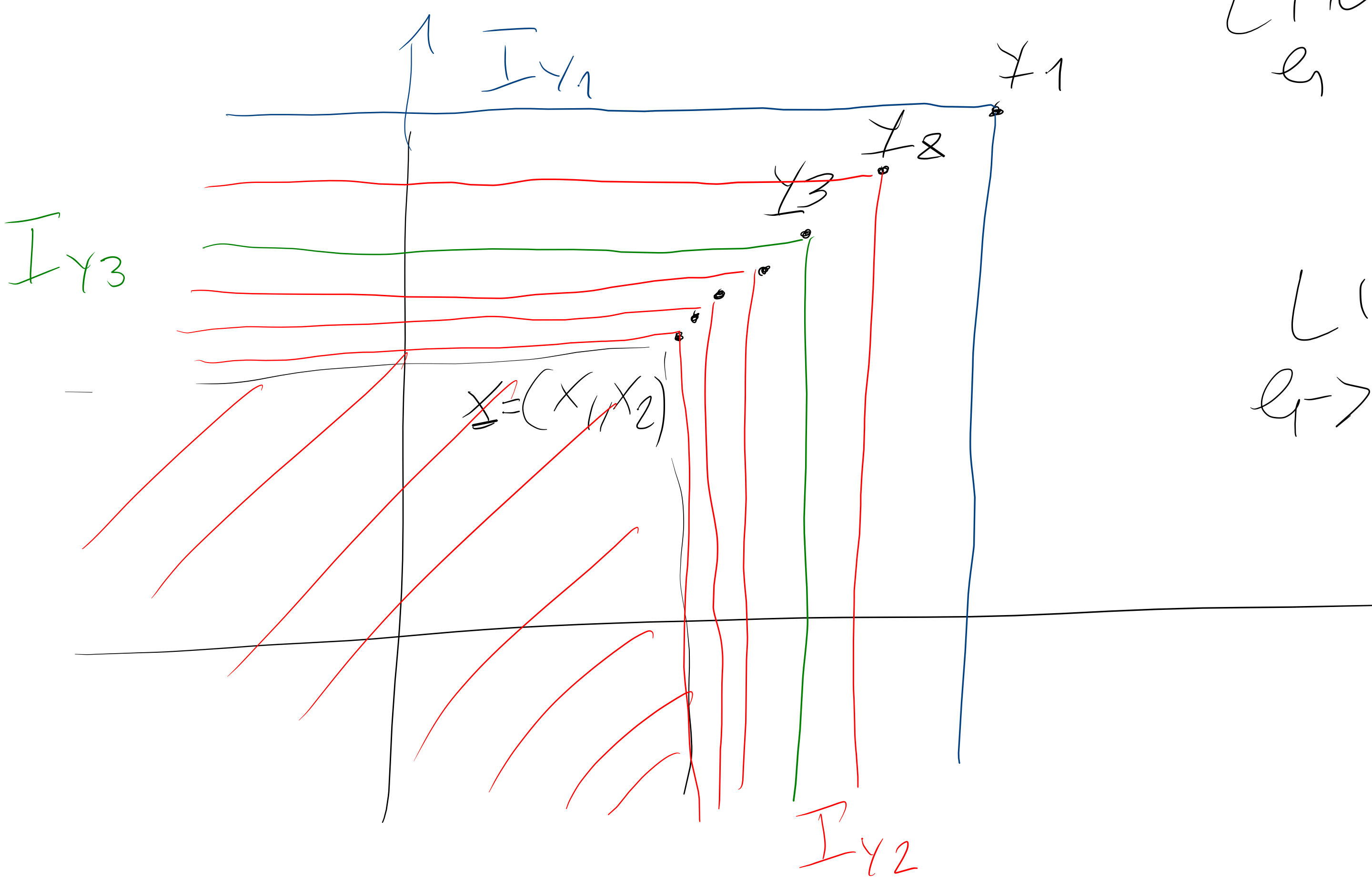
$$n \neq m \quad \underline{X}^{-1}(B_n) \cap \underline{X}^{-1}(B_m) = \underline{X}^{-1}(\underbrace{B_n \cap B_m}_{=\emptyset}) = \underline{X}^{-1}(\emptyset) = \emptyset$$

$$= \sum_{\ell_1} P_{\underline{X}}(B_{\ell_1})$$

$\underline{x} = (x_1 \ldots x_n) \in \mathbb{R}^n$

$\underline{y}_k = (y_{1k}, \ldots, y_{nk})$

CON $y_{ik} \downarrow x_i \quad \left(x_i < y_{ik} < y_{i(k-1)}, \ \lim_{k} y_{ik} = x_i \right)$



$$\lim_{k \to +\infty} \underbrace{F_{\underline{X}}(\underline{y}_k)}_{P(\underbrace{\underline{X} \in I_{y_k}}_{A_k})} = F_{\underline{X}}(\underline{x})$$

$$A_1 \supset A_2 \supset A_3 \supset \dots \supset A_n \supset A_{n+1} \supset \dots$$

