Image Processing for Physicists

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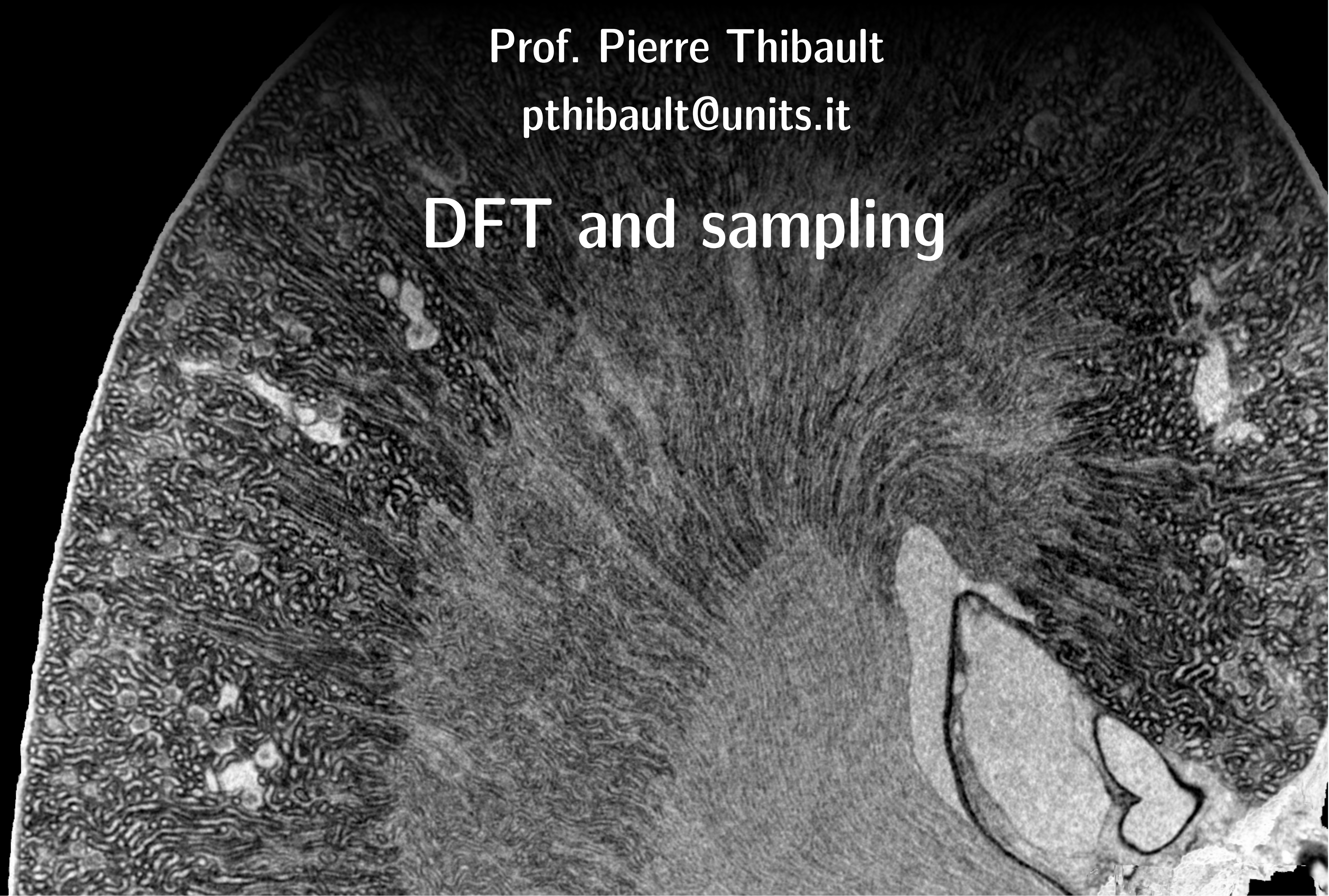
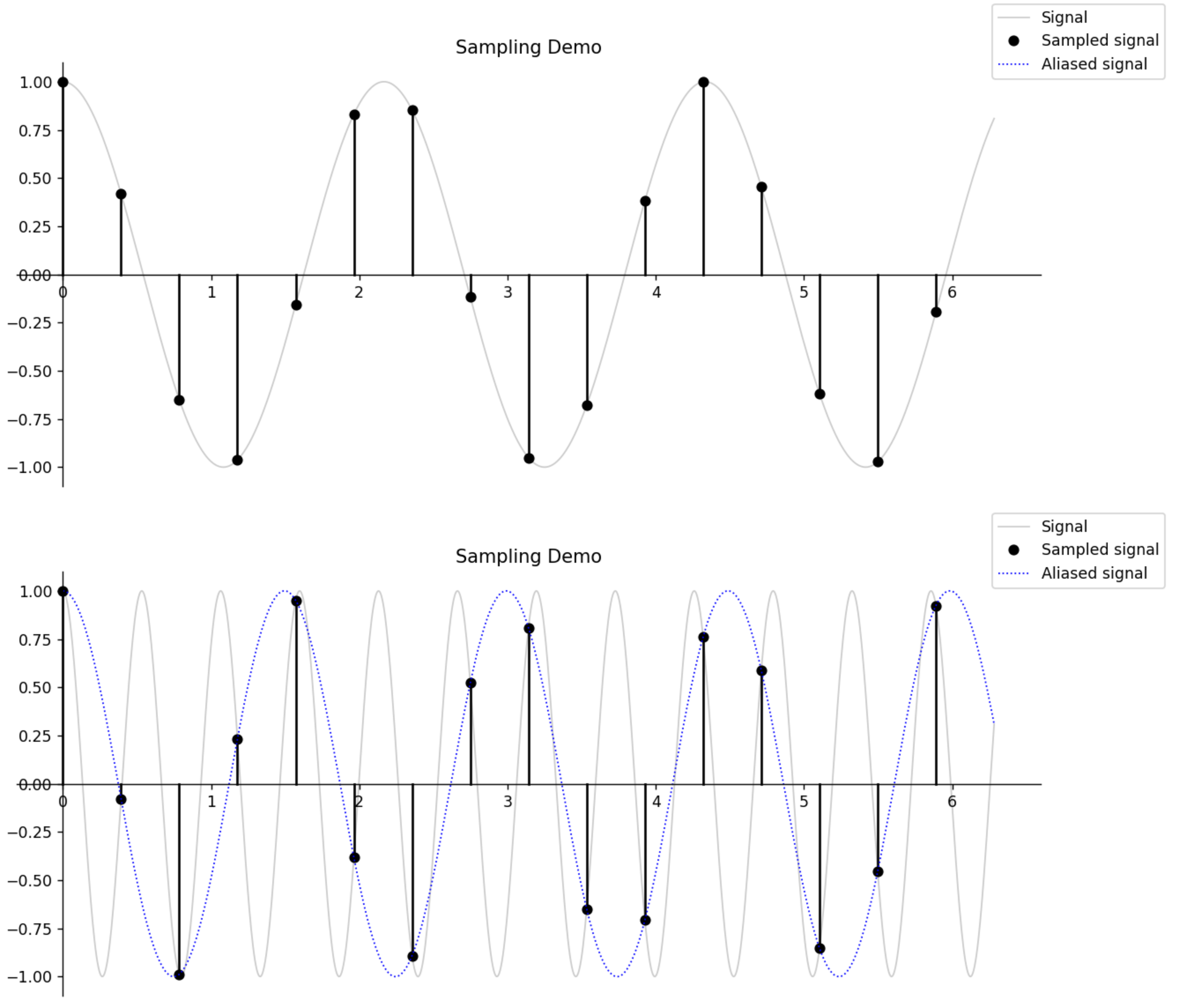
DFT and sampling



Overview

- Sampling
  - Nyquist theorem
- Discrete Fourier transform
  - Undersampling and Aliasing
- Interpolation (resampling)

Sampling


Sampling Demo
Signal
Sampled signal
Aliased signal
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
0
1
2
3
4
5
6
Sampling Demo
Signal
Sampled signal
Aliased signal
1.00
0.75
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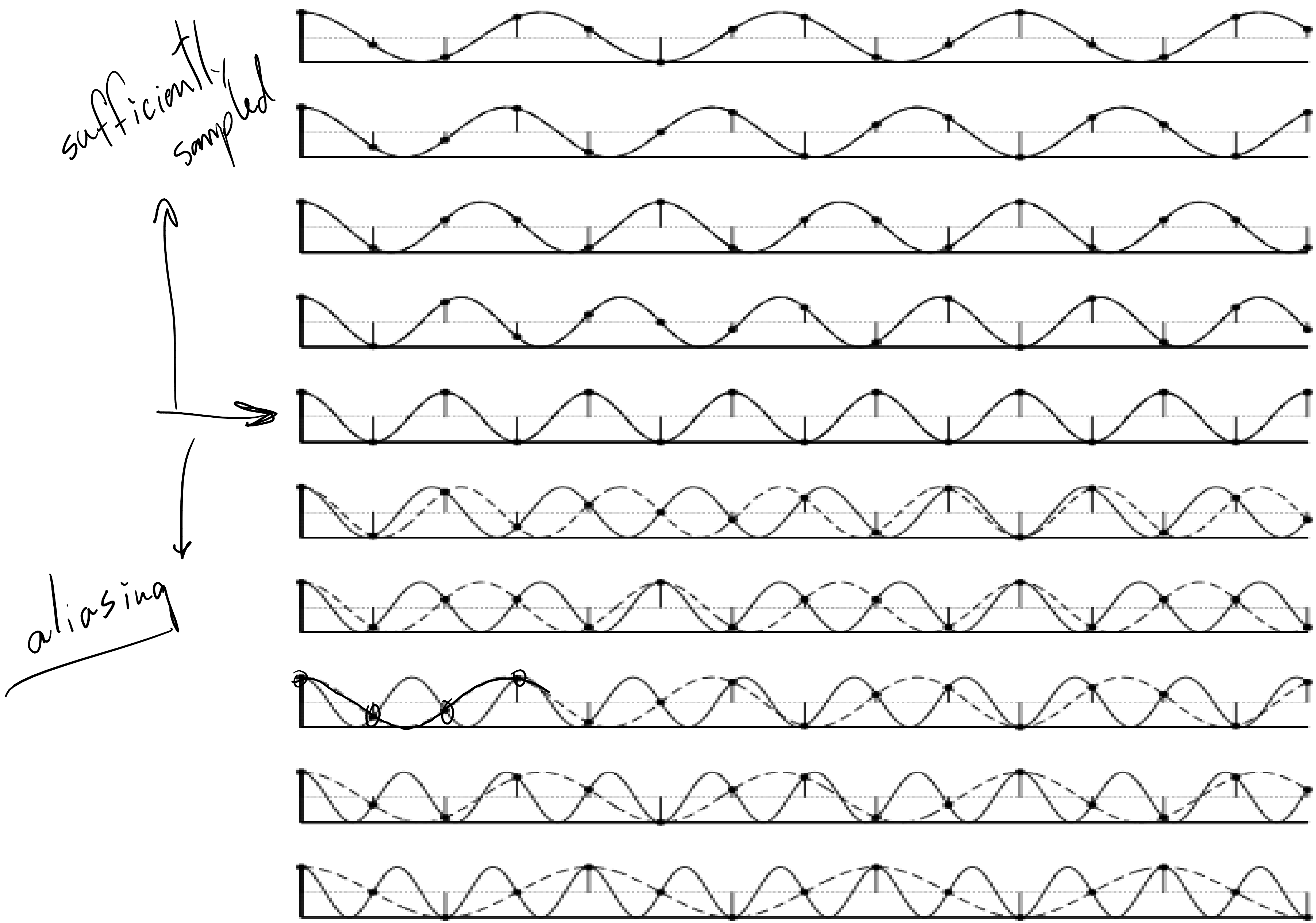
0.25
0.00
−0.25
−0.50
−0.75
−1.00
0
1
2
3
4
5
6

Undersampling and aliasing



The Nyquist-Shannon sampling theorem

"The largest frequency that can be represented in a signal sampled at intervals s is 1/2s"

sampling frequency
sampling rate } $f = \frac{1}{s}$

maximum frequency $= \frac{f}{2}$

Periodic signals

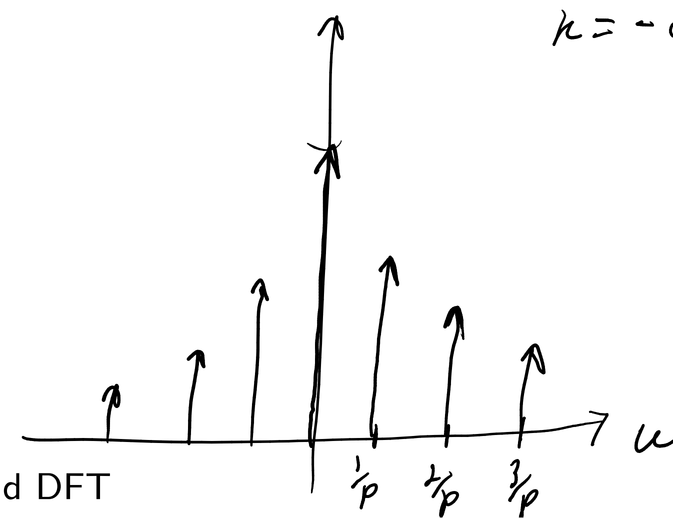
$f(x)$: periodic with period p

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i x k/p} \quad \leftarrow \text{Fourier series}$$

What is the continuous Fourier transform of f?

