Image Processing for Physicists

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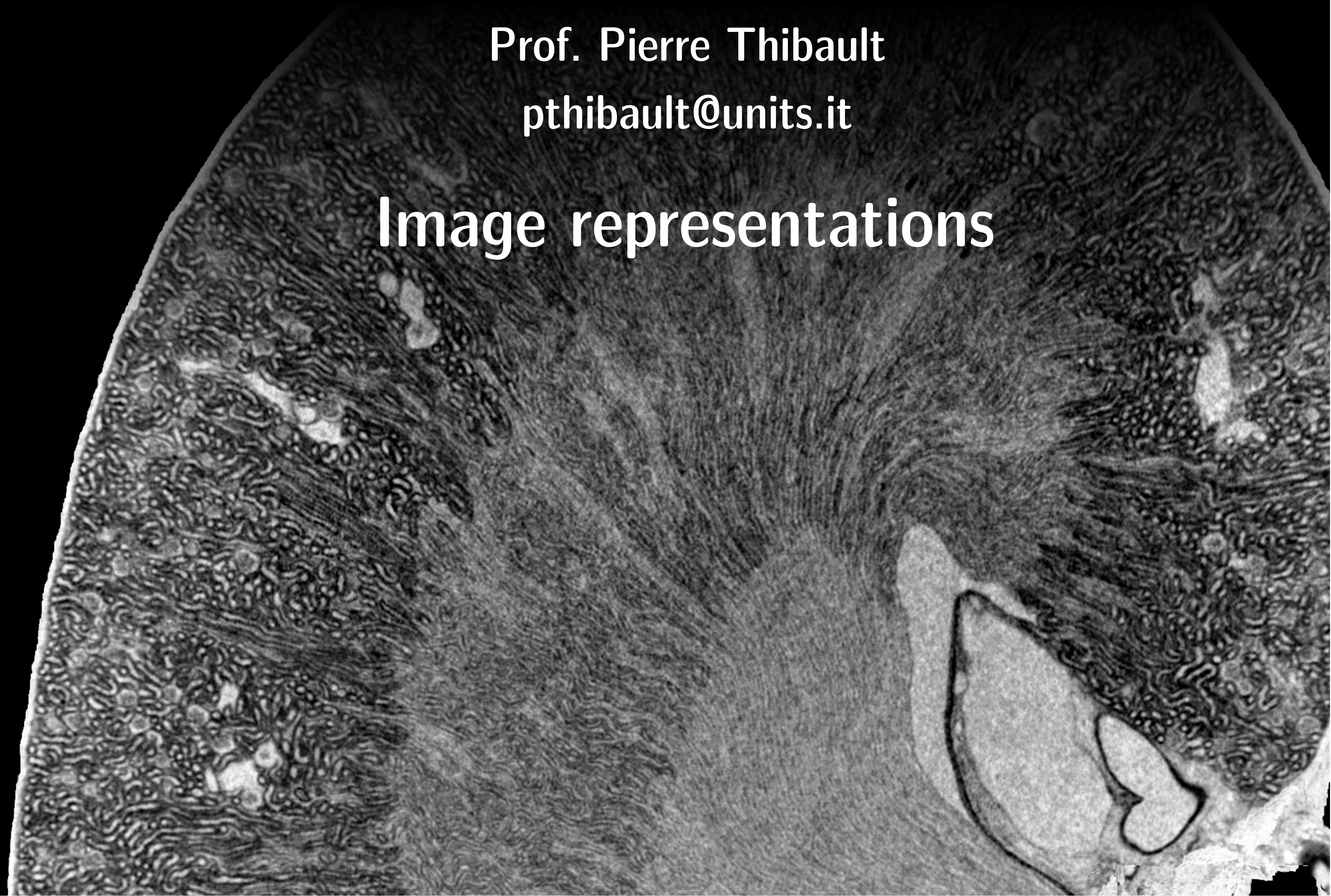
Image representations



Overview

- The Discrete Fourier Transform as a change of basis
- Discrete Cosine Transform
- Windowed Fourier Transform
- Wavelet Transform
- (many others omitted!)

Image representations

$$f(x,y) = \sum_n c_n B_n(x,y)$$

c_n: coefficients

B_n: basis functions

⇒ convenient; orthonormal

DFT:

$$f(m,n) = \sum_{k,l} F_{kl} \underbrace{e^{2\pi i\left(\frac{mk}{M}+\frac{nl}{N}\right)}}_{B_{kl}(m,n)\ \text{DFT basis function}}$$

shape of image is (M, N)

$$\left(\begin{array}{l} \langle kl | mn \rangle \\ \langle \vec{q} | \vec{x} \rangle = e^{i\vec{q}\cdot\vec{x}} \end{array} \right)$$