SUCCESSIONE NON CRESCENTE

$$P\left(\lim_n A_n\right) = \underbrace{\lim_n P(A_n)}_{\lim_n F_X(x_n)}$$

$$\Downarrow$$

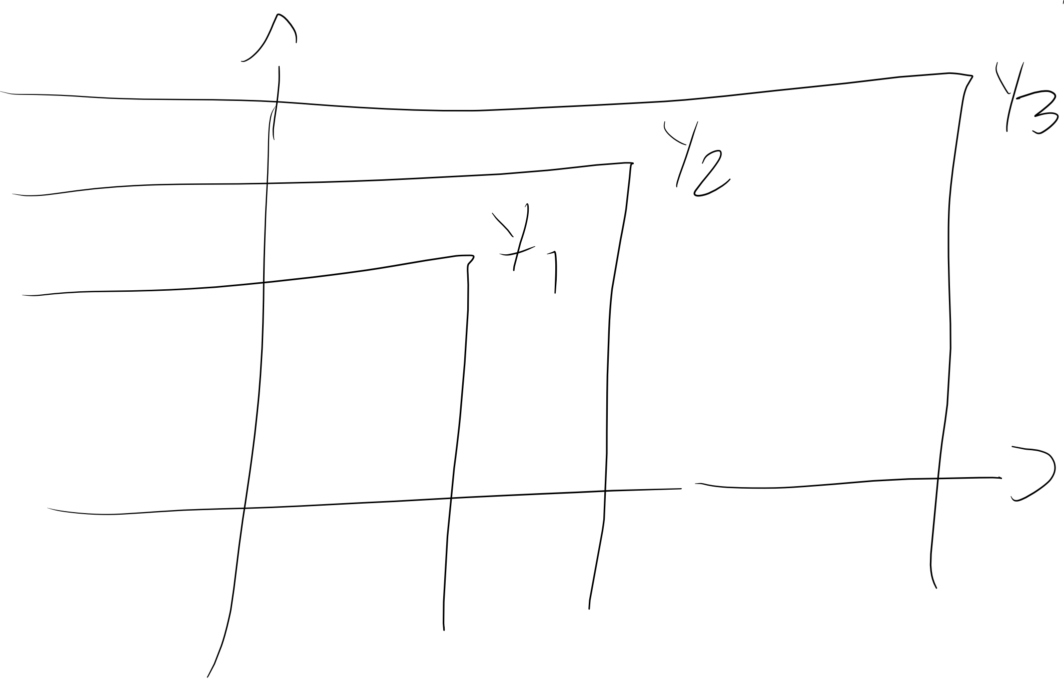
$$P\left(\underbrace{\bigcap_n A_n}\right) \qquad \bigcap_n \{X_1 \le y_{1n}, \dots, X_n \le y_{nn}\}$$

$$P(\{X_1 \le x_1, \dots, X_n \le x_n\}) = F_X(x)$$

$\underline{y}_k = (y_{1k} \ldots y_{nk})$ con $y_{ik} < y_{i(k+1)}$

e $\lim_k y_{ik} = +\infty$

$A_k = \{\underline{X} \in I_{\underline{y}_k}\} = \{x_1 \le y_{1k}, \ldots, x_n \le y_{nk}\}$