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i u x} dx = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_k e^{2\pi i x k/p} e^{-2\pi i u x} dx$$

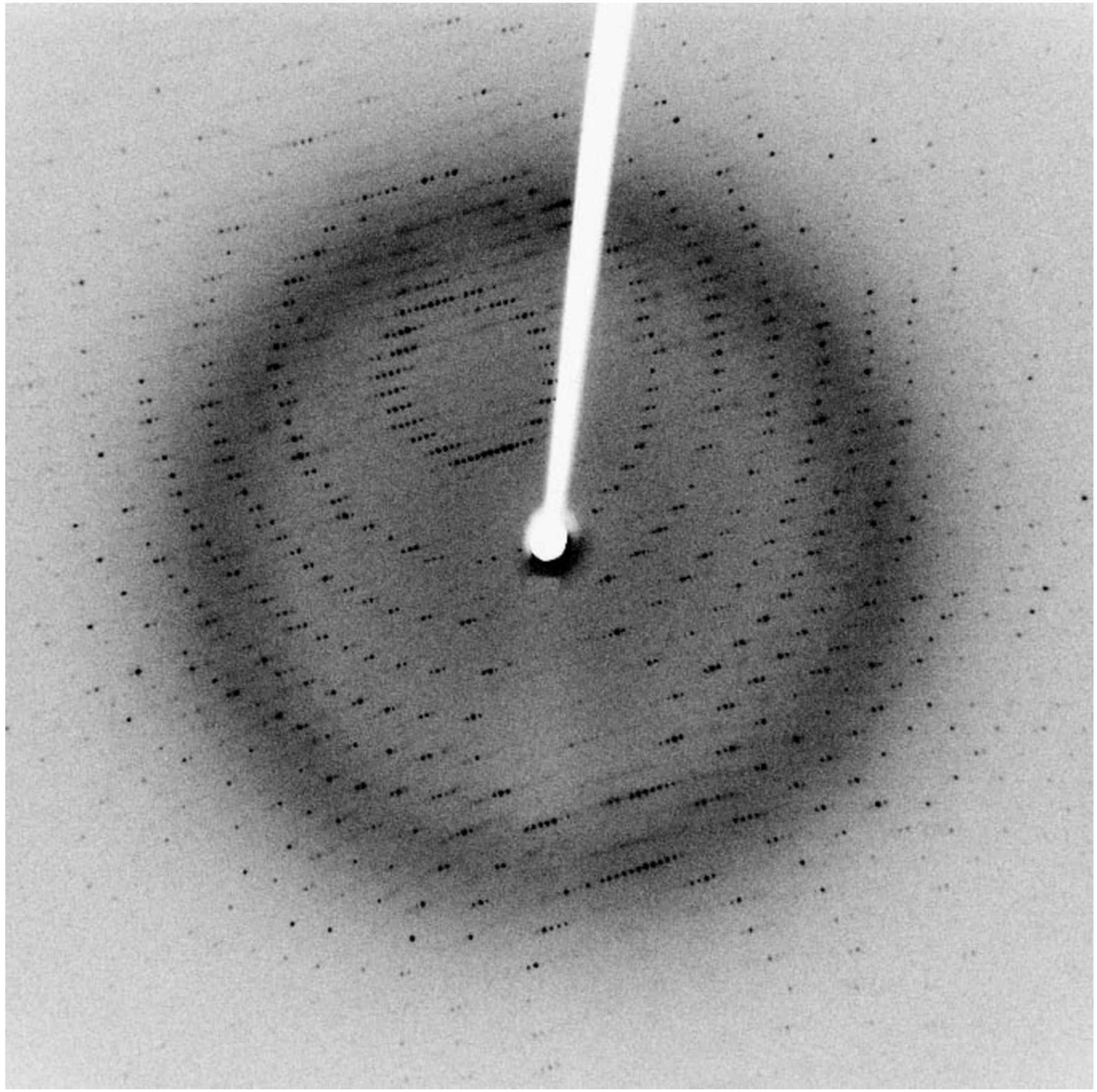
$$= \sum_{k=-\infty}^{\infty} c_k \int_{-\infty}^{\infty} e^{2\pi i x (k/p - u)} dx = \sum_{k=-\infty}^{\infty} c_k \delta\left(u - \frac{k}{p}\right)$$



F.T. of a periodic signal has non-zero amplitudes at discrete values of u located at multiples of $\frac{1}{p}$

Periodic signals

X-ray diffraction by a crystal

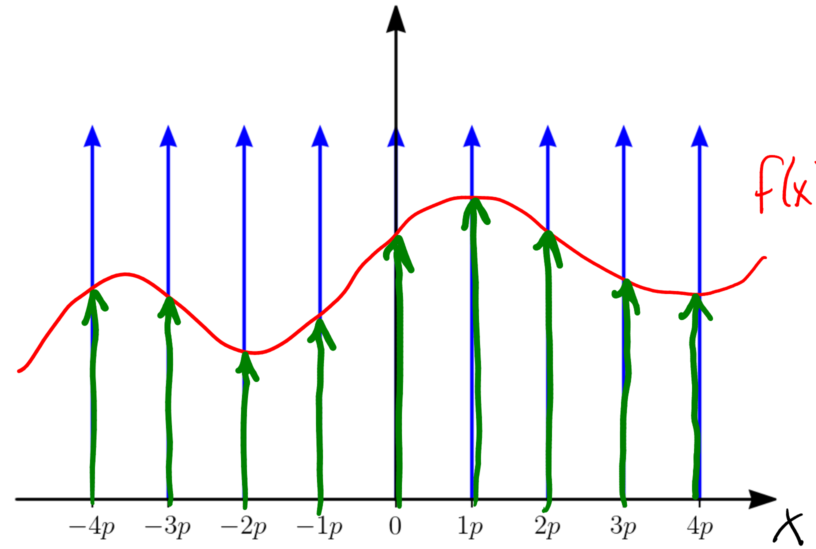


Bragg peaks

Sampling with the Dirac comb

A periodic function made of Dirac functions

$$\Delta_p(x) = \sum_{n=-\infty}^{\infty} \delta(x - np)$$



$\Delta_p(x)$ can be used to represent the sampling process. For a function $f(x)$

$$f(x)\,\Delta_p(x) = \sum_{n=-\infty}^{\infty} f(x)\,\delta(x-np) = \sum_{n=-\infty}^{\infty} f(np)\,\delta(x-np)$$

Fourier transform of a Dirac comb

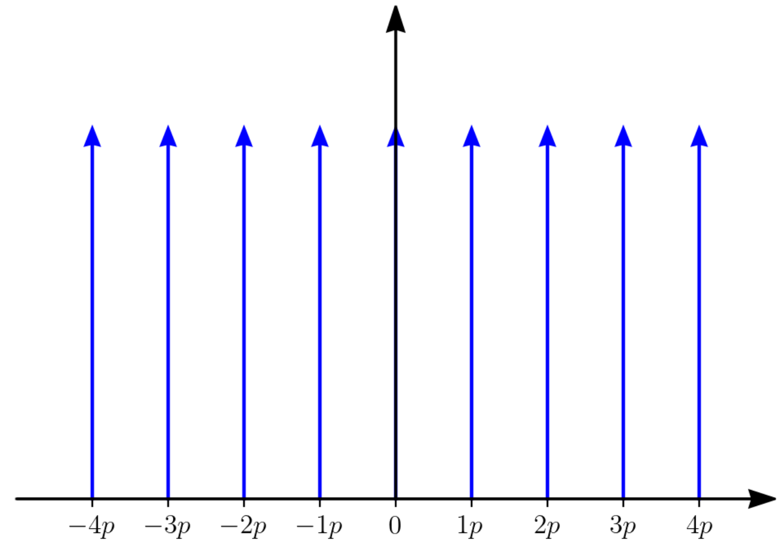
Fourier series of $\Delta_p(x)$

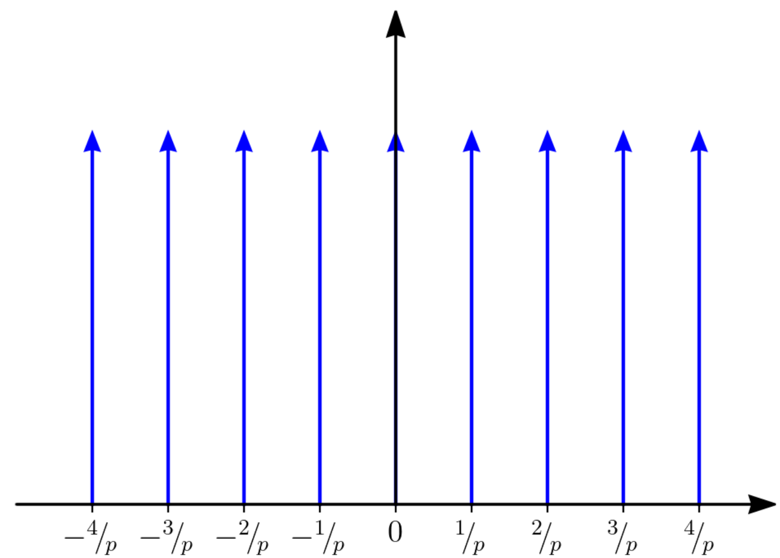
$$c_k = \int_{-p/2}^{p/2} \Delta_p(x)\, e^{-2\pi i k x/p}\, dx$$

$$= 1$$

$$\Rightarrow \mathcal{F}\{\Delta_p(x)\} = \Delta_{\frac{1}{p}}(u)$$

Fourier transform of a Dirac comb is another Dirac comb

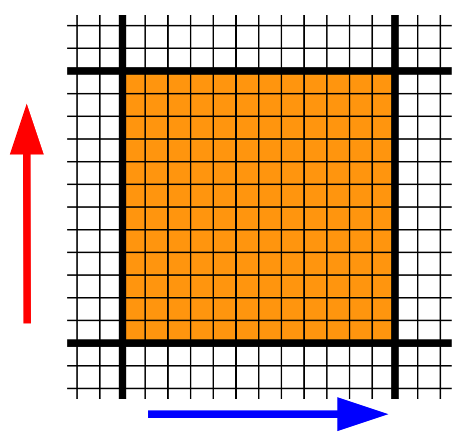


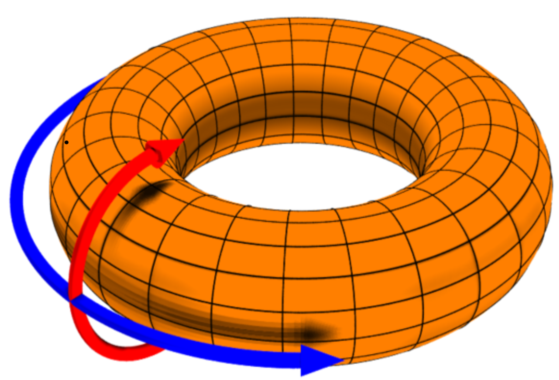


Discrete Fourier Transform

- A **periodic** function has a **discrete** spectrum in the Fourier domain;
- A function with **discrete** values in the spatial domain is **periodic** in the Fourier domain;

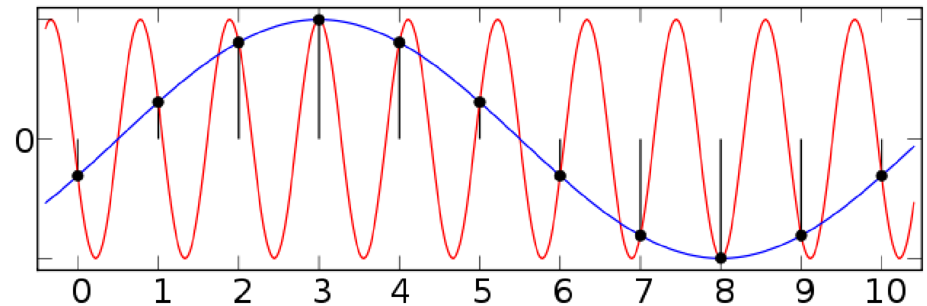
$\Rightarrow$ A periodic and discrete function has a periodic and discrete Fourier transform.



