October 31

Given E and F two normed spaces in $E \times F$ we can introduce

$$\|(x,y)\| := \|x\|_E + \|y\|_F \qquad \sqrt[p]{\|x\|_E^p + \|y\|_F^p} \quad 1 \le p < +\infty$$

$$\sup\{\|x\|_E, \|y\|_F\}$$

<u>Def</u> If X is a vector space two norms on X $\|\cdot\|_1, \|\cdot\|_2$, are said to be equivalent if $\exists\, C \ge 1$ s.t.

$$\frac{1}{C}\|x\|_2 \le \|x\|_1 \le C\|x\|_2 \qquad \forall x \in X.$$

<u>Exercise</u> If X is topological vector space whose topology is induced by a norm $\|\cdot\|_1$ and if $\|\cdot\|_2$ is another norm which induces on X the same topology, then the two norms are equivalent.

Def If X is a set and d_1 and d_2 are two metrics on X we say that d_1 and d_2 are equivalent if $\exists\; C \ge 1$ st.

$$\frac{1}{C} d_2(x,y) \le d_1(x,y) \le C\, d_2(x,y) \qquad \forall x,y \in X$$

<u>**Remark**</u> If X is a TVS whose topology is induced by a metric d_1 and if d_2 is another metric inducing the same topology the two metrics are not necessarily equivalent.

Theorem Let E and F be B-space and $T: E \longrightarrow F$ linear.
Then T is continuous iff $G(T)$, = the graph of T, is closed in $E \times F$

Pf If T is continuous $\Longrightarrow$ $G(T)$ is closed

because
$$E \times F \xrightarrow{L} F$$
$$(x, y) \longrightarrow Tx - y \in F$$

is continuous $\ker L = G(T)$ is necessarily closed
$L^{-1}(0)$

Suppose $G(T)$ is closed.

$$\begin{array}{ccc} E \times F & \xrightarrow{\pi_2} & F \\ \downarrow \pi_1 & & \\ E & & \end{array} \qquad \begin{array}{c} (x, y) \\ \downarrow \\ x \end{array}$$

$\pi_1 : E \times F \longrightarrow E$

$\pi_1|_{G(T)} : G(T) \longrightarrow E$

is an onto and is continuous and is also 1-1 map

$$\|\pi_1(x, Tx)\|_E = \|x\|_E \leq \|x\|_E + \|Tx\|_F = \|(x, Tx)\|_{E \times F}$$

Since $G(T)$ is closed in $E \times F \Longrightarrow G(T)$ is a complete Banach space for the norm $\| \ \|_{E \times F}$

is an open map and the inverse is continuous.

$\pi_1 : (x, Tx) \longrightarrow x$

That is $x \in E \longrightarrow (x, Tx) \in G(T)$ is continuous

$\Longrightarrow \exists\ C > 0$ s.t.

$$\|(x, Tx)\|_{G(T)} = \|x\|_E + \|Tx\|_F \leq C\|x\|_E \quad \forall x \in E$$

$$\Downarrow$$

Obviously here we can take $C > 1$

$$\|Tx\|_F \leq (C - 1)\|x\|_E \quad \forall x \in E$$

$\Longrightarrow$ $T \in \mathcal{L}(X, Y)$

Example $F = C^0([0,1])$ with the $L^\infty([0,1])$ norm

$E = C^1([0,1])$ with the $L^\infty([0,1])$ norm. E is not a closed subspace of $L^\infty([0,1])$, $(E, \|\cdot\|_\infty)$, $(F, \|\cdot\|_\infty)$

$T : E \longrightarrow F$

$f \in E \longrightarrow Tf = \frac{d}{dx} f$ is a linear operator

Is not continuous because

$$T x^n = \frac{d}{dx} x^n = n\, x^{n-1}$$

$$\|T x^n\|_{L^\infty[0,1]} = \|n\, x^{n-1}\|_{L^\infty[0,1]} = n$$

$$\|x^n\|_{L^\infty[0,1]} = 1$$

Since $\|T x^n\|_{L^\infty[0,1]} = n\, \|x^n\|_{L^\infty[0,1]}$ $\forall n \Rightarrow$ if T was bounded ~~we~~ we would

$\Rightarrow \|T\|_{\mathcal{L}(E,F)} \geq n \quad \forall n \in \mathbb{N}$

So $T : E \longrightarrow F$ is unbounded.

Yet $G(T) \subseteq E \times F$ is closed

If $(f_n, Tf_n) = (f_n, f_n') \xrightarrow{n \to +\infty} (f, g)$ in $E \times F$

$\Rightarrow g = f'$ and so $(f, g) \in G(T)$

$x \in [0,1]$

$$f(x) = \lim_{n \to +\infty} f_n(x) = \lim_{n \to +\infty} \left[f_n(0) + \int_0^x f_n'(x')\, dx' \right]$$

$$f(x) = f(0) + \int_0^x g(x')\, dx' \qquad \forall x \in [0,1]$$

where $g \in C^0([0,1]) =: F$

By the Fundamental Theorem of Calculus $f'(x) = g(x)$ $\forall x \in [0,1] \Rightarrow f \in C^1([0,1]) =: E$

$\Rightarrow (f, g) \in G(T)$

Def a ~~closed~~ vector subspace F of a TVS E is complemented in E if $\exists$ a closed vector subspace G of E s.t. $E = F \oplus G$.