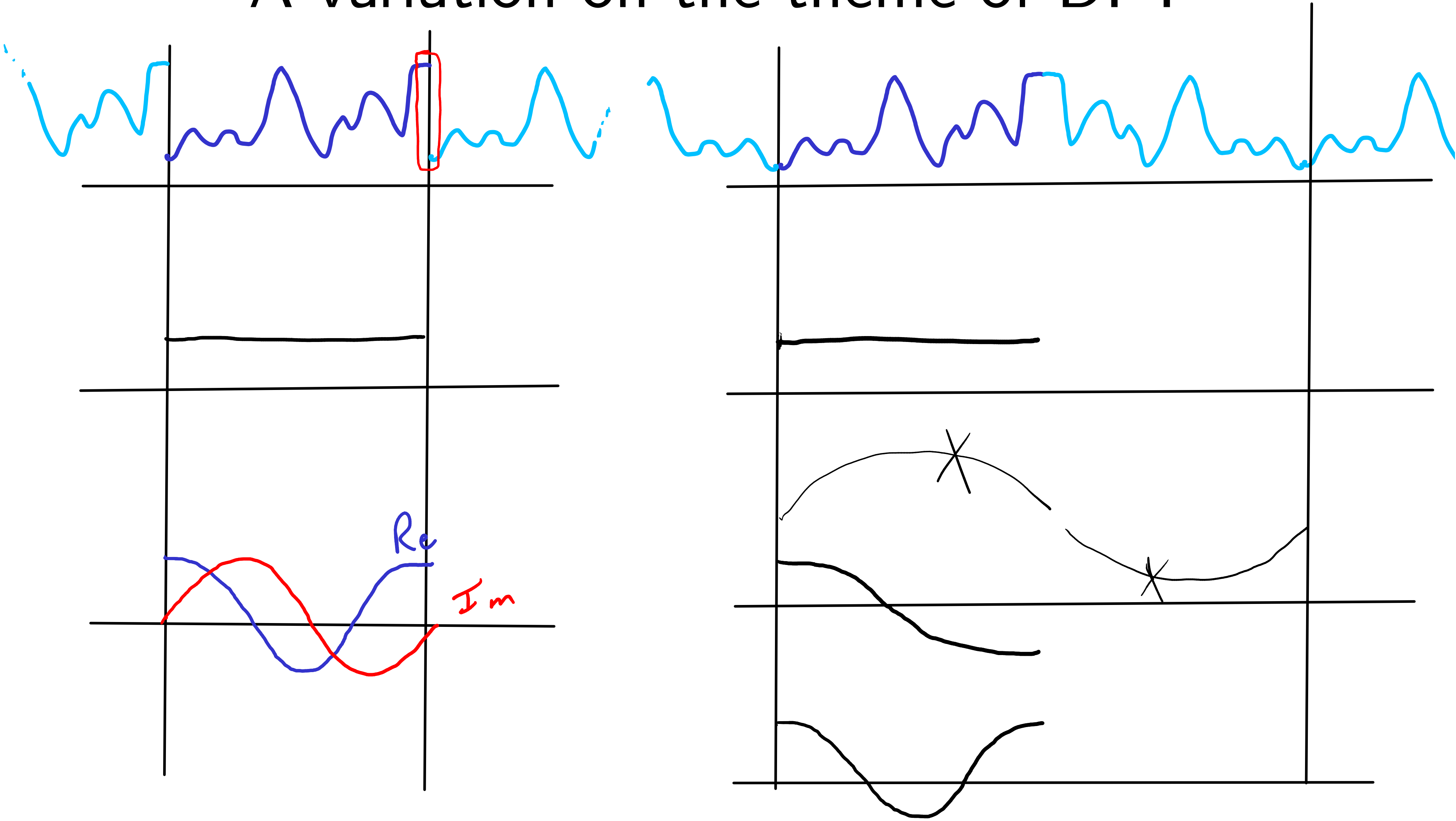
DFT (1D): $Z = e^{2\pi i/N}$

$$\begin{bmatrix} \\ \\ f \\ \\ \\ \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & z & z^2 & \cdots & z^{N-1} \\ 1 & z^2 & z^4 & \cdots & \\ \vdots & \vdots & & \ddots & \\ 1 & z^{N-1} & & & \end{bmatrix} \begin{bmatrix} \\ \\ F \\ \\ \\ \end{bmatrix}$$

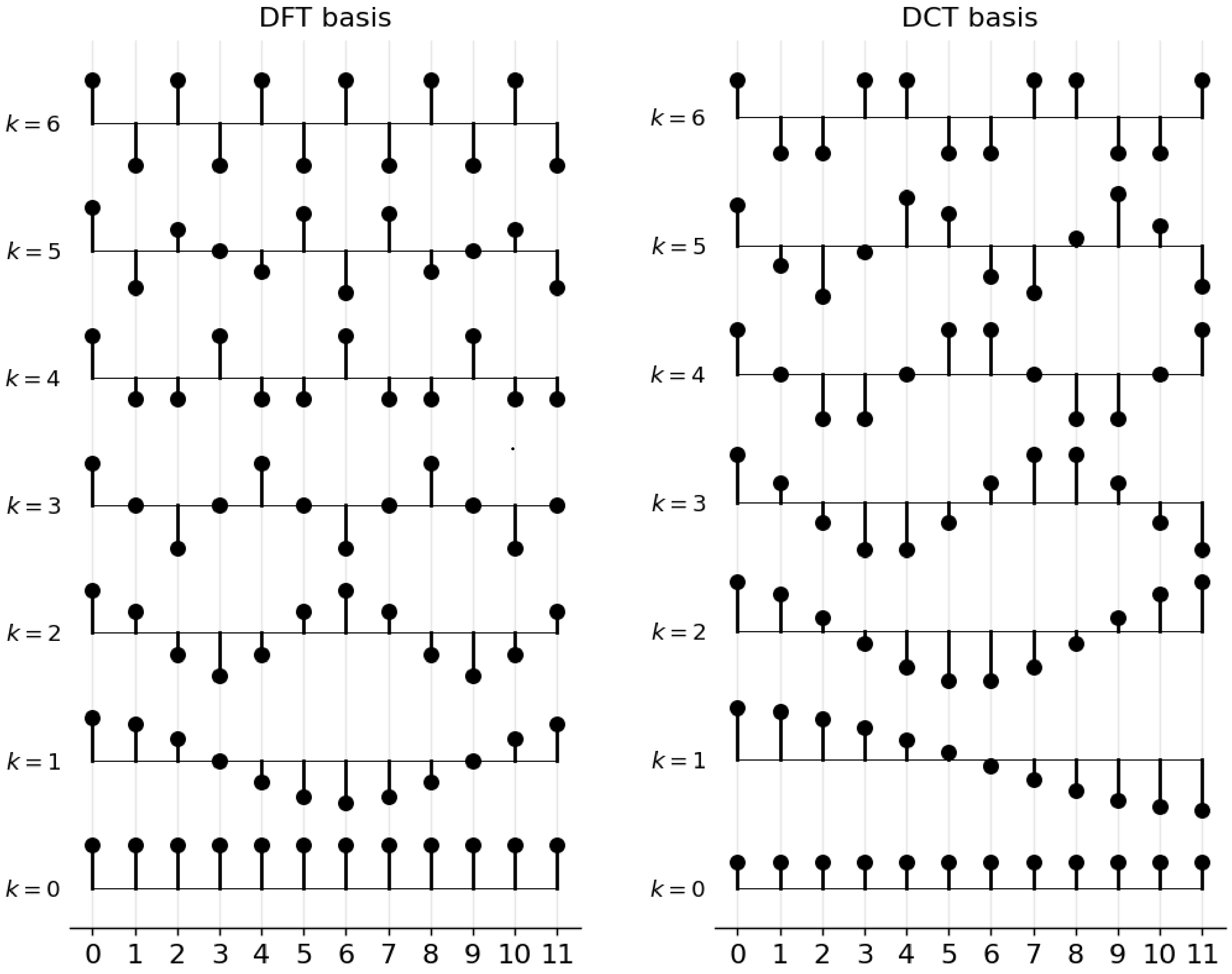
$$f_n = \sum_k F_k z^{nk}$$ "z-transform"