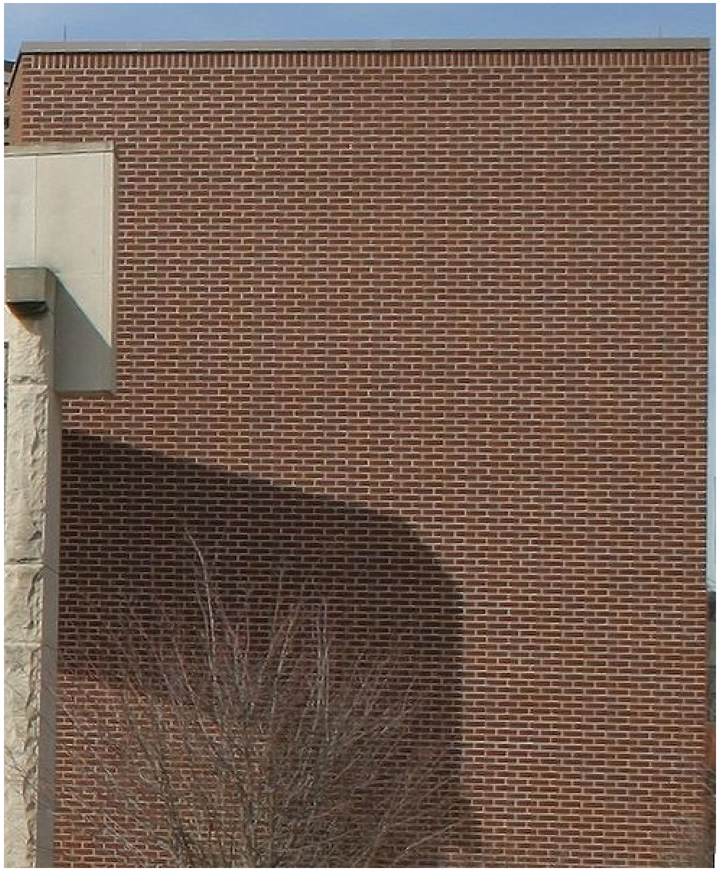


Aliasing

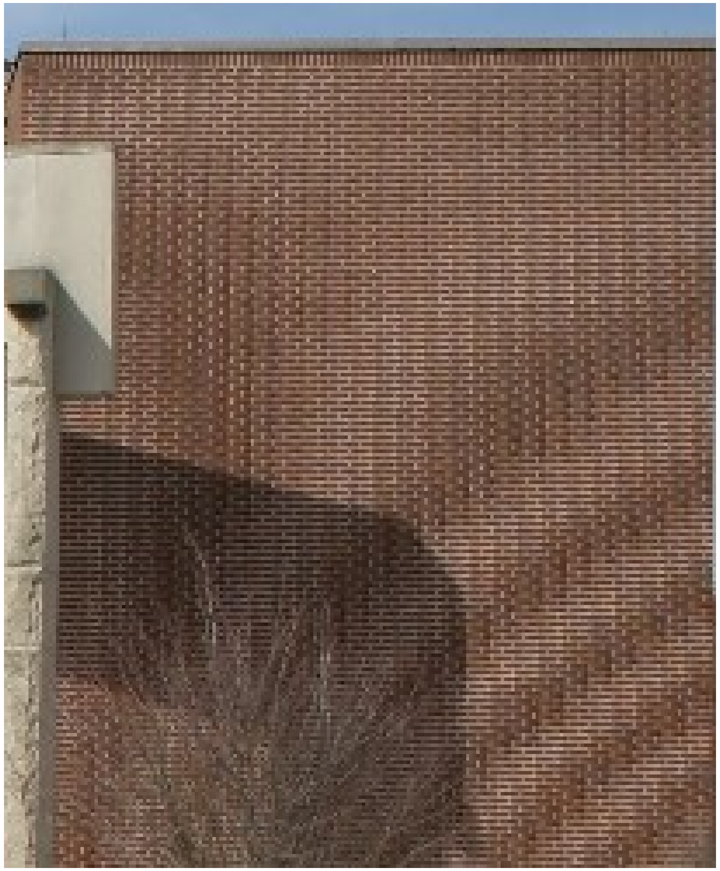
Moiré: after resampling, high spatial frequencies appear as low spatial frequencies



high resolution

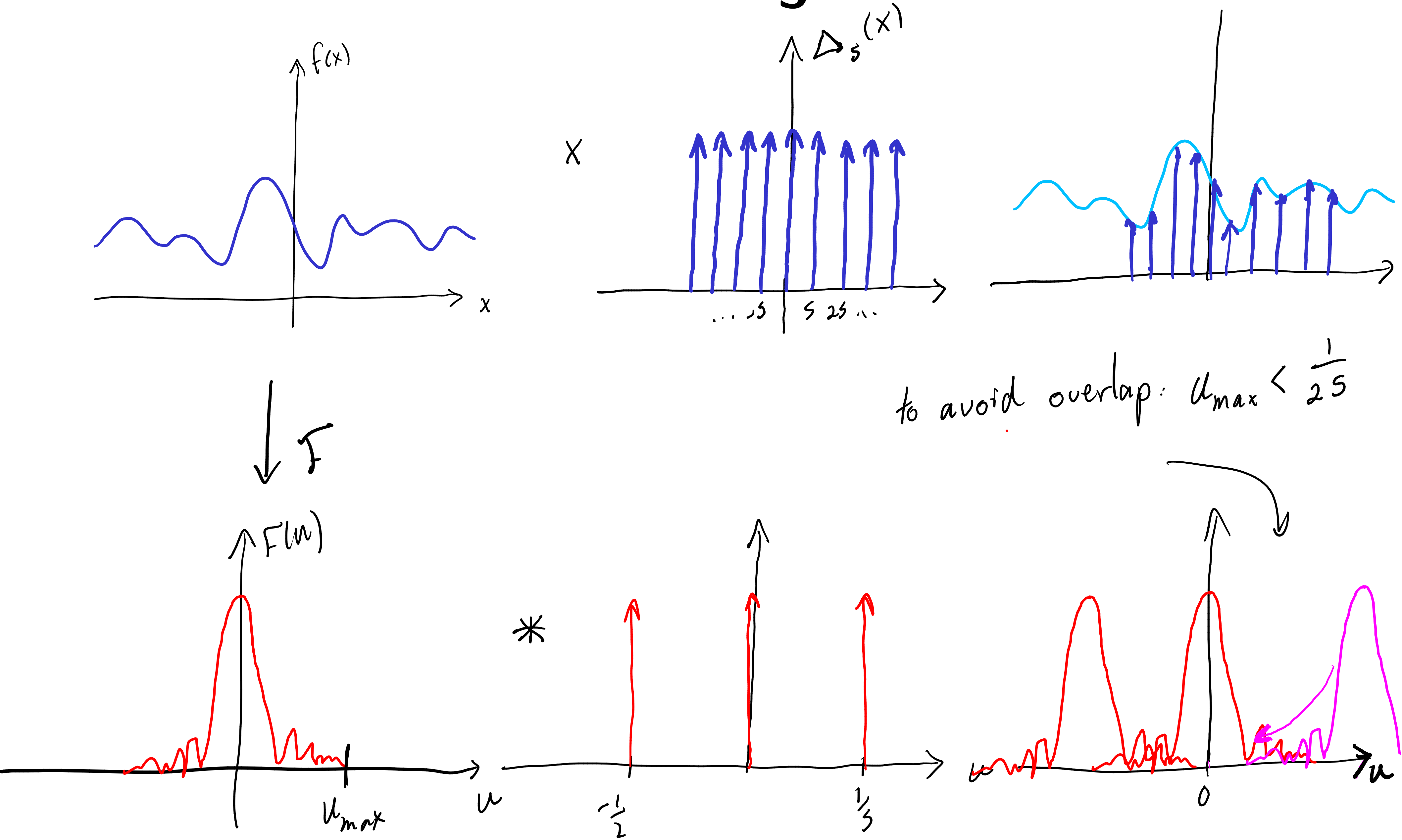


low resolution



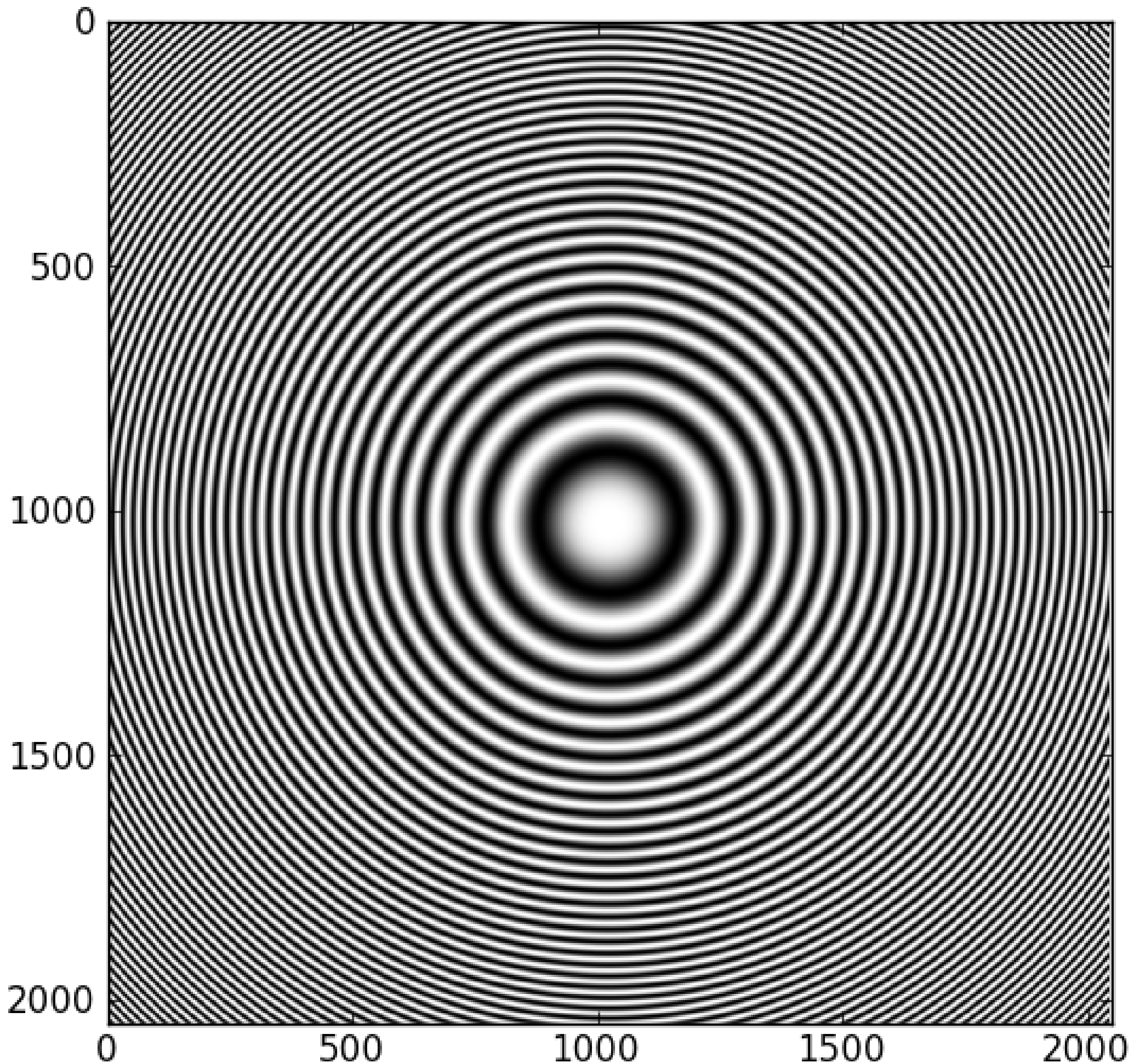
source: http://wikipedia.org

Aliasing


f(x)
x
X
$\Delta_s(x)$
... -s
s 2s ...
F
F(u)
u
u_{max}
*
$-\frac{1}{2}$
$\frac{1}{3}$
0
u
to avoid overlap: $u_{max} < \frac{1}{2s}$