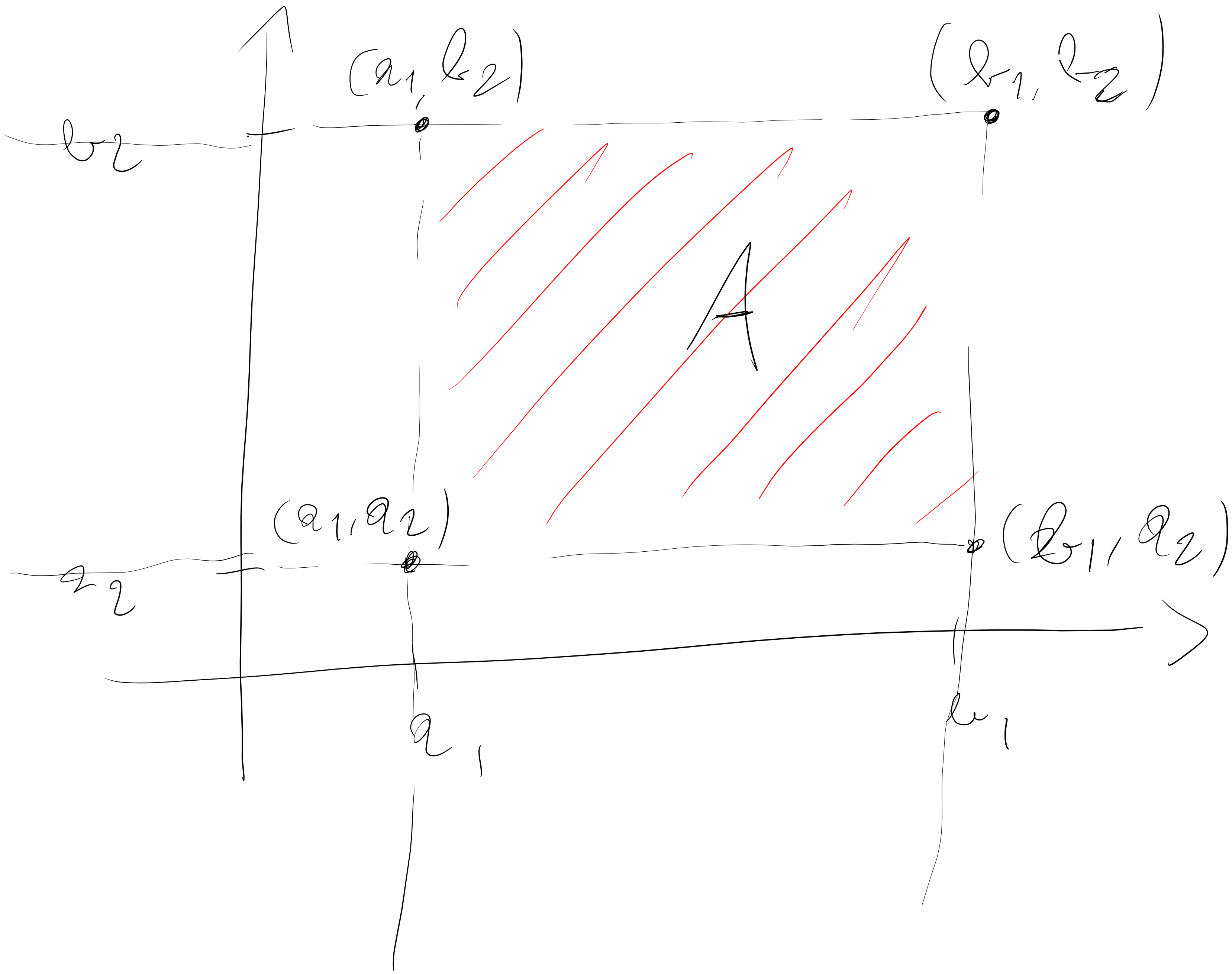


$A_k \subset A_{k+1}$

$\bigcup A_k = \Omega$

$\underbrace{P(\lim A_k)}_{=1} = \lim_k \underbrace{P(A_k)}_{F_{\underline{X}}(\underline{y}_k)}$