Discrete Cosine Transform

A variation on the theme of DFT



Discrete Cosine Transform

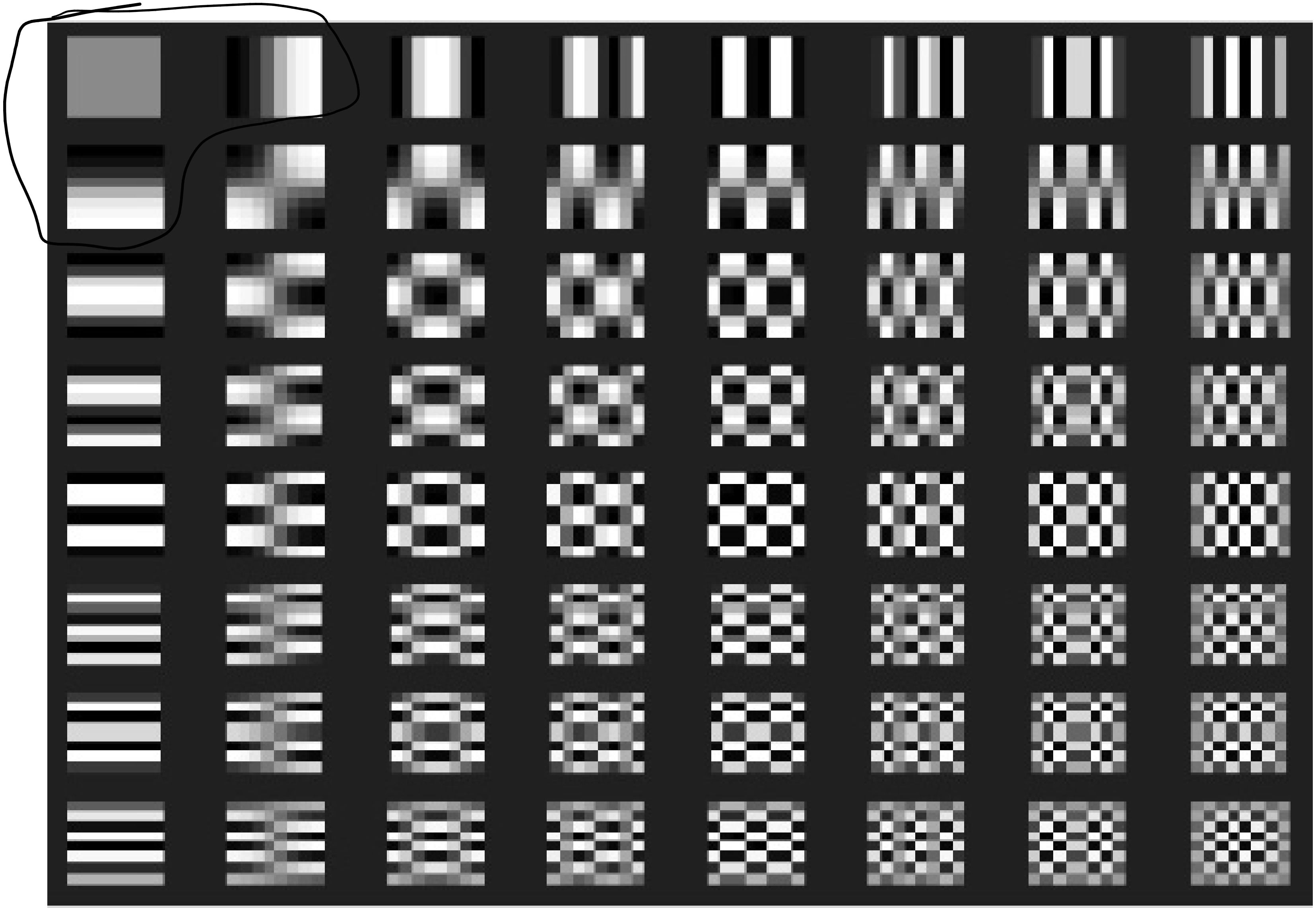


7 first basis functions

N = 12

Discrete Cosine Transform

64 DCT basis vectors for 8x8 image



Discrete Cosine Transform

Image compression

1:1 bit rate

8:1 bit rate

JPEG

← keeping $\frac{1}{8}$ of the coefficients

32:1 bit rate

128:1 bit rate

Basis functions for DCT:

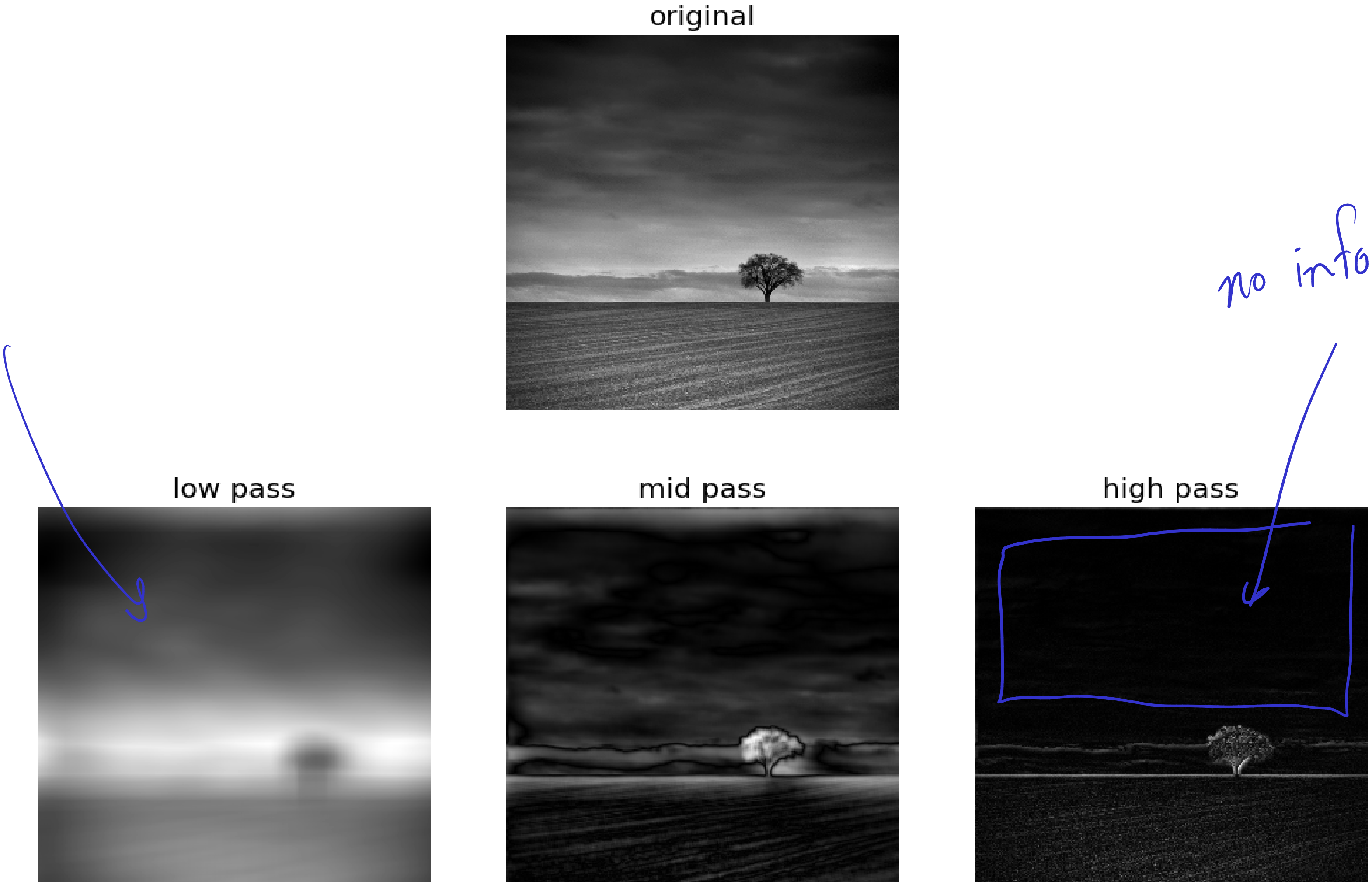
$$B_k(n) = \cos\left(\left(n+\frac{1}{2}\right)\frac{k\pi}{N}\right)$$

$$B_0(n) = \frac{1}{\sqrt{2}}$$

Historical overview

- 1822 Fourier: Fourier transform
- 1946 Gabor: “Gabor transform”, Short-time Fourier transform (STFT)
- 1974 Ahmed, Natarajan & Rao: Discrete Cosine Transform
- 1980s Morlet, Mallat, Daubechies, … : Wavelets