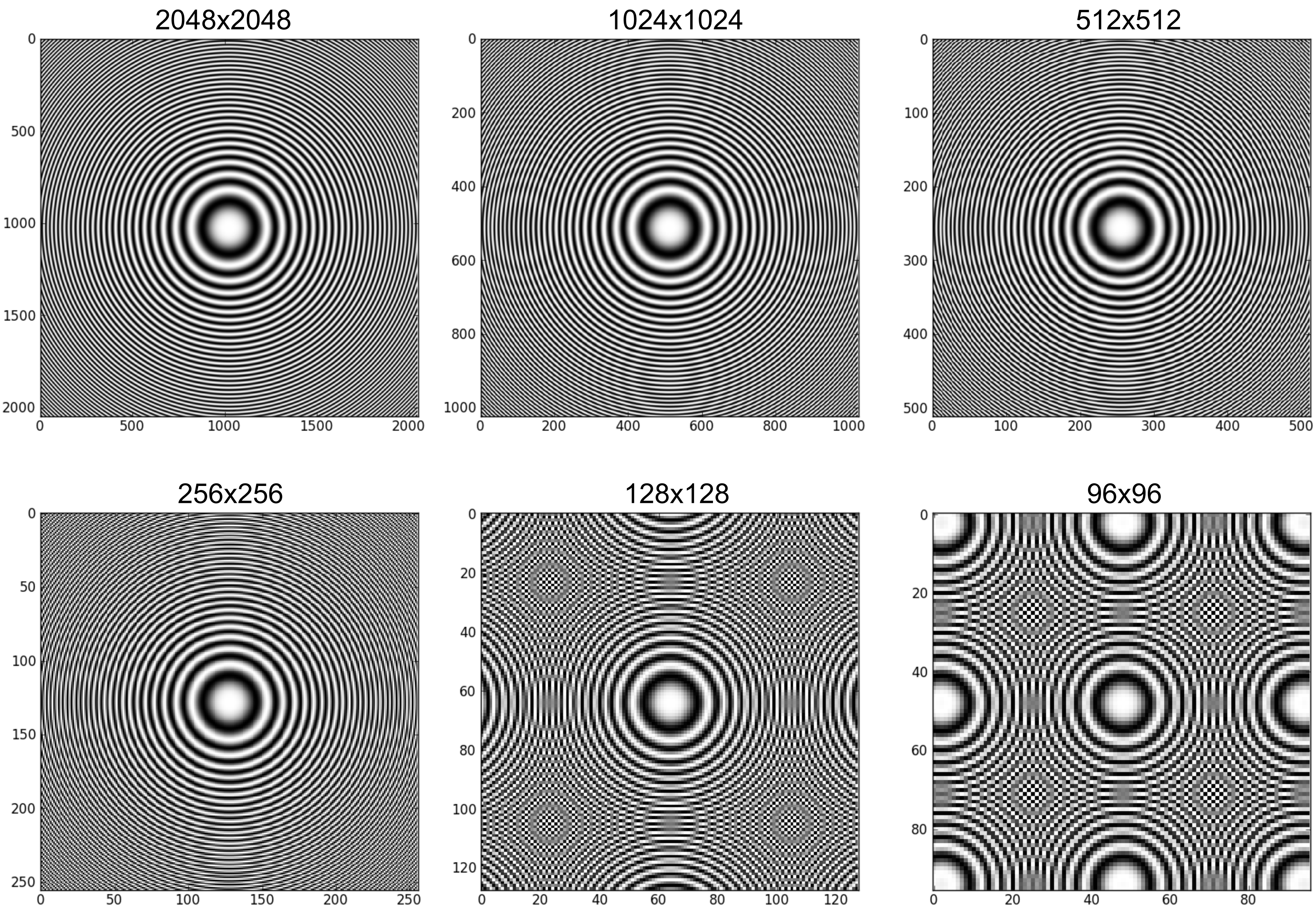
Undersampling

"Fresnel zone" test pattern: radial linear increase in spatial frequency



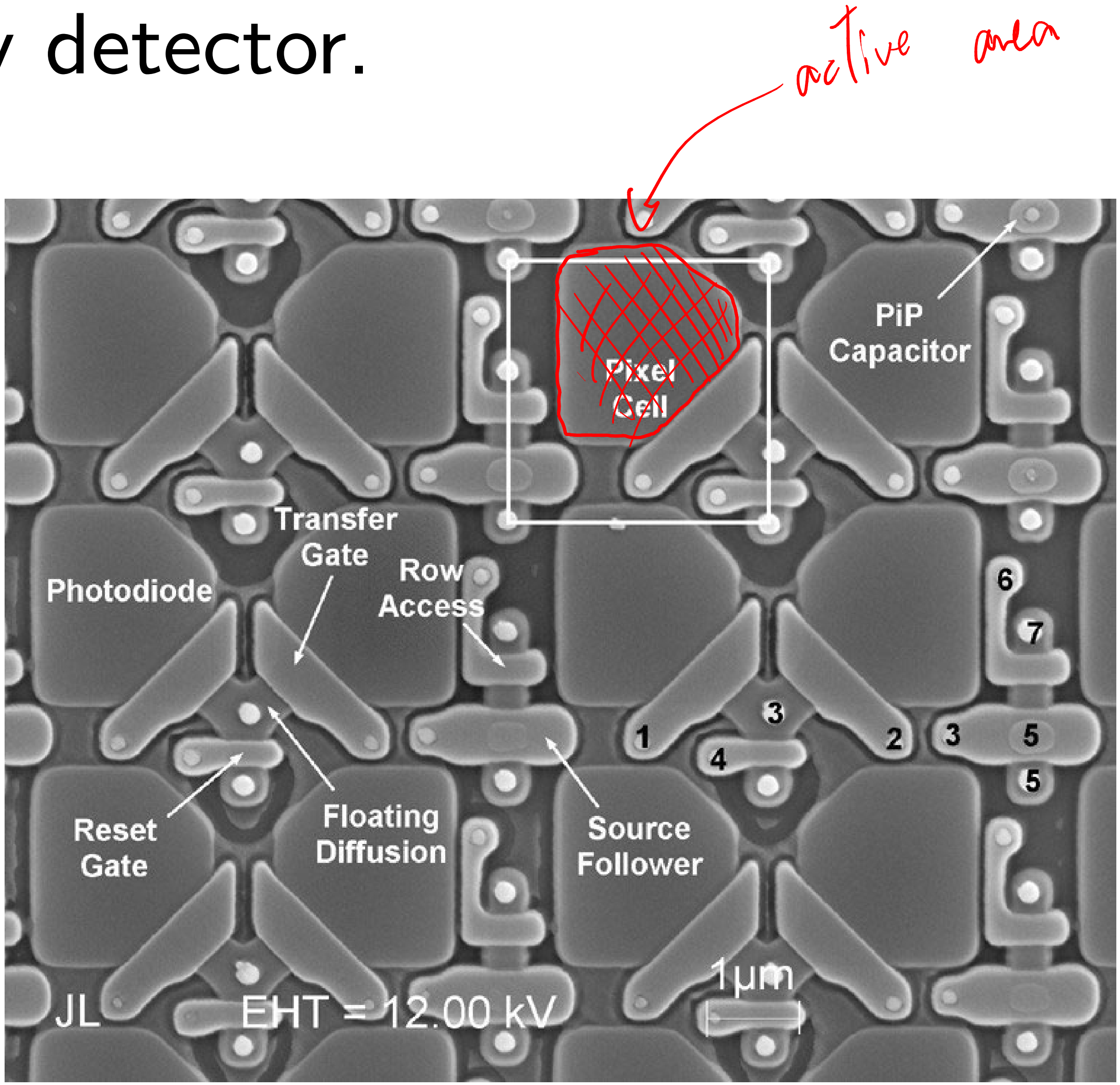
"radial chirp"

Undersampling & aliasing



Sampling with a pixel-array detector

- A 2D light field is sampled with a 2D pixel-array detector.



Discrete Fourier Transform

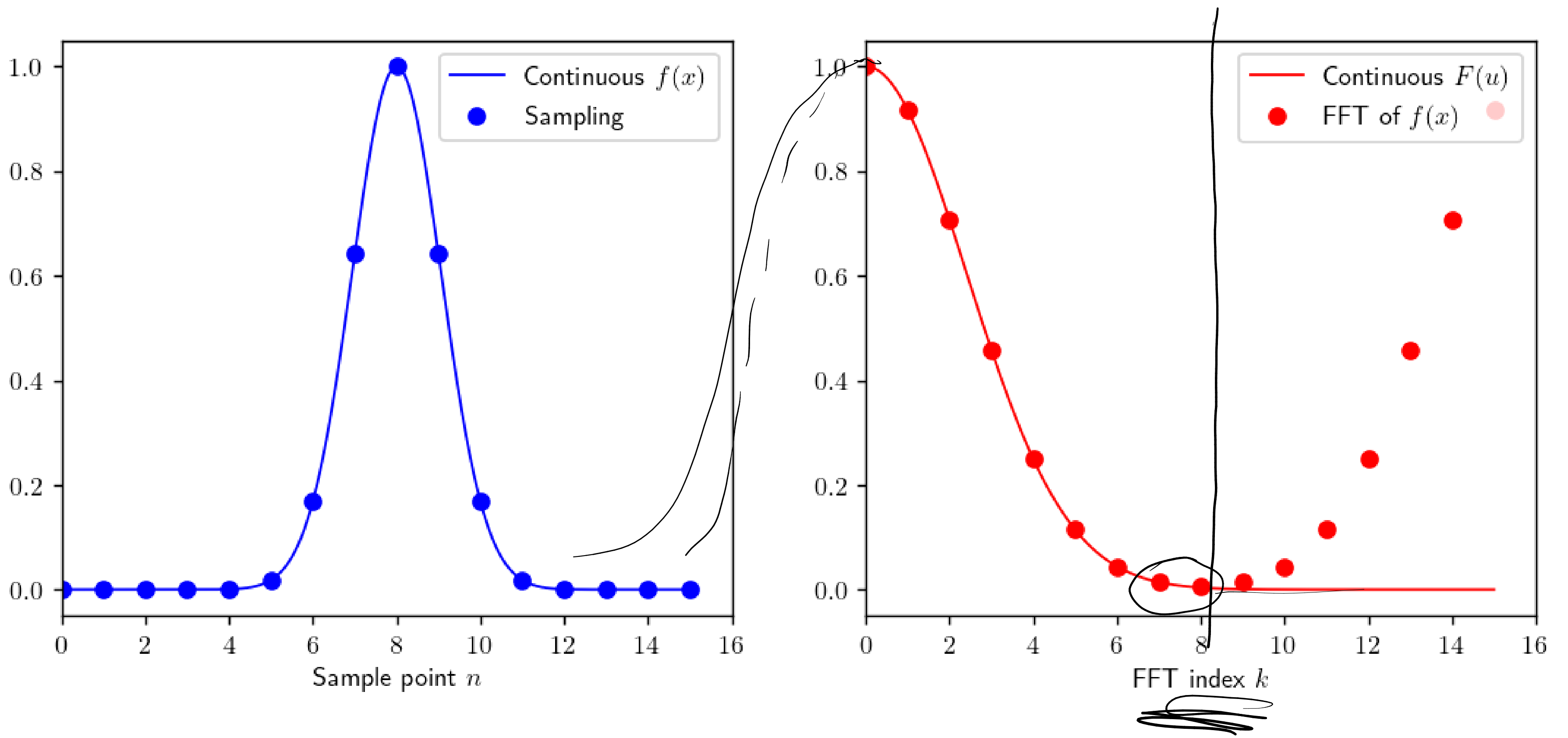
$$F_k = \sum_{n=0}^{N-1} f_n e^{-2\pi i kn/N}$$

- On computers, the signal is both sampled and of finite extent $\rightarrow$ using DFT on this signal means that is it assumed to be periodic.

- If the signal is sufficiently sampled, then the DFT can be interpreted as a sampled version of the continuous Fourier Transform.

DFT example

- Example: relation between space, sampling and frequency



zero frequency component is in the top left corner output array.

FT to DFT conversion

* look at the exp() argument.

continuous: $e^{2\pi i u x}$

discrete: $e^{2\pi i n k / N}$

$$ux = \frac{nk}{N}$$

* function sampling $f(x) \longrightarrow f_n$

L: "physical size" of signal

step size s : $x = ns$

$$ux = uns = \frac{nk}{N} \Longrightarrow u = \frac{k}{Ns} = \frac{k}{L}$$

Observation: $F_{k+N} = \sum_{n=0}^{N-1} f_n e^{-2\pi i n (k+N)/N}$

$$= \sum_{n=0}^{N-1} f_n e^{-2\pi i n \left(\frac{k}{N} + 1\right)} = F_k$$

F_k is periodic

Fourier space translation

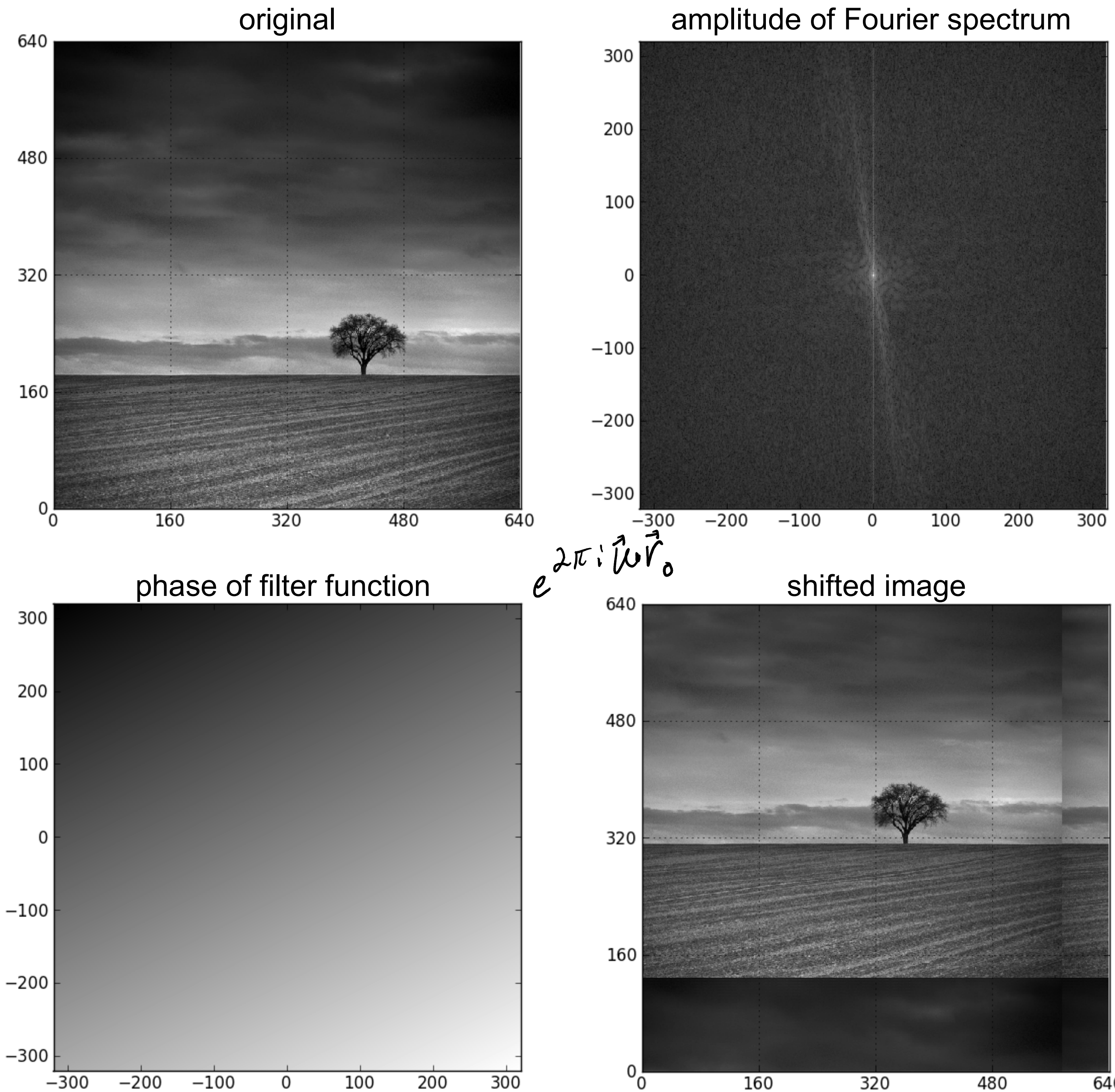
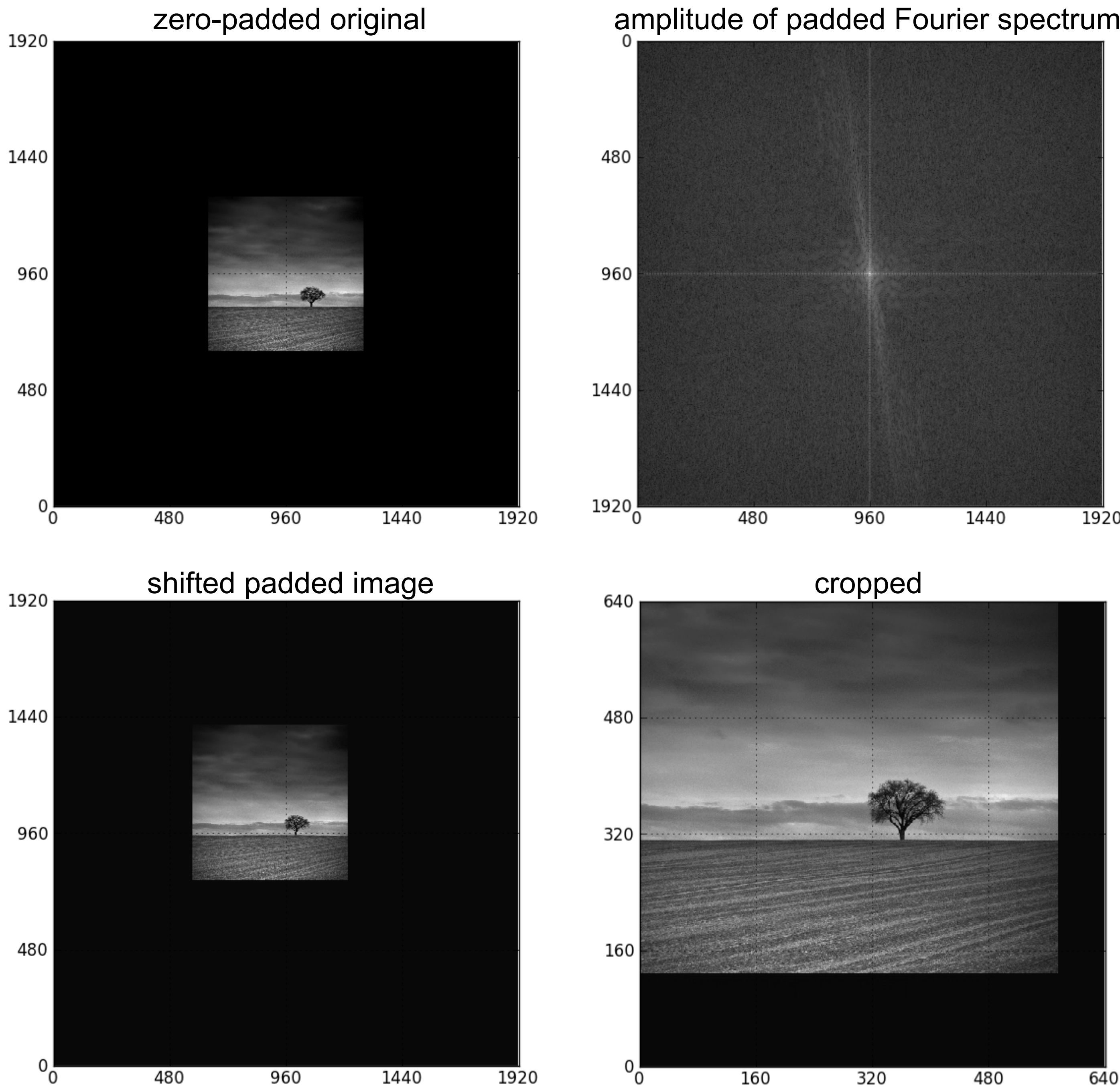