Theorem If E B space $\dim F < +\infty$ then F is complemented.

Pf Let $f_1, \dots, f_n$ be a basis of F. Then $\forall x \in F$ we can write $x = \lambda_1 f_1 + \dots + \lambda_n f_n$ $\lambda_1, \dots, \lambda_n \in \mathbb{R}$.

For $j = 1, \dots, n$

$\phi_j : F \longrightarrow \mathbb{R}$ $\phi_j x = \lambda_j$ is continuous

By Hahn-Banach $\phi_j \in E'$

$G \doteq \bigcap_{j=1}^{n} \ker \phi_j$. G is closed

$F \cap G = 0$ because if $x \in F \cap G$

$\Rightarrow x = \lambda_1 f_1 + \dots + \lambda_n f_n = \underbrace{\phi_1(x)}_{0} f_1 + \dots + \underbrace{\phi_n(x)}_{0} f_n = 0$

We want to show that any $z \in E$ can be expressed as a sum $z = x + (z - x)$ where $x \in F$ and $z - x \in G$ $\quad (E = F + G)$

$\exists \lambda_j \in \mathbb{R}$ s.t. $\phi_j(z) = \lambda_j$ $\forall j = 1, n$. Set $x = \lambda_1 f_1 + \dots + \lambda_n f_n \in F$

Notice that $\phi_j(x) = \lambda_j$ $\forall j$.

$\Rightarrow \phi_j(z - x) = 0$ $\forall j$ $\Rightarrow z - x \in G$.

__Theorem__ E Banach. If F is closed with $\operatorname{codim} F < +\infty$ then F is complemented.

__Dim__ $\dim E/F < +\infty$. Then $\exists$ $g_1, \dots, g_n \in E$ s.t. their equivalent classes are a basis of E/F.

$G = \operatorname{span}\{g_1, \dots, g_n\}$. It is closed and $E = F \oplus G$.

__Remark__ If E is a Hilbert space any $F \subseteq E$ is complemented $E = F \oplus F^\perp$ so every closed F is complemented.

If E is a B space which is not isomorphic to a Hilbert space then $\exists$ F closed subspace which is not complemented in E.

For example $c_0(\mathbb{N})$ is not complemented in $\ell^\infty(\mathbb{N})$

Def Let $\cancel{X}$ be a B-space. $P \in \mathcal{L}($

$P \in \mathcal{L}(X)$ is a projection if $P^2 = P$.

Exercise P projection $\Rightarrow$ $1-P$ projection

$$(1-P)^2 = (1-P)(1-P) = 1-2P+P^2 = 1-2P+P = 1-P$$

Exercise Let $E = F \oplus G$

$\forall z \in E$
$\exists! (x,y) \in F \times G \quad z = x + \boxed{y}$

Define $Pz := x$. Then P is a projection.

$P^2 z = P x = x = Pz \qquad x \in F \qquad x = x + 0$

$P^2 z = Pz \quad \forall z \Rightarrow P^2 = P$

$Qz = y \qquad Q = 1-P$

$\boxed{P \in \mathcal{L}(E)}$

$P : E \longrightarrow F \subseteq E \qquad P \in \mathcal{L}(E,F)$

P is continuous if $\forall$ closed subspace $C \subseteq F$ we have the inverse image is closed.

$C = F \cap Y \qquad Y \subseteq E$ closed

C is closed in E because ~~also~~ F is also closed

$P^{-1}C = C$;

Spectral projection

Let $A \in \mathcal{L}(X)$ and let

$$\sigma(A) = \Sigma_1 \cup \Sigma_2$$

Σ_1 and Σ_2 are disjoint.

Let γ be ~~the~~ simple closed curve in $\rho(A)$ (is a topological circle) with Σ_1 in the interior

$$R_A(z) = (A - z)^{-1} \qquad R_A \in C^{\omega}(\rho(A), \mathcal{L}(X))$$

$$P = -\frac{1}{2\pi i} \int_\gamma R_A(z)\, dz \qquad \gamma : I \longrightarrow \rho(A)$$

$$\gamma \in C^1(I)$$

$$= -\frac{1}{2\pi i} \int_I R_A(\gamma(t))\, \gamma'(t)\, dt$$

P is a projection $\quad P^2 = P$

$$X = R(P) \oplus \ker(1-P)$$

$$A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \qquad B = PA$$

$$C = (1-P)A$$

$$\sigma(B) = \Sigma_1, \quad \sigma(C) = \Sigma_2$$