Bandpass filtering



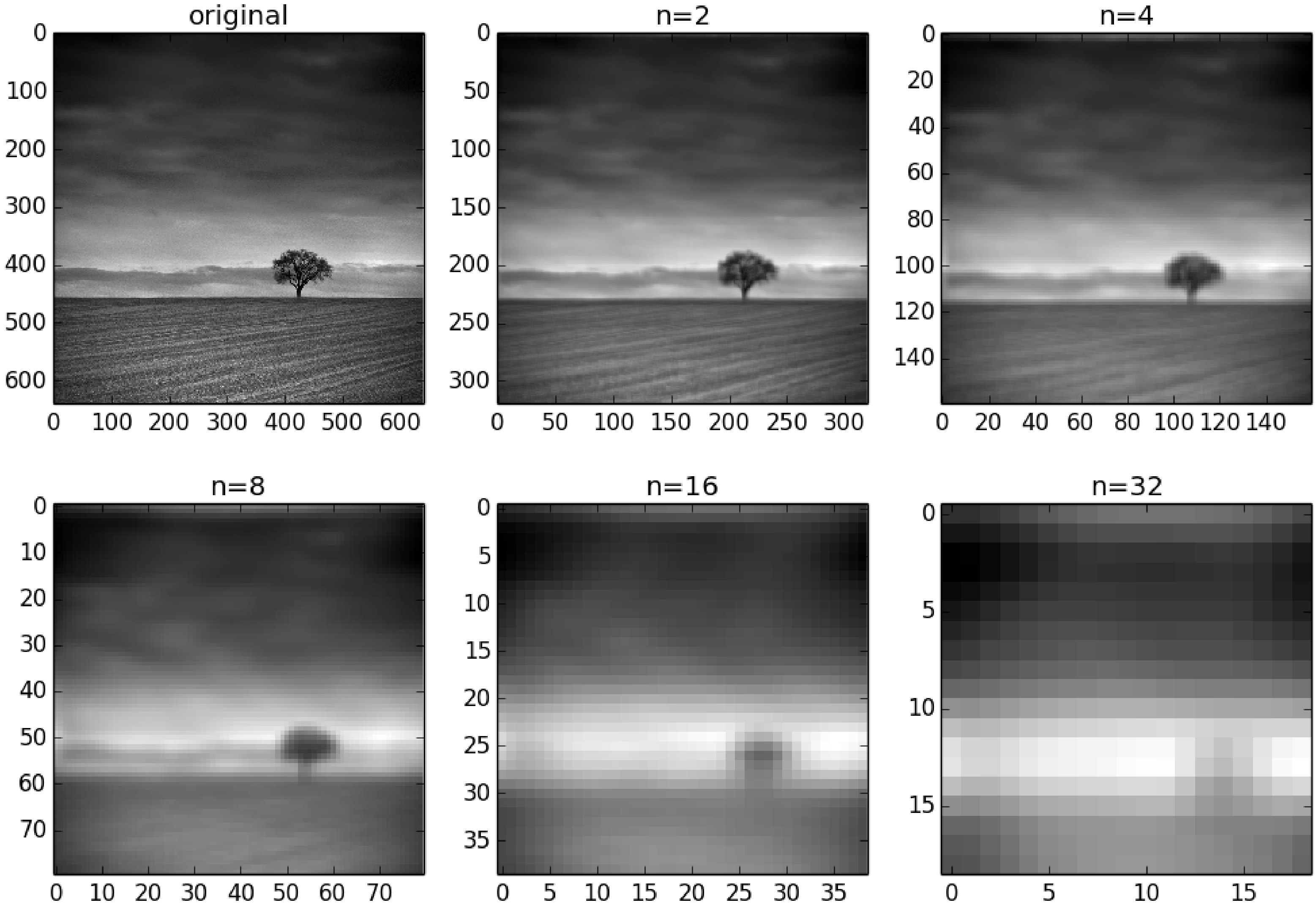
no information

Don't need high spatial resolution

Need high spatial resolution

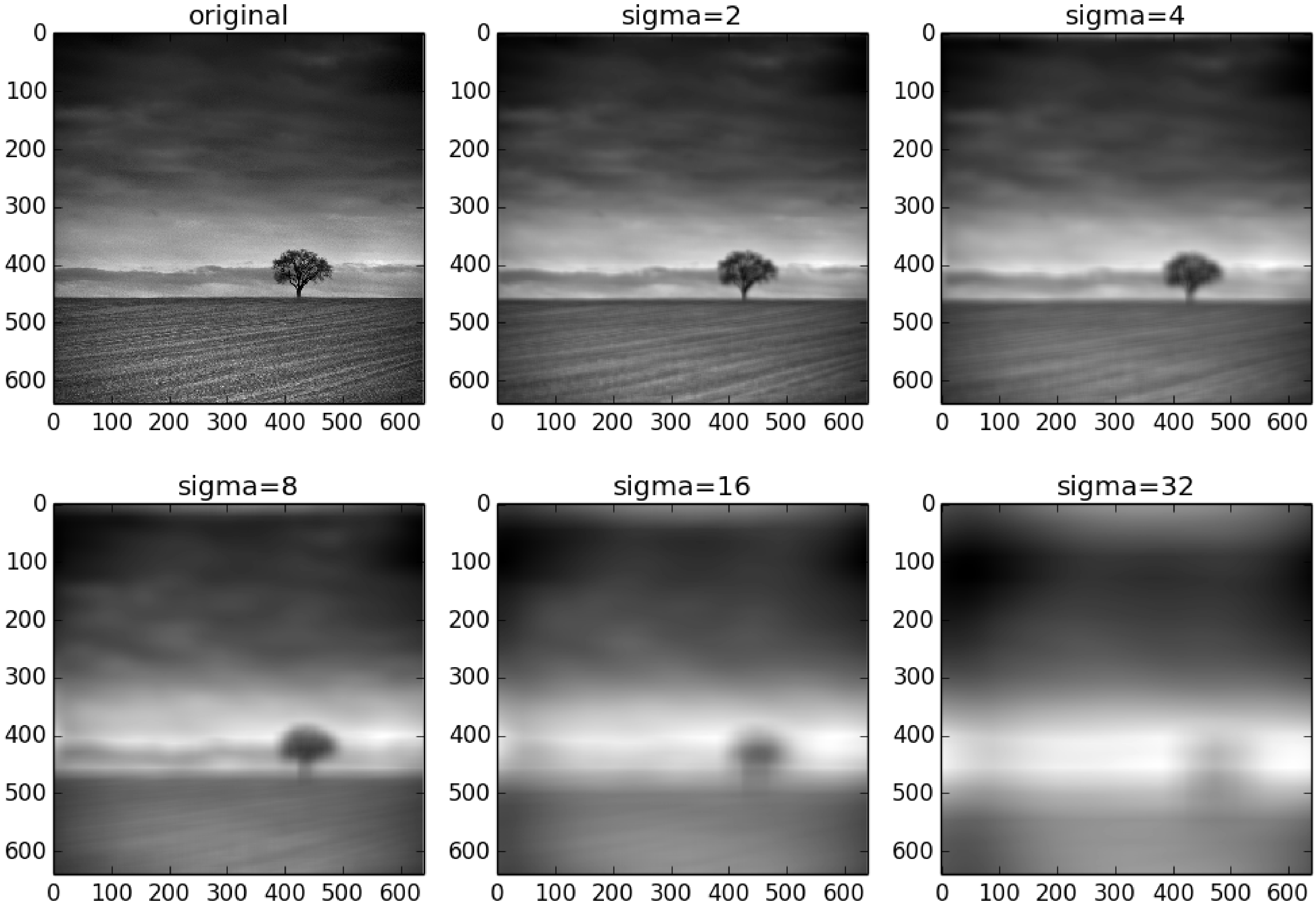
Multiresolution analysis

Subsampling (taking every nth pixel) successively reduces high frequency content



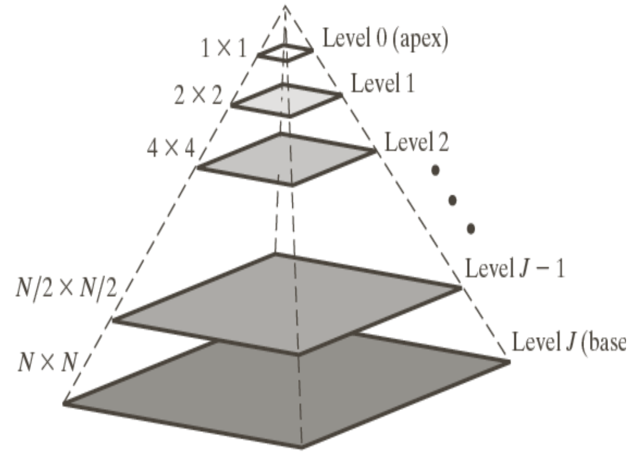
Multiresolution analysis

Multiple filtering with Gaussian filters, sigma determines resolution