Image shifting using shifting property of FT

Image gets wrapped around

Zero-padding

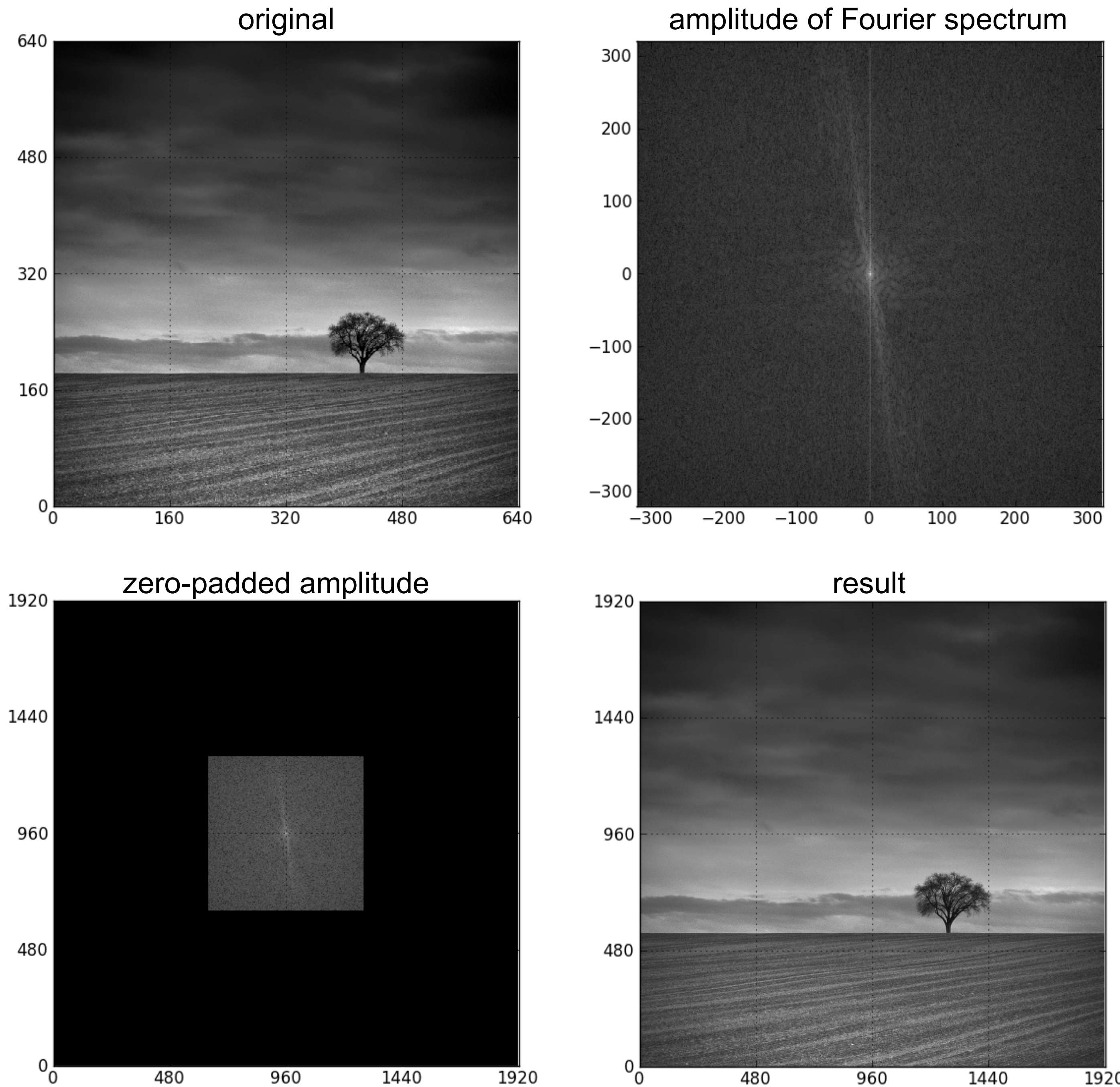


1. Add zeros around original image (*zero-padding*)

2. Shift using FT

3. Crop result

Zero-padding in Fourier space



Result: increased sampling!

↓

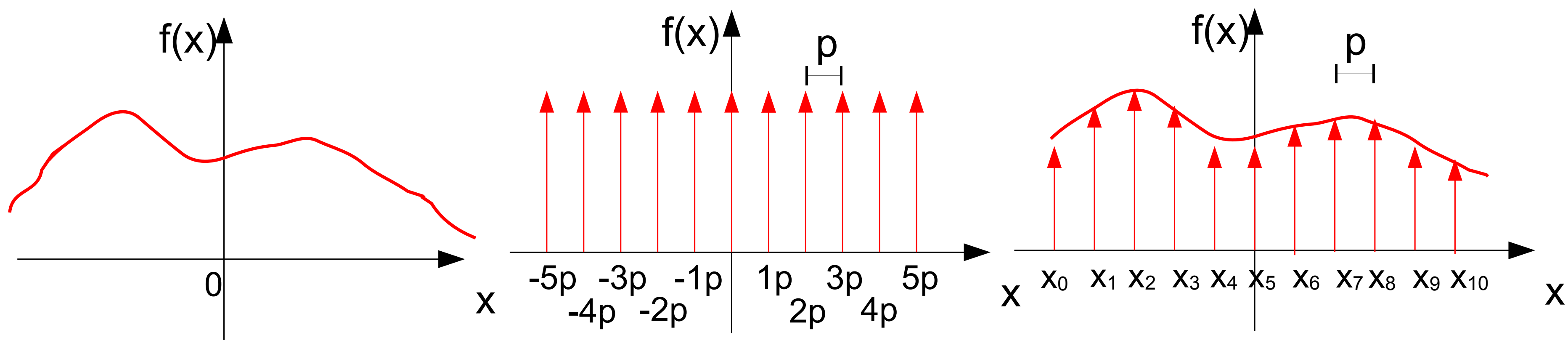
interpolation

"upscaling"

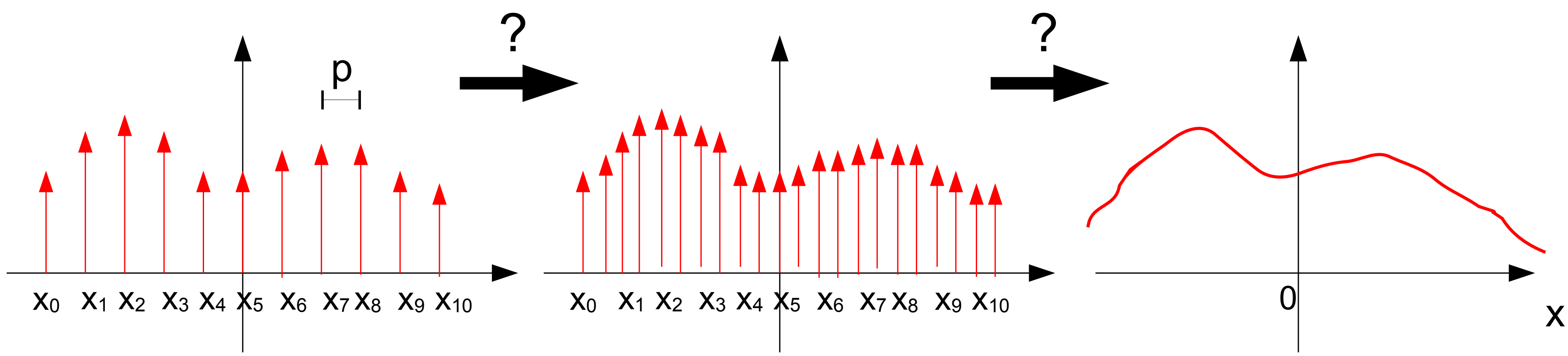
"upsampling"

Interpolation

- Discrete sampling of a continuous function



- Reconstruct original function from sampled data?

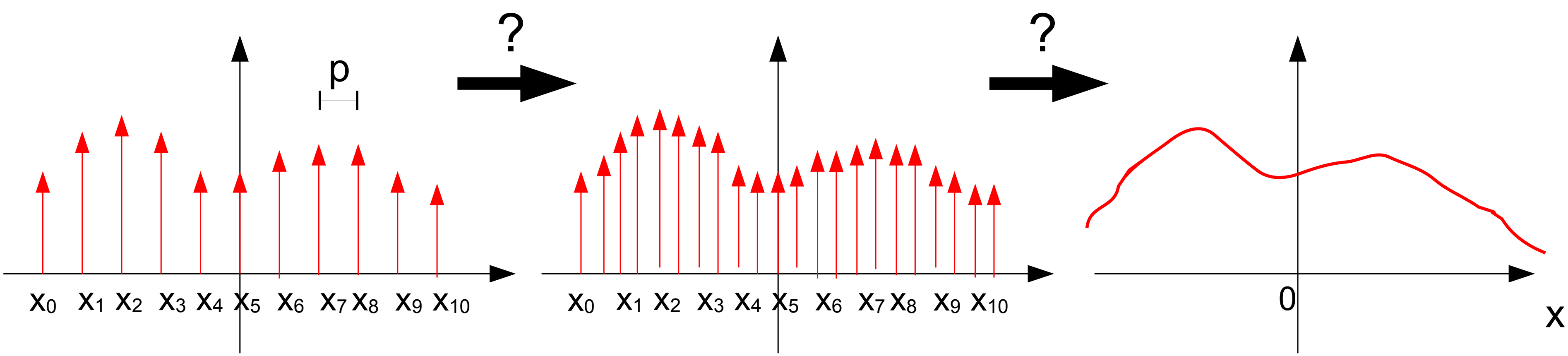


Interpolation

Finding unknown points between known ones

- wide field, many different approaches
- closely related to approximation theory and curve fitting

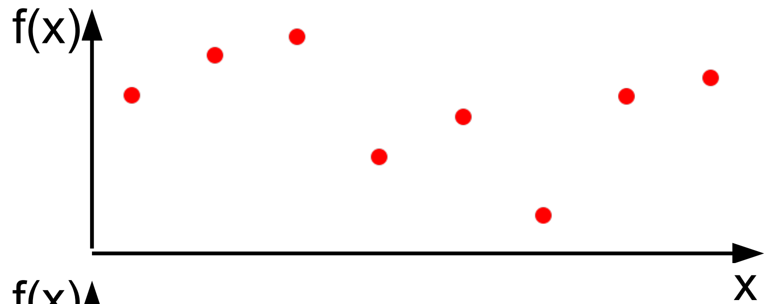
difference: interpolated curve has to pass through all known samples