$a_1 < b_1 \qquad a_2 \le b_2$

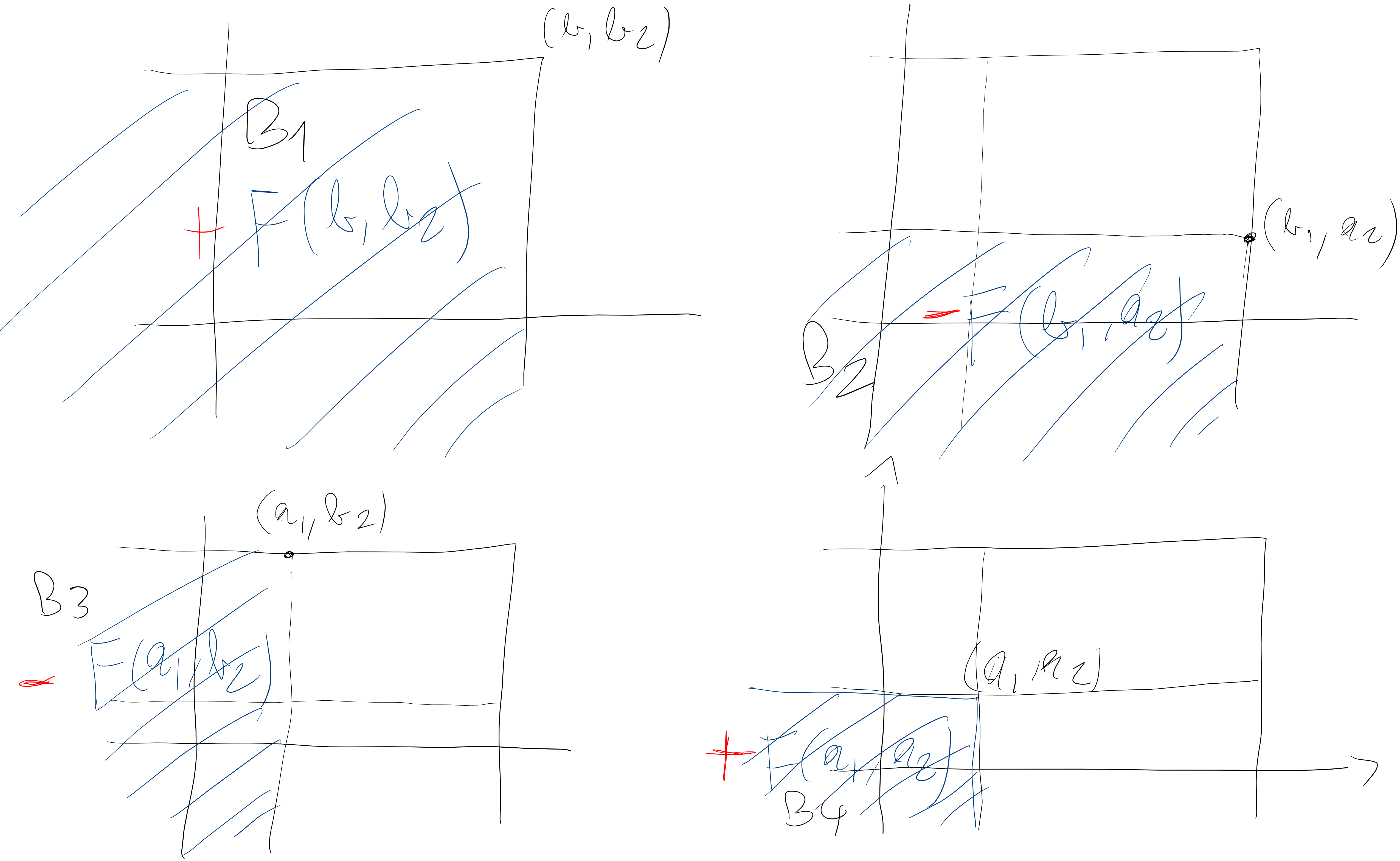


$$F_{(x_1,x_2)}^{(x_1,x_2)} = F^{(x_1,x_2)}$$

$$\Delta_{a_1,b_1}^{(1)} \underbrace{\Delta_{a_2,b_2}^{(2)} F}_{F(x_1,b_2)-F(x_1,a_2)} =$$

$$= \{F(b_1,b_2) - F(b_1,a_2)\} - \{F(a_1,b_2) - F(a_1,a_2)\}$$

$$= F(b_1,b_2) - F(b_1,a_2) - F(a_1,b_2) + F(a_1,a_2) \geq 0$$


(b_1, b_2)
B_1
$+F(b_1, b_2)$
(b_1, a_2)
B_2
$-F(b_1, a_2)$
(a_1, b_2)
B_3
$-F(a_1, b_2)$
(a_1, a_2)
B_4
$+F(a_1, a_2)$

$$B_1 = B_2 \cup B_3 \cup A$$

$$\{X_1 \le b_1, X_2 \le b_2\} = \overbrace{\{X_1 \le b_1, X_2 \le a_2\}}^{B_2} \cup \overbrace{\{X_1 \le a_1, X_2 \le b_2\}}^{B_3} \cup$$

$$\cup \underbrace{\{a_1 < X_1 \le b_1, a_2 < X_2 \le b_2\}}_{A}$$

$$\underbrace{P(B_1)}_{F(b_1, b_2)} = P(B_2 \cup B_3 \cup A) =$$

$$= P(B_2) + P(B_3) + P(A)$$

$$- P(\overbrace{B_2 \cap B_3}^{B_4}) - P(\overbrace{B_2 \cap A}^{\emptyset}) - P(\overbrace{B_3 \cap A}^{\emptyset})$$

$$+ P(\overbrace{B_2 \cap B_3 \cap A}^{\emptyset})$$