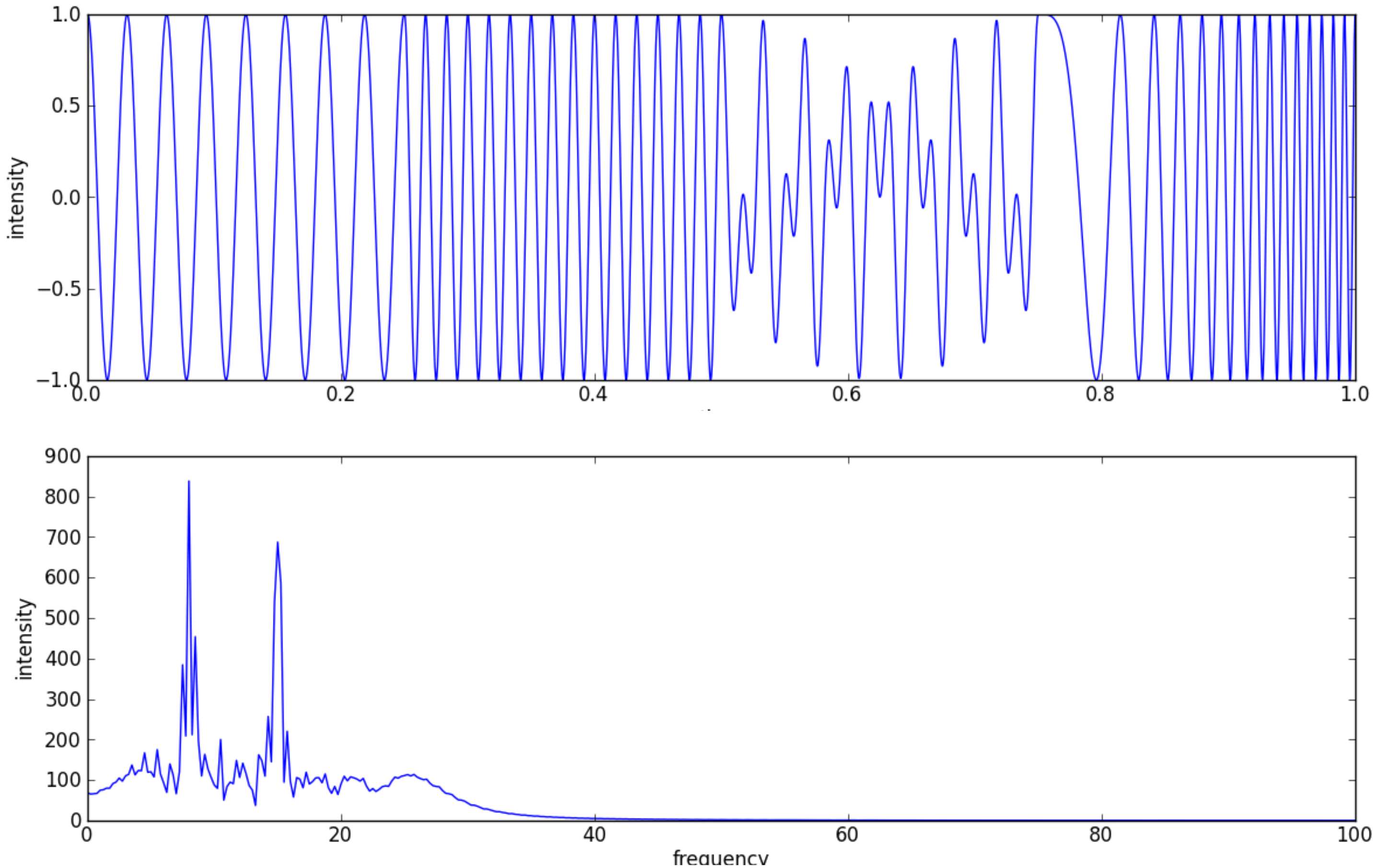
Pyramid representation

Scale-space representation, pyramidal representation



Stationary vs. non-stationary signals

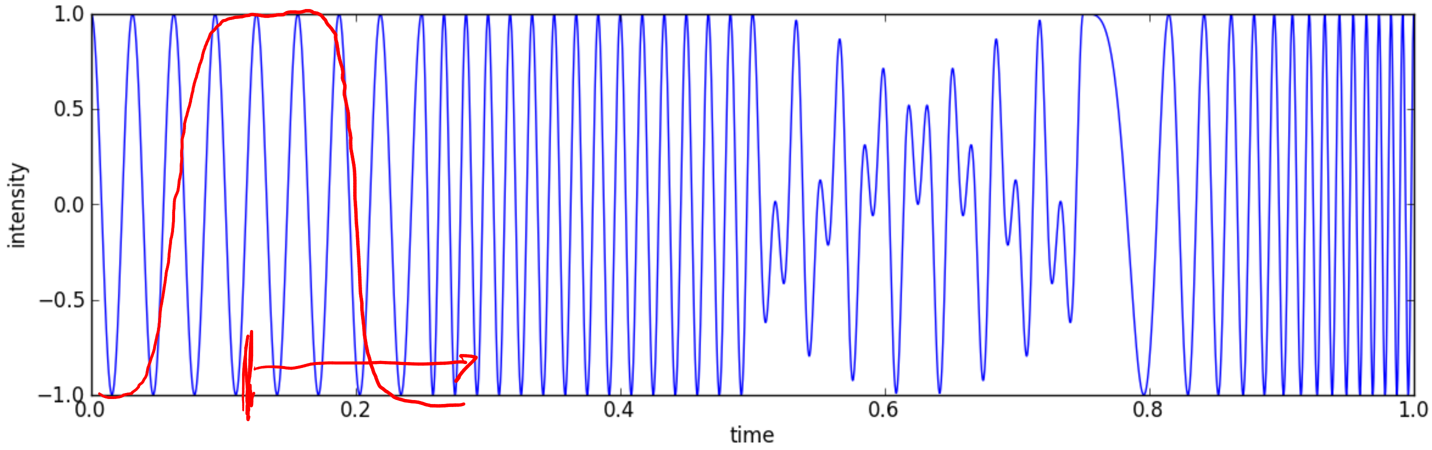
- Stationary signals: frequency doesn't change over time (spatially over the image)
- Non-stationary signals: frequency changes over time (spatially over the image)
- Examples of non-stationary signals: speech, most images