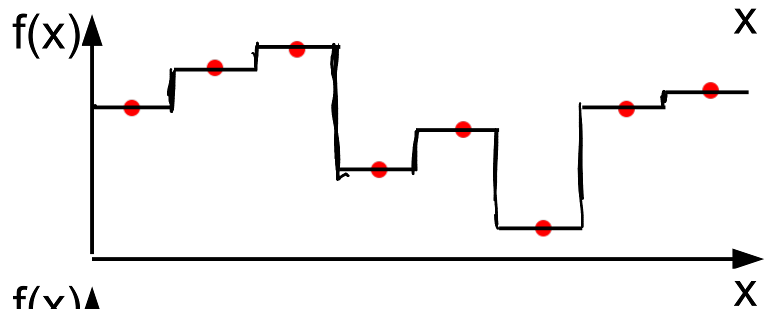
Interpolation

Various “classical” interpolation methods available

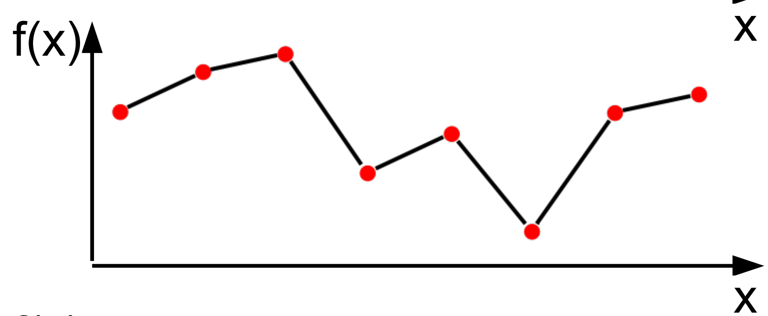
sampled points



nearest neighbor



linear



cubic spline

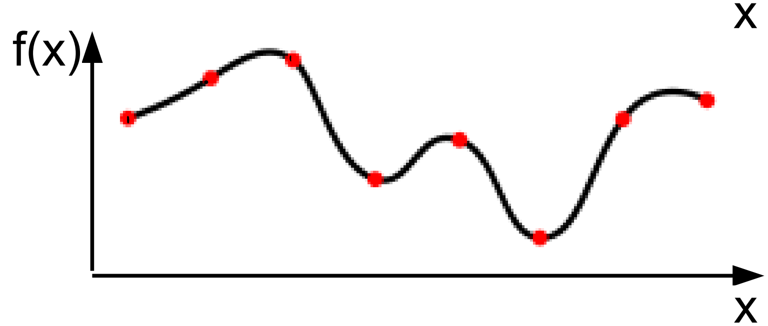
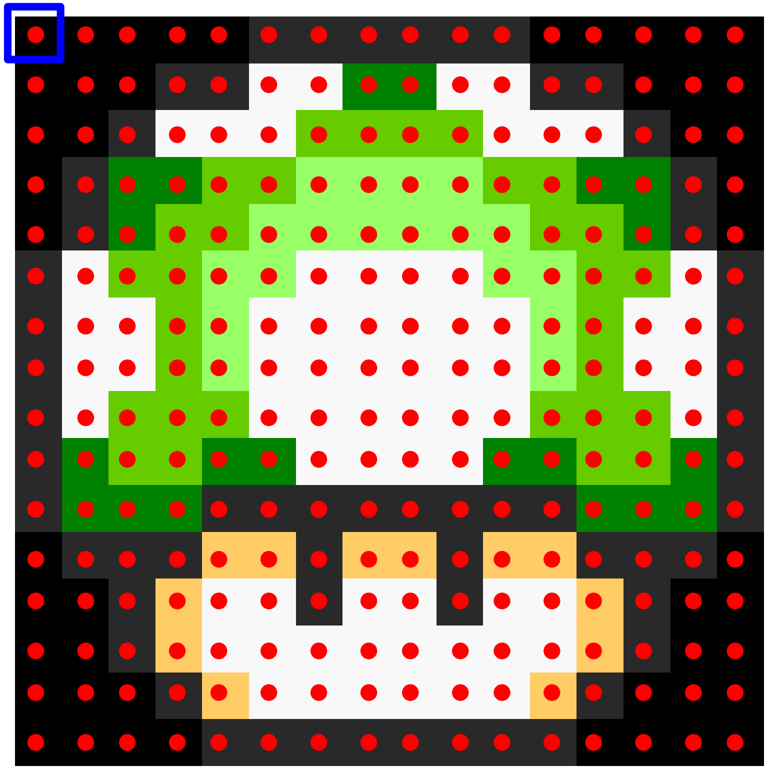


Image pixels

Images are discrete samples of a continuous function

- ...with coordinates
- ...and values (voltage at coordinate, integral over pixel area, ...)
- ...represented by pixel basis functions on a sampling grid



Linear interpolation

- Interpolation as an operator

$$f(x) = \mathcal{L}\{f_n\}$$

- Linear interpolation

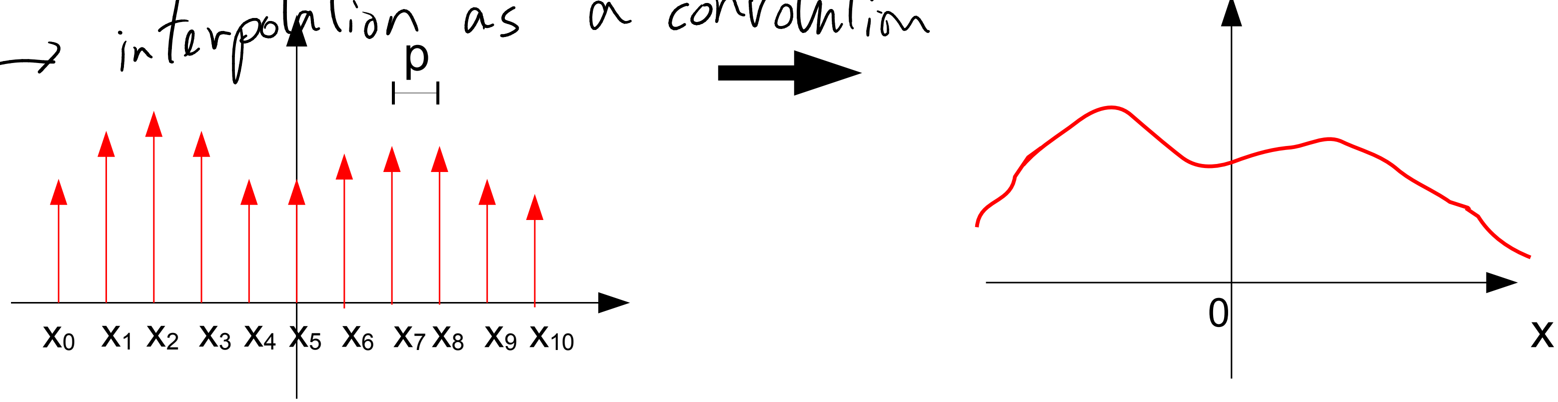
$$\mathcal{L}\{f_n + g_n\} = \mathcal{L}\{f_n\} + \mathcal{L}\{g_n\}$$

- Shift invariance

$$\mathcal{L}\{f_{n+n_0}\} = f(x + n s_0)$$

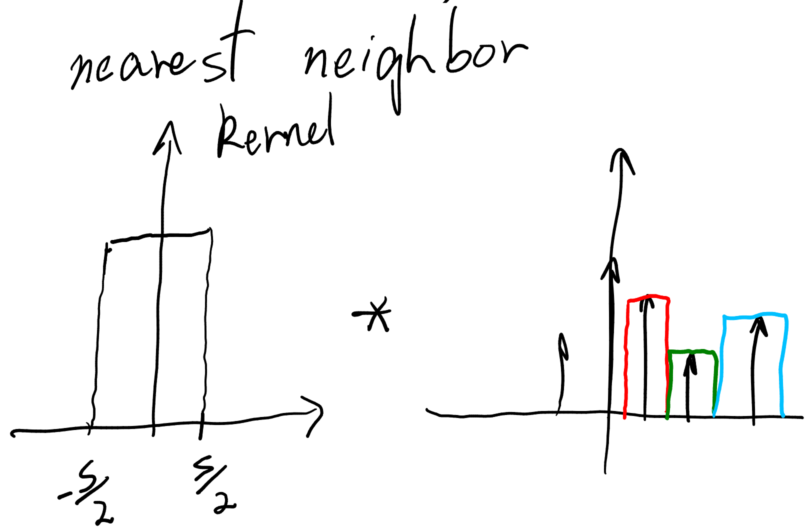
- Kernel

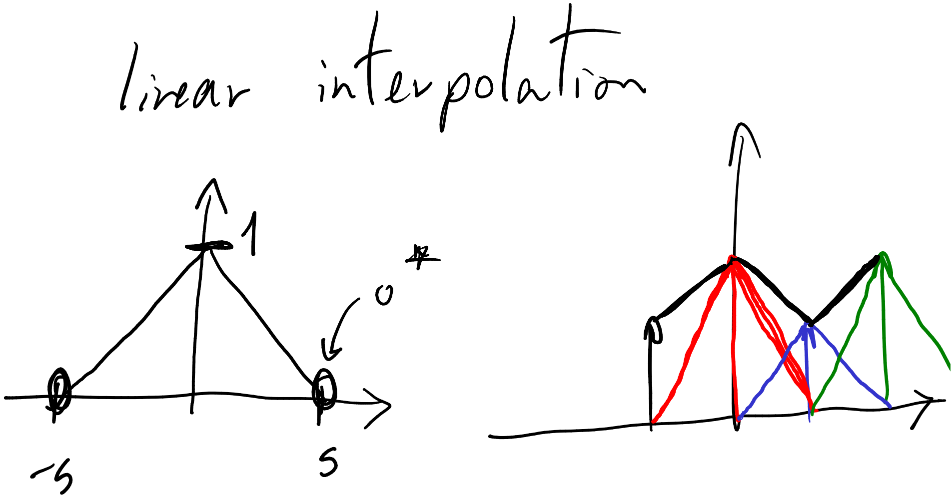
↳ interpolation as a convolution

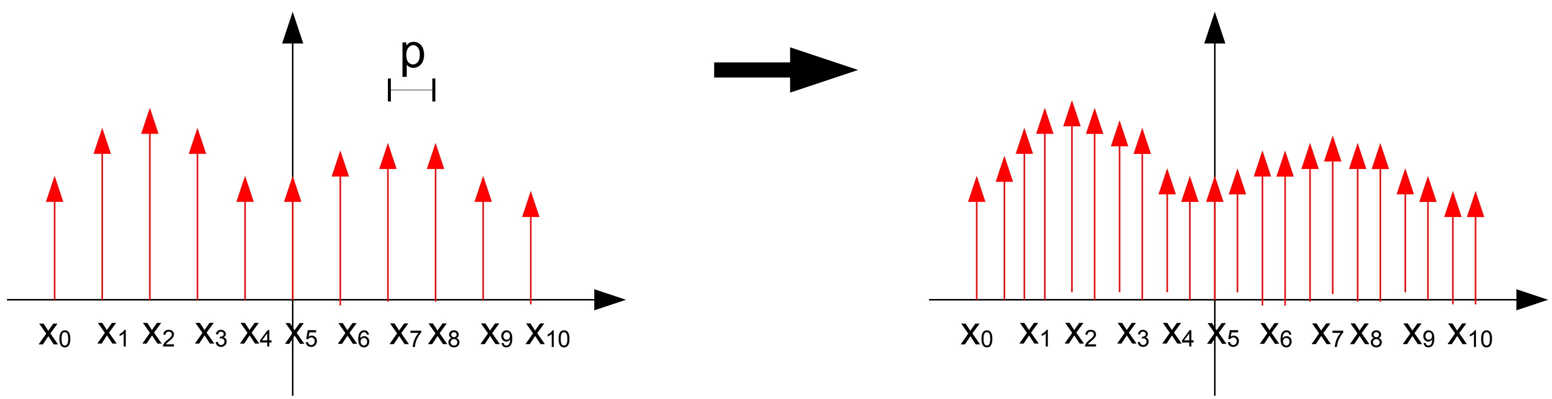


Linear interpolation

- Linear interpolation can be written as a convolution with a kernel (e.g. a basis function)

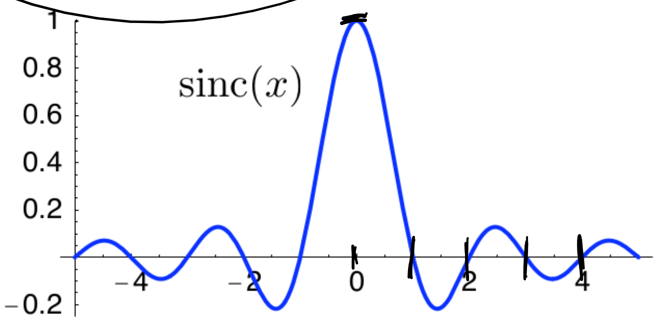




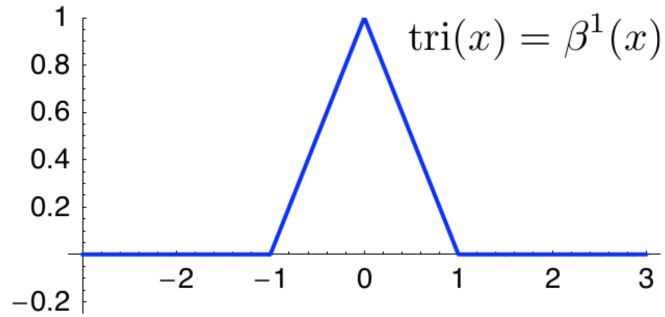


Linear interpolation

- Bandlimited



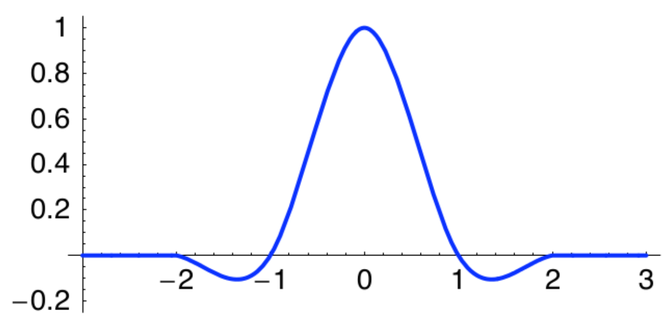
- Piecewise linear



Interpolation condition:

$$\varphi_{\text{int}}(k) = \delta_k = \begin{cases} 1, & k = 0 \\ 0, & \text{otherwise} \end{cases}$$