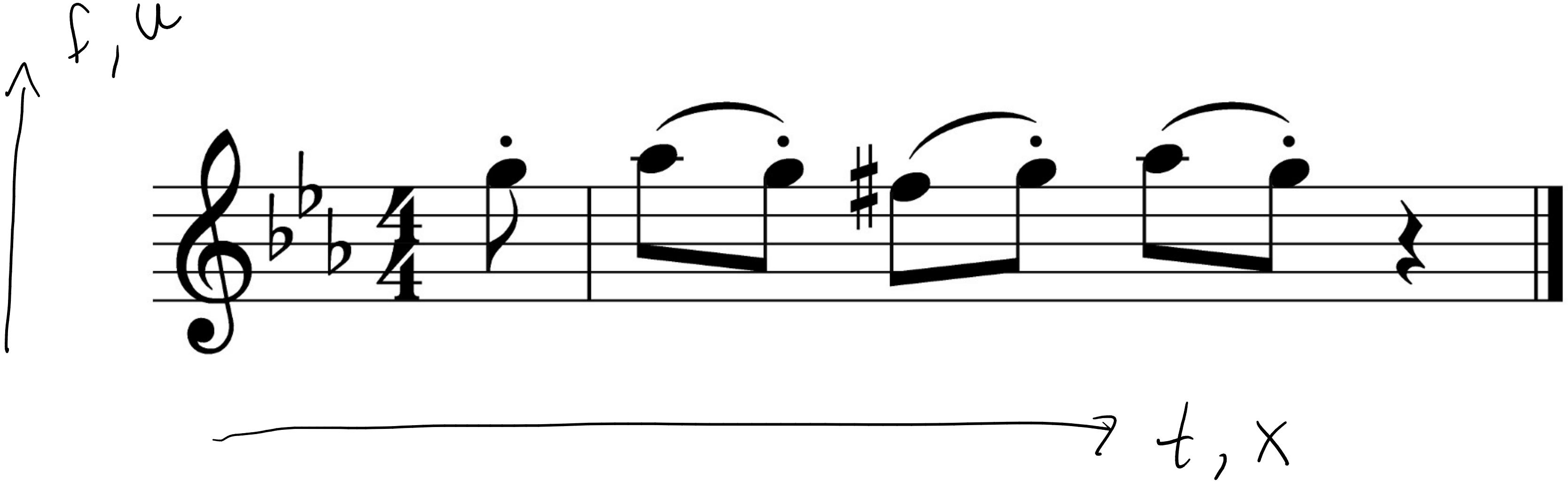
FT insufficient to localize the frequencies in our signal (image)

Windowed Fourier transform

- Windowed Fourier transform is part of the field of “time-frequency analysis”
- Also known as Short-time Fourier Transform (STFT)
- Time-frequency representations are used in many different contexts (Audio, image processing/optics, quantum mechanics)
- Idea: slice up signal into small parts, analyze each separately
  - Multiply with window function w (of width d) at position x0
  - Take Fourier transform of result
  - Slide window to new position
  - repeat

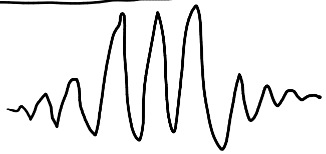


Analogy to audio signals



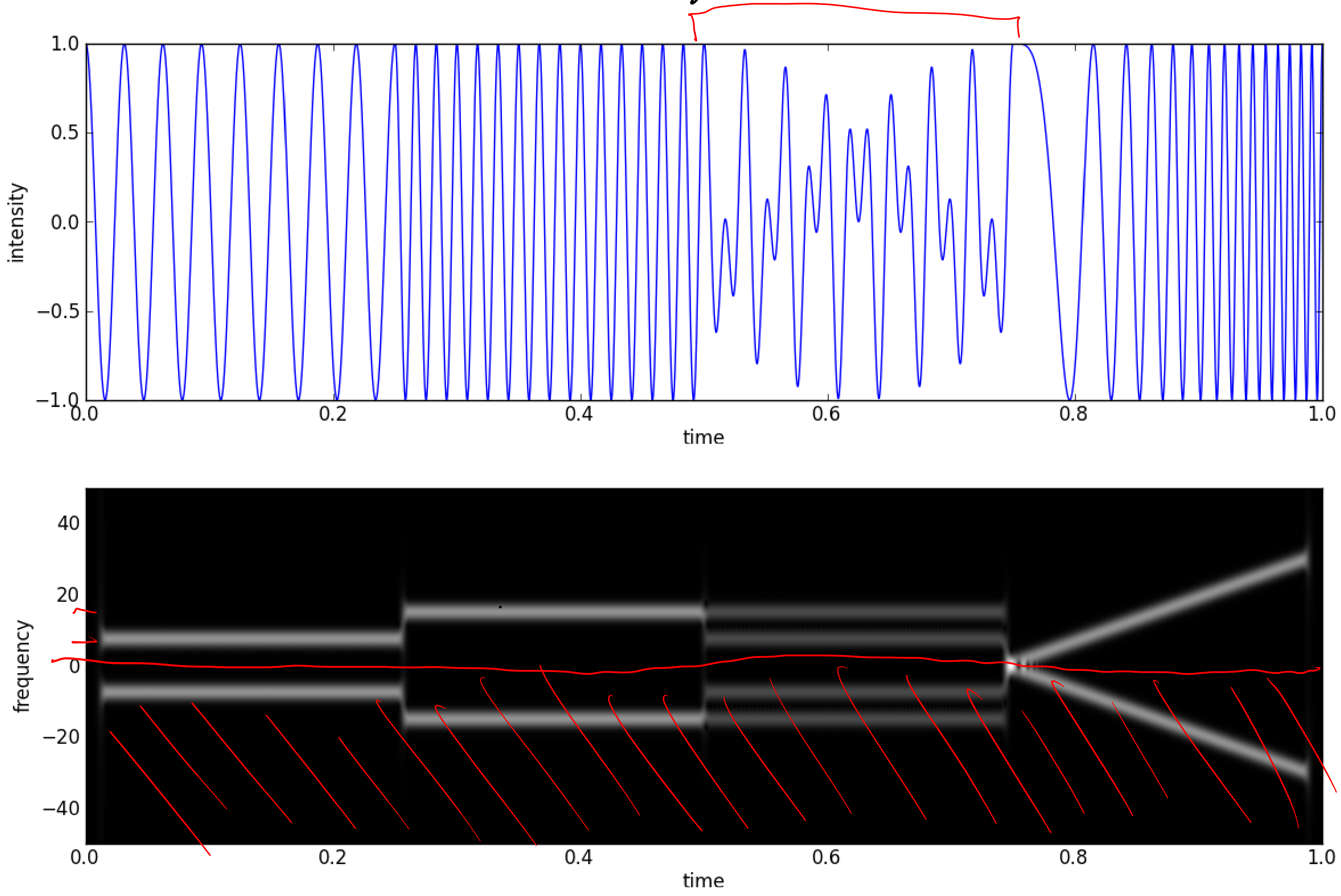
spectrogram

$$S(u, x_0) = \int_{-\infty}^{\infty} f(x) \underbrace{w(x - x_0) e^{-2\pi i u x}} dx$$



Spectrogram

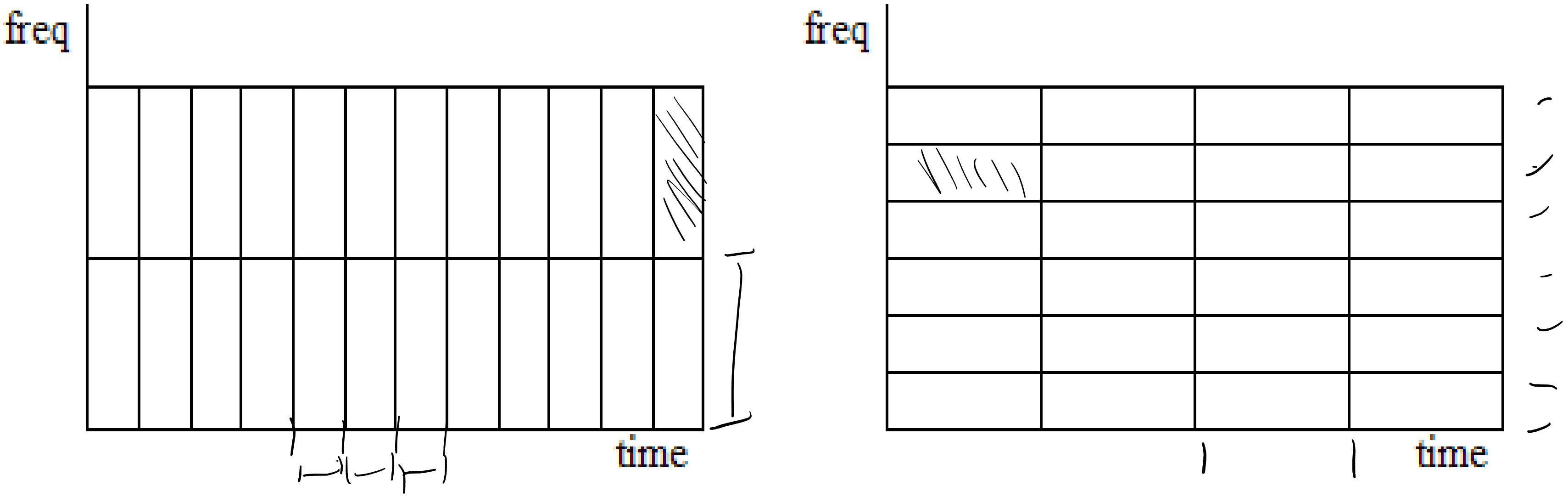
$\cos f_1 t$ $\cos f_1 t$ $\frac{1}{2}(\cos f_1 t + \cos f_2 t)$ $\cos(\alpha t^2)$



Uncertainty relation

$$\sigma_s \sigma_f \geq \frac{1}{4\pi}$$

- Finite area in the time-frequency plane



- This is limitation of WFT and hence development of **wavelets**

Continuous wavelet transform (WT)

- Parameters: translation and scaling

position

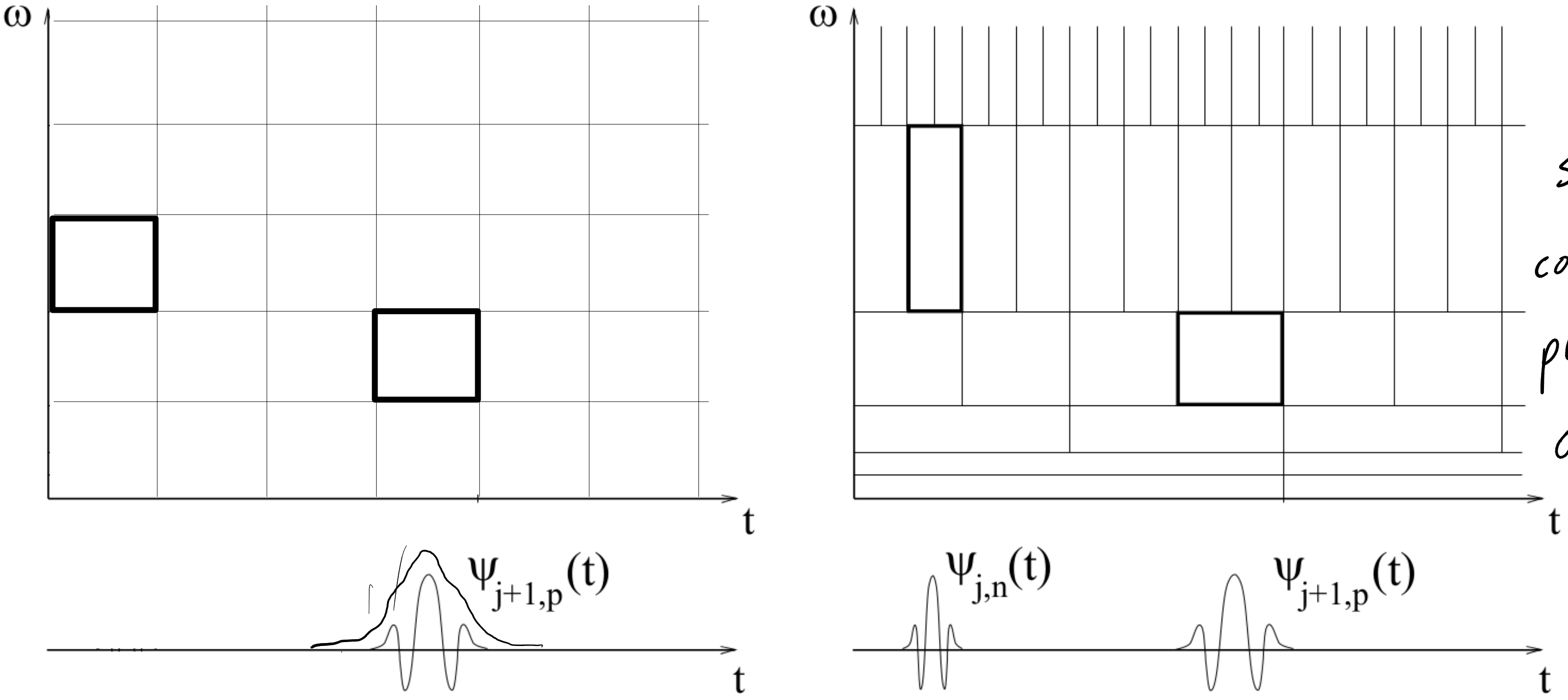
wavelets

$$WT_{s,x_0}\{f\} = \int_{-\infty}^{\infty} f(x)\, \psi_{\underset{\uparrow\ \text{scale}}{s},\ \overset{\text{position}\ \downarrow}{x_0}}(x)\, dx \qquad \overset{\text{wavelets}\ \swarrow}{\psi_{s,x_0}} = \frac{1}{\sqrt{s}} \underset{\uparrow\ \text{mother wavelet}}{\psi}\left(\frac{x-x_0}{s}\right)$$

scale

mother wavelet

- Analyze signal at <u>different scales</u> instead of <u>different frequencies</u>