- Cubic convolution



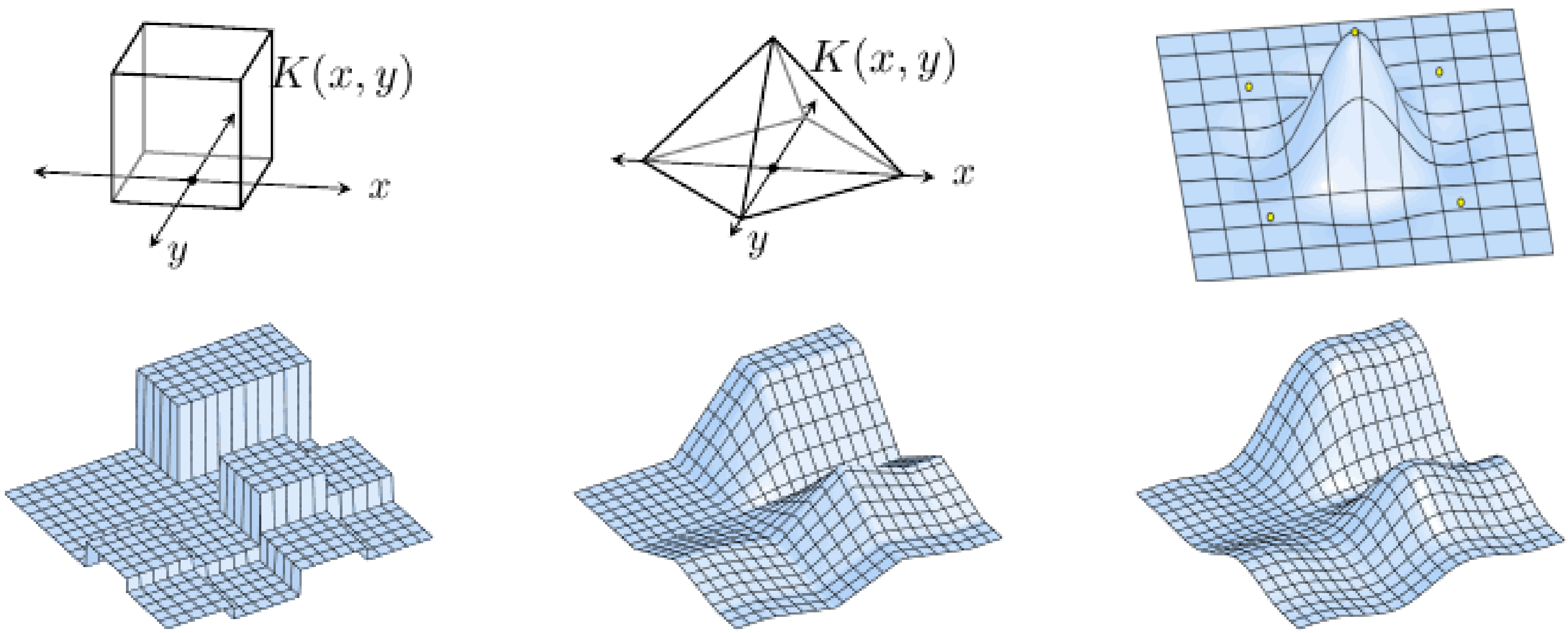
[Keys, 1981; Karup-King 1899]

source: http://bigwww.epfl.ch/tutorials/unser_isbi_06_part1

2D interpolation

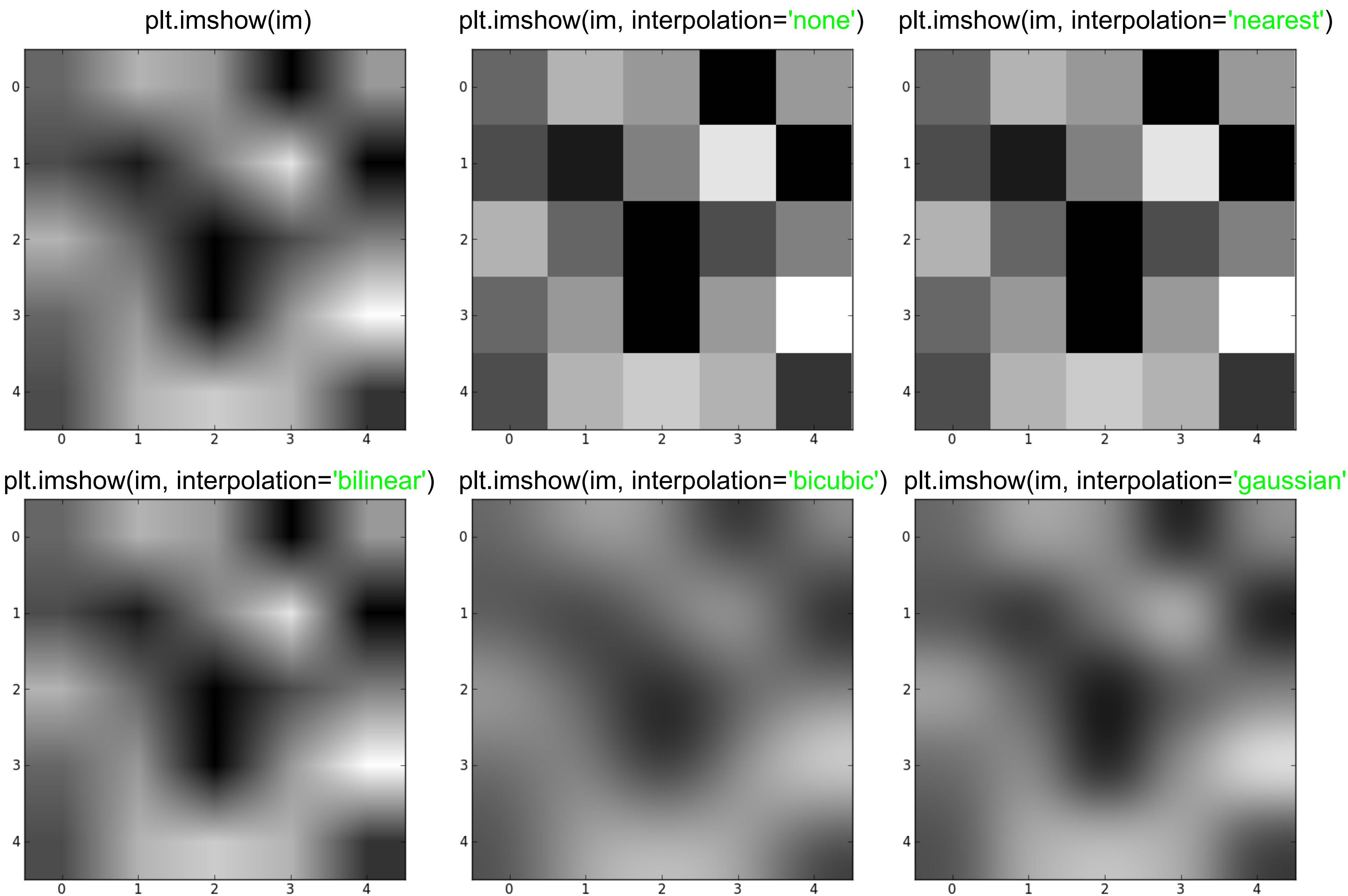
- Make 2D interpolation linear in each variable

nearest neighbor bilinear bicubic

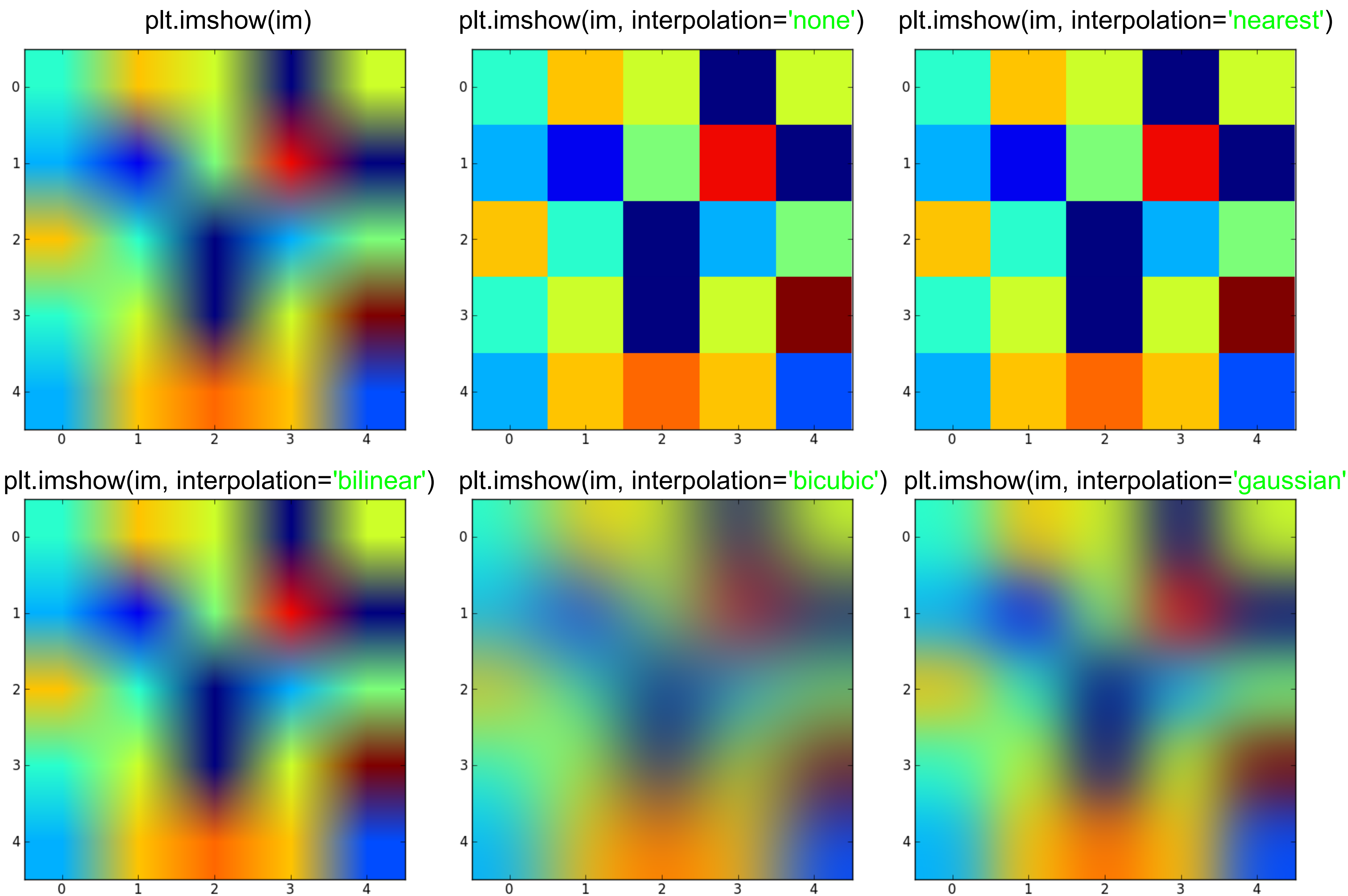


source: http://www.ipol.im/pub/art/2011/g_lmii/

Python plotting

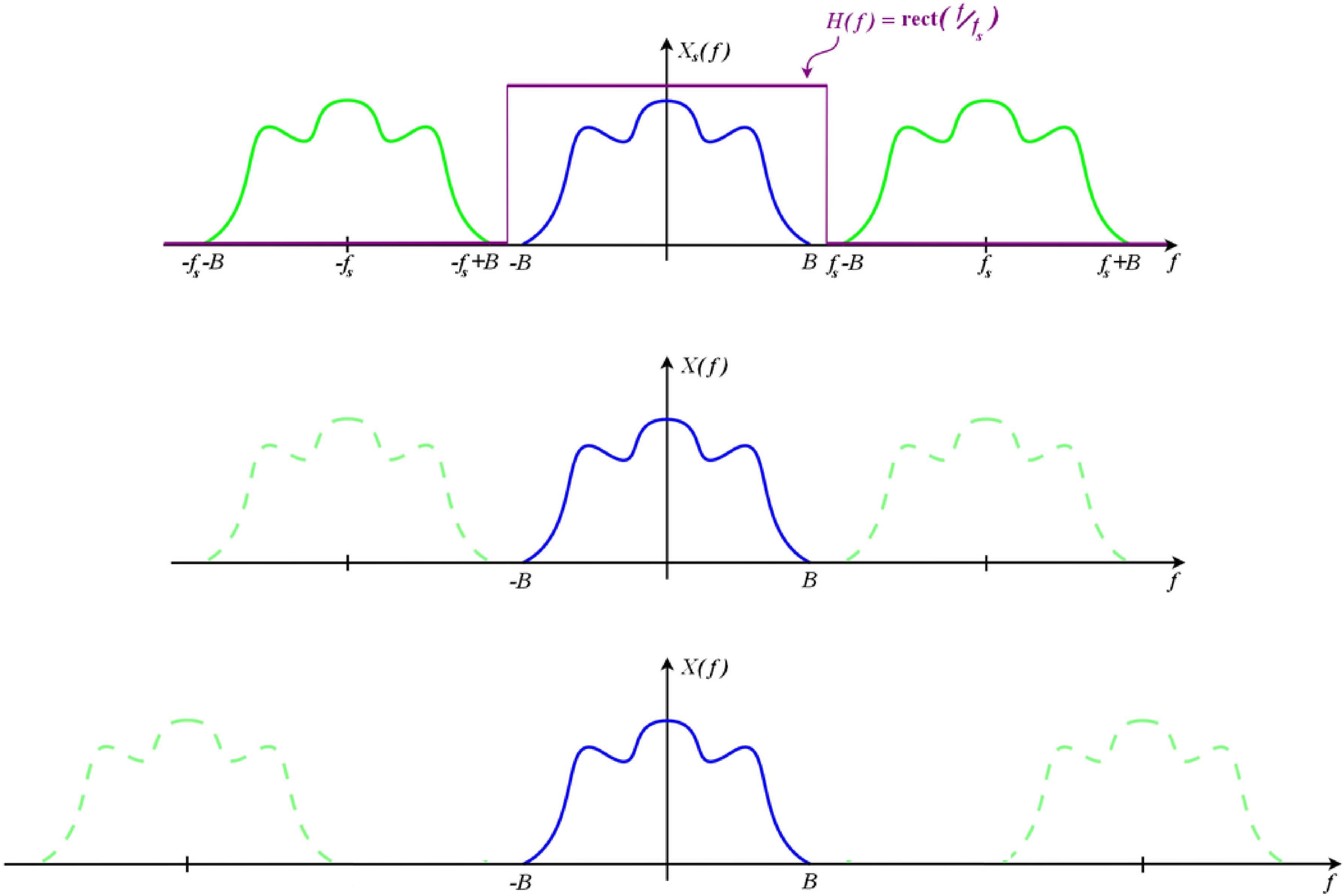


Python plotting



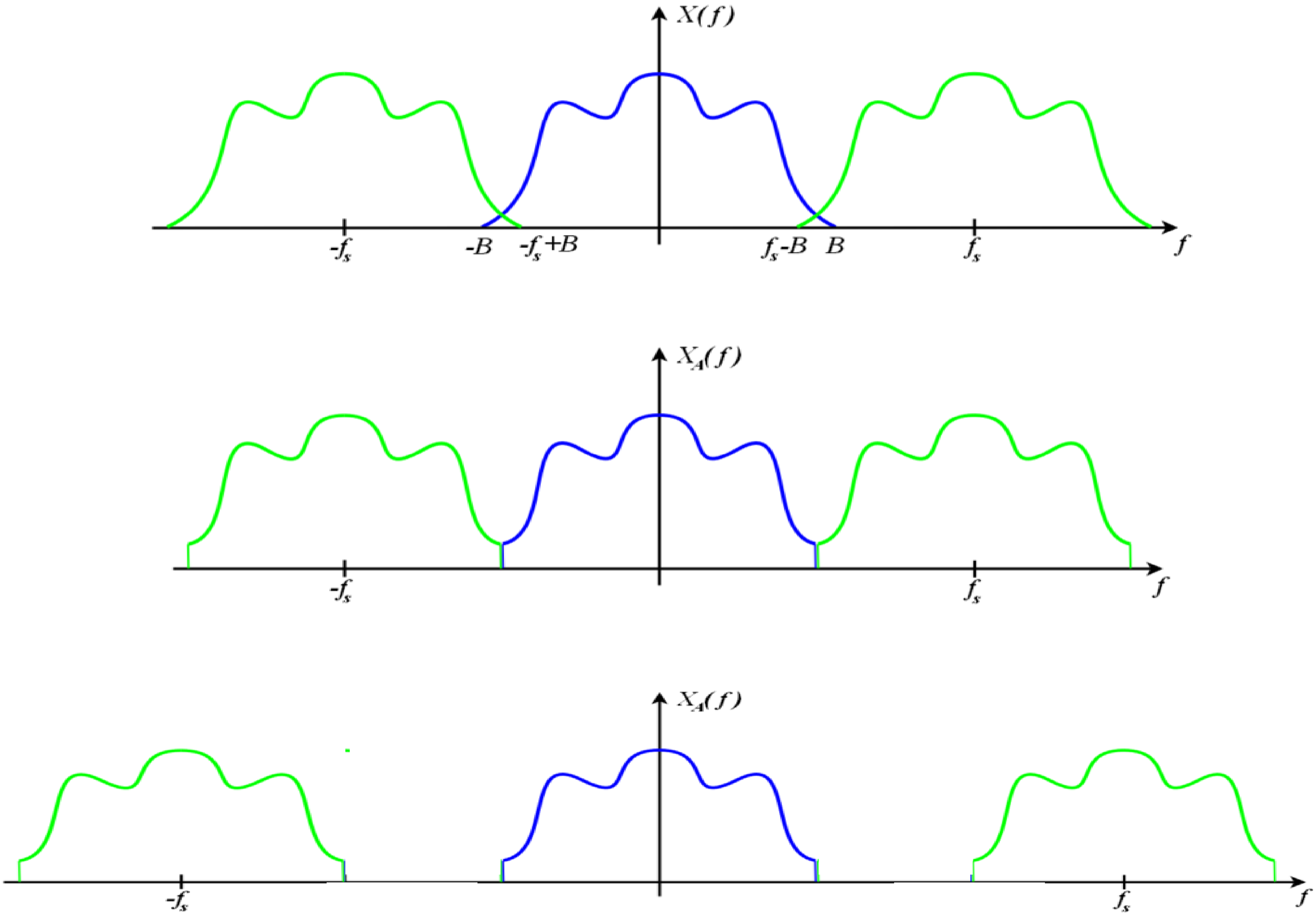
Sinc interpolation and zero-padding

Also known as “Whittaker–Shannon interpolation”



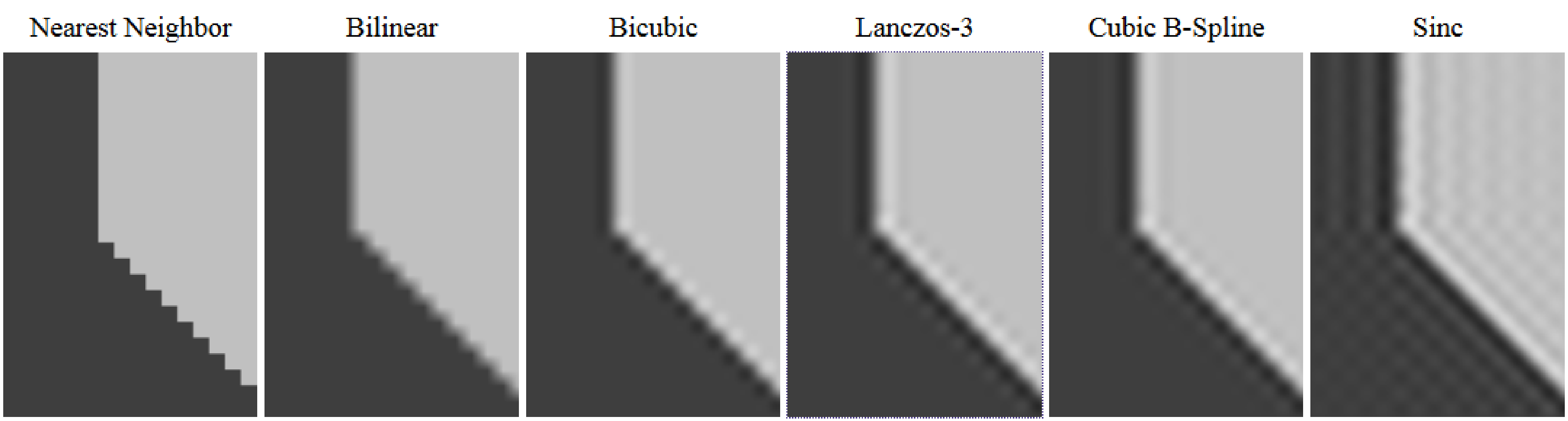
Sinc interpolation and zero-padding

Also known as “Whittaker–Shannon interpolation”



Reconstruction from samples

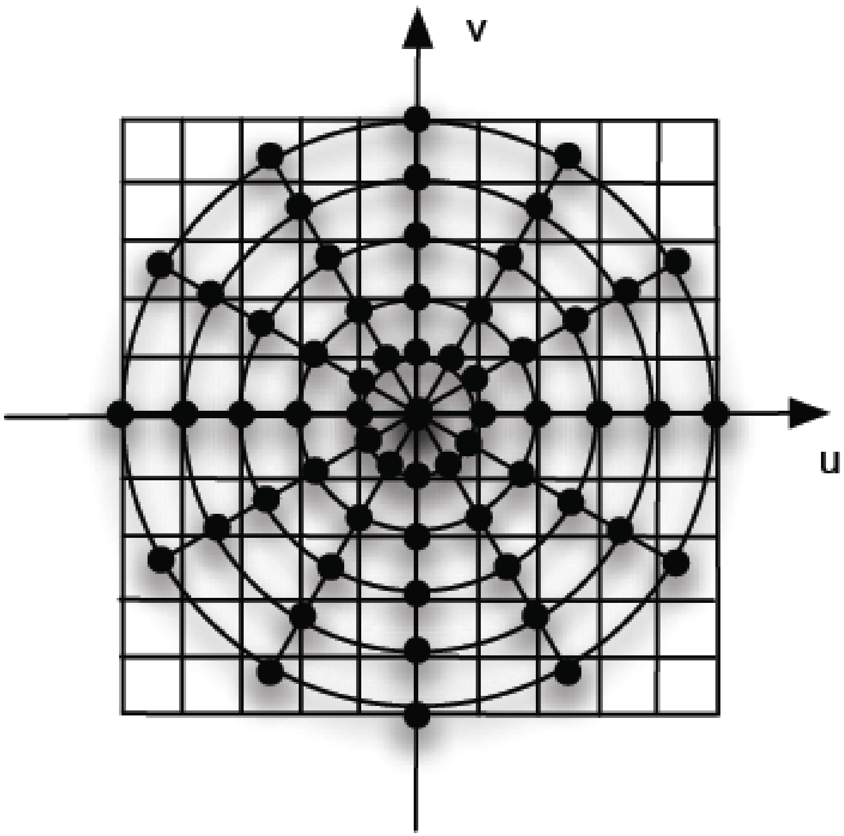
- Sinc interpolation can perfectly reconstruct a function from its samples if
  - sampled at a rate higher than Nyquist rate
  - bandlimited up to Nyquist frequency
  - no aliasing
- Sinc interpolation introduces ringing otherwise, due to leakage of aliased frequencies



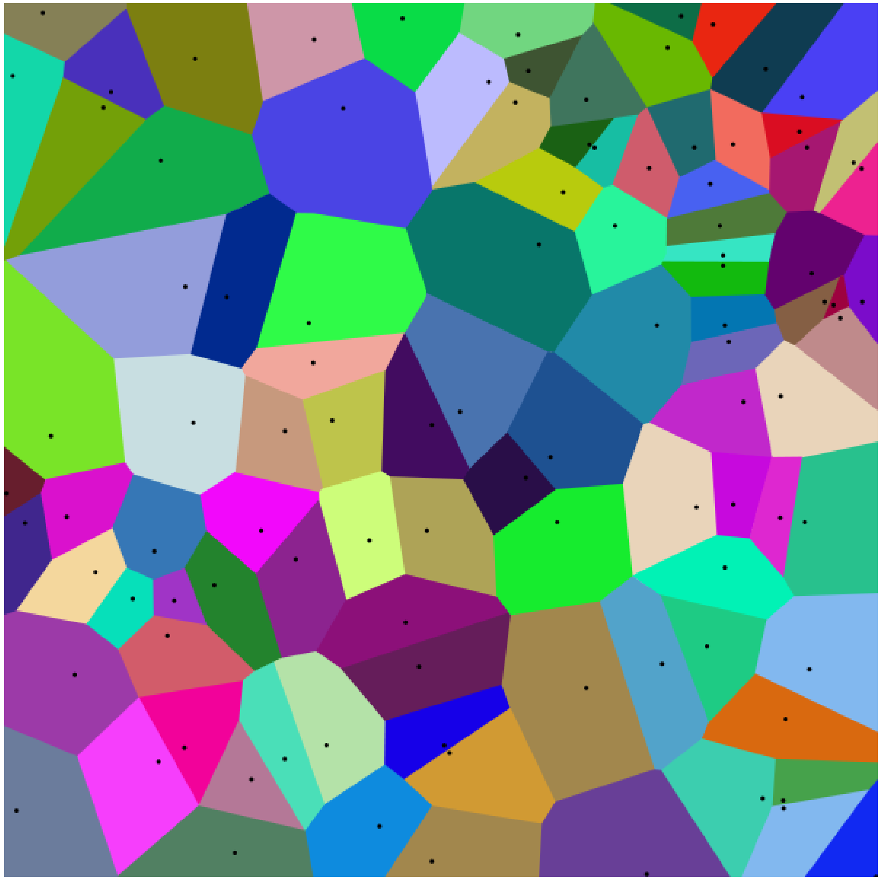
Linear interpolation of a step edge: a balance between staircase artifacts and ripples.

Other Interpolation

- Change from polar to cartesian grid
- Linear, but not translation invariant



polar vs. cartesian sampling



irregular sampling

Summary

- Images can be represented as a sampling grid and pixel basis functions
- Need for interpolation arises when changing the grid
- Linear and translation invariant interpolation can be written as a convolution with an interpolation kernel function
- Typical interpolation kernels include nearest neighbor, linear, cubic and higher B-spline interpolation
- Zero-padding in one domain equals sinc interpolation in the other
- “ideal” sinc interpolation may lead to ringing artifacts