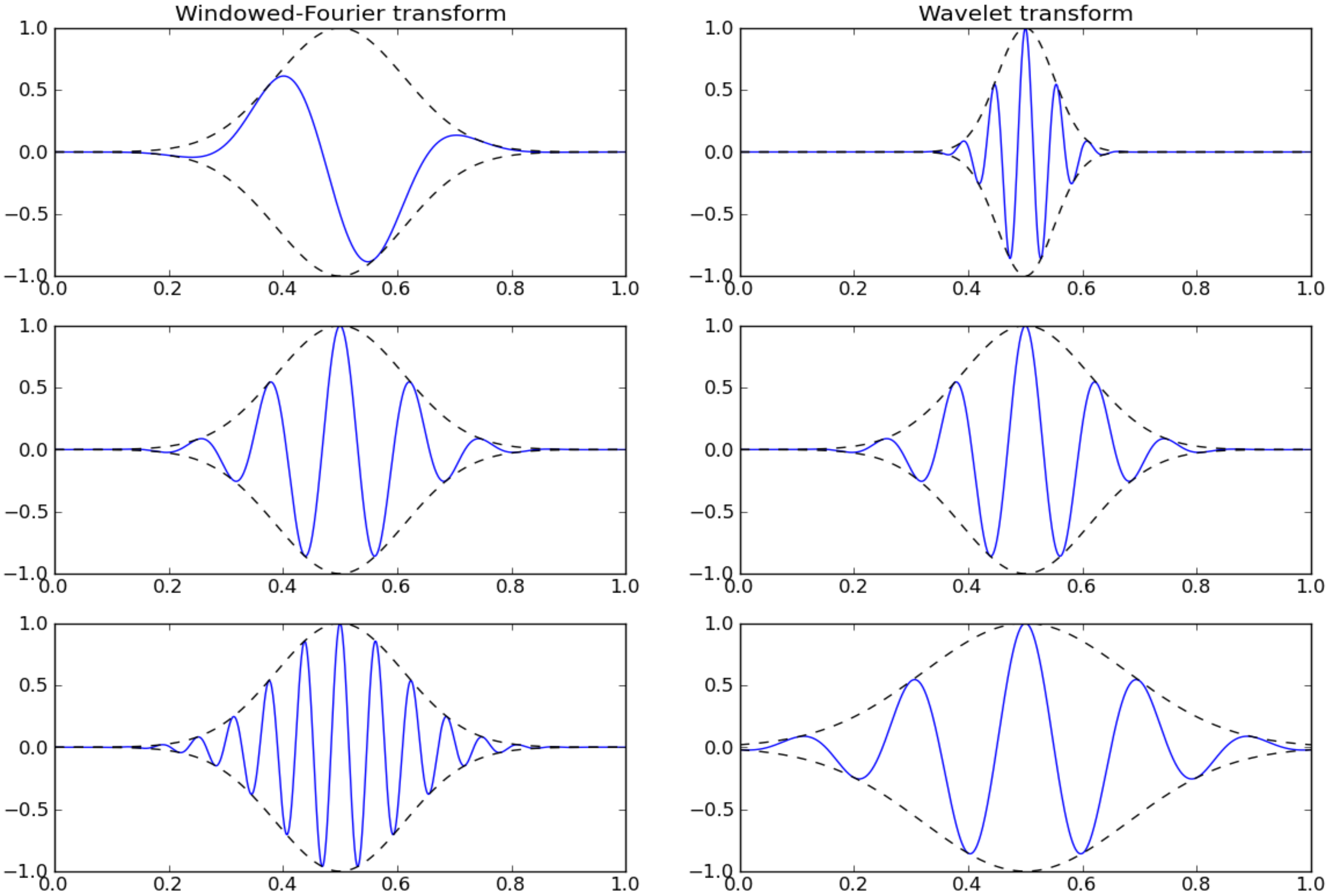
scaling gives convenient partitioning of phase space

Source: Mallat, "A wavelet tour of signal processing"

WFT vs WT

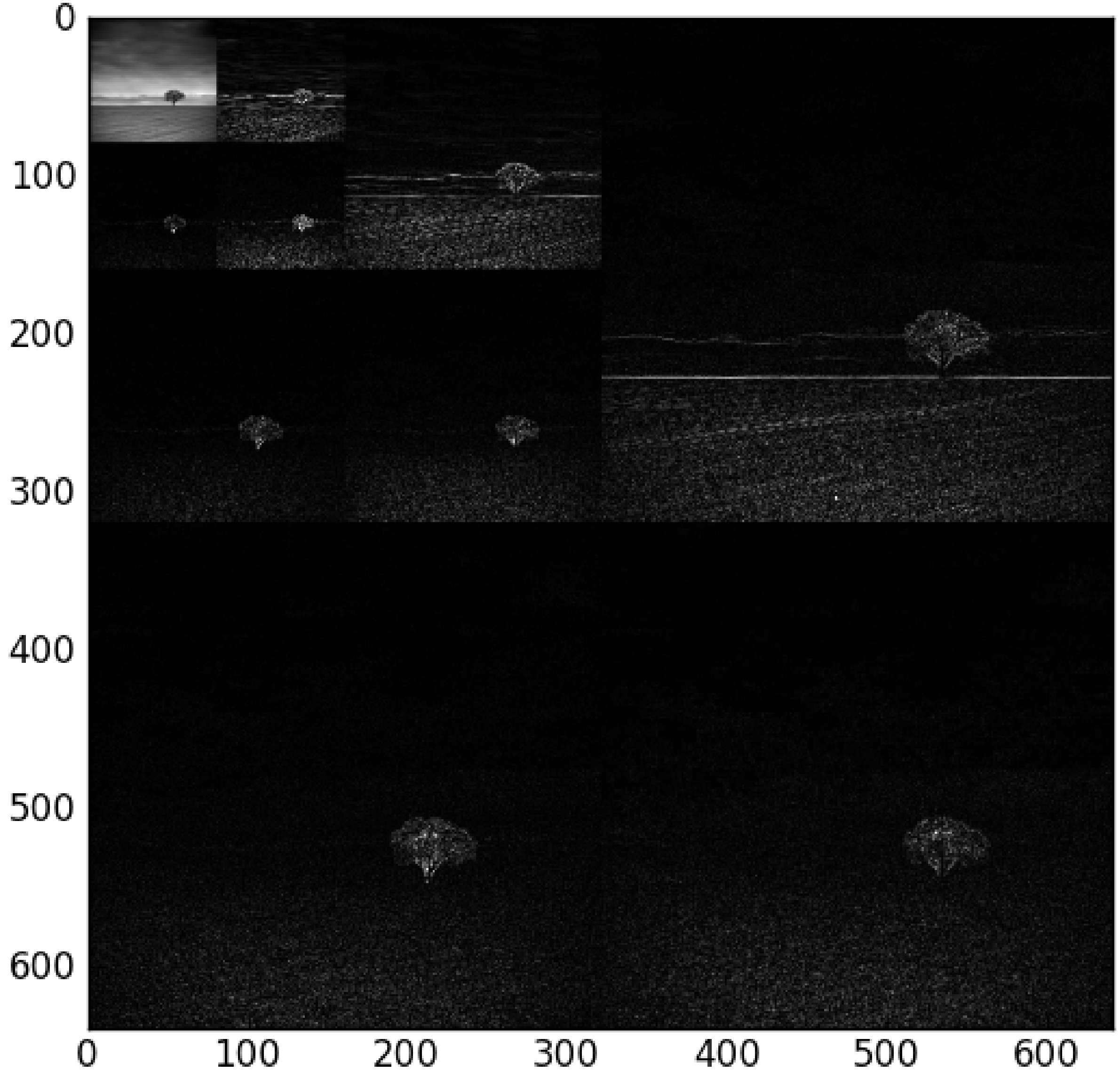
WFT - keep window width constant
- change modulation

Wavelet - keep shape constant
- change scale



Discrete Wavelet decomposition of image

- Perform each DWT, collect and tile all coefficients
- Here: 3 level decomposition



used for instance
as a way to
impose sparsity
of a signal/image

Summary

- Images can be represented by different basis functions.
- Fourier basis: localized in frequency, delocalized in real space.
- Windowed Fourier Transform: localized – to some extent – in both spaces
- Wavelet analysis decomposes a signal in position and scale (instead of position and frequency as for WFT).
- Sparse representations are representations in which the image content is represented by a few relevant coefficients, while the other (pixels) are close to zero

  values
- Sparse representations have advantages for compression, denoising, ...

  regularizers, ...