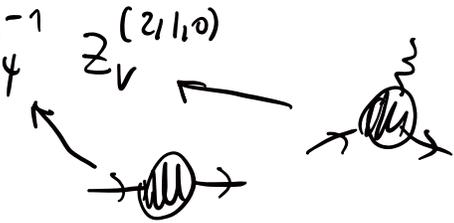
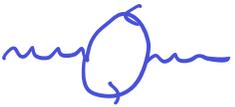


β -function

$$g_B = g_r \mu^{2-\omega} Z_A^{-1/2} Z_4^{-1} Z_V^{(2,1,0)}$$

cost. necessaria per cancellare l'AS in 



Per trovare la relazione tra g_B e g_r dobbiamo calcolarci tre ampiezze 1 PI (in QED, grazie a id. Ward, $Z_V^{(2,1)} = Z_4 \Rightarrow$ bastava calcolare n , che a 1 loop è )

$$Z_A \leftrightarrow \text{diagram} \quad (4 \text{ diag. a 1-loop})$$

$$Z_4 \leftrightarrow \text{diagram} \quad (1 \text{ diag. a 1-loop})$$

$$Z_V^{(2,1,0)} \leftrightarrow \text{diagram} \quad (2 \text{ diag. a 1-loop})$$

n° di FLAVOURS

Prendiamo una teoria con N_f fermioni di Dirac nella rappresentaz. R di G ($i \bar{\Psi}^I \not{D} \Psi^I = i \bar{\Psi}^I (\not{D} + i A_\mu^a \gamma^a t_R^a) \Psi^I$)

$$Z = 1 + \delta Z \quad (\text{Vedi sotto per calcolo } \delta Z)$$

simmetria globale ("di flavor") è $SU(N_f)$

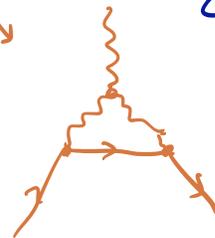
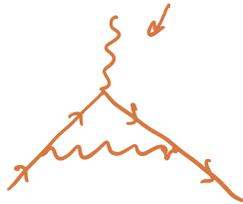
$$\delta Z_A = - \frac{g_r^2}{16\pi^2} \left(\frac{4}{3} N_f c(R) - \frac{5}{3} c_2(G) \right) \frac{1}{2-\omega}$$



$$\delta Z_4 = - \frac{g_r^2}{16\pi^2} c_2(R) \frac{1}{2-\omega} \rightarrow$$



$$\delta Z_V^{(2,1,0)} = - \frac{g_r^2}{16\pi^2} (c_2(R) + c_2(G)) \frac{1}{2-\omega}$$



$\delta Z_V^{(2,1,0)} \neq \delta Z_4$
 contribution
 alla QED

$$g_B = g_r \mu^{2-\omega} \underbrace{\left(1 - \frac{1}{2} \delta Z_A\right)}_{\equiv Z_A^{-1/2}} \underbrace{\left(1 - \delta Z_4\right)}_{\equiv Z_4^{-1}} \underbrace{\left(1 + \delta Z_V^{(2,1,0)}\right)}_{Z_V^{(2,1,0)}}$$

conto perturbativo \rightarrow

$$= g_r \mu^{2-\omega} \left(1 - \frac{g_r^2}{16\pi^2} \left\{ -\frac{2}{3} N_f c(R) + \frac{5}{6} c_2(G) - \cancel{c_2(R)} + \cancel{c_2(R)} + c_2(G) \right\} \frac{1}{2-\omega} \right)$$

$$= g_r \mu^{2-\omega} \left(1 - \frac{g_r}{16\pi^2} \underbrace{\left\{ \frac{11}{6} c_2(G) - \frac{2}{3} N_f c(R) \right\}}_{\equiv \Delta} \frac{1}{2-\omega} \right)$$

$\epsilon \equiv 2-\omega$

non dip. da μ

$$g_B = g_r(\mu) \mu^\epsilon \left(1 - \frac{g_r(\mu)^2}{16\pi^2} \frac{\Delta}{\epsilon} \right) \quad (0)$$

Vogliamo usare questa relazione per calcolare la dipendenza di g_r da μ . Si sceglie μ dell'ordine delle scale a cui si vuole fare conto (o esperimento) affinché $\ln(P^2/\mu^2)$ siano $\mathcal{O}(1)$; $g_r(\mu)$ corrispondente un'altra se siamo in regime perturbativo o no (cioè $g_r(\mu) \ll 1$ o $\gg 1$?).

Prendiamo $\mu \frac{d}{d\mu} (0)$

$$0 = \mu \frac{d}{d\mu} g_r \cancel{\mu^\epsilon} \left(1 - \frac{g_r^2}{16\pi^2} \frac{\Delta}{\epsilon} \right) + \epsilon g_r \cancel{\mu^\epsilon} \left(1 - \frac{g_r^2}{16\pi^2} \frac{\Delta}{\epsilon} \right) - g_r \cancel{\mu^\epsilon} \frac{g_r}{8\pi^2} \frac{\Delta}{\epsilon} \mu \frac{dg_r}{d\mu} + \text{higher order in } g_r$$

g_r e ϵ
 $\mu \frac{dg_r}{d\mu}$ sono
 finiti in $\epsilon \rightarrow 0$

$$= \epsilon g_r + \mu \frac{dg_r}{d\mu} - \frac{g_r^3}{16\pi^2} \Delta - \frac{3g_r^2}{16\pi^2} \frac{\Delta}{\epsilon} \mu \frac{dg_r}{d\mu} + \dots$$

A ordine zero otteniamo

$$\mu \frac{dg_r}{d\mu} = -\epsilon g_r + \underbrace{\beta(g_r)}_{\text{subleading}} \quad (\epsilon \rightarrow \mu \frac{dg_r}{d\mu} \text{ in } \epsilon \rightarrow 0)$$

$$\Rightarrow 0 = \cancel{\epsilon g_r} - \cancel{\epsilon g_r} + \beta(g_r) - \frac{g_r^3}{16\pi^2} \Delta + \frac{3g_r^3}{16\pi^2} \Delta + \text{higher order in } g_r$$

$$\Rightarrow \beta(g_r) = -\frac{2g_r^3 \Delta}{16\pi^2} = -\frac{g_r^3}{16\pi^2} \left(\frac{11}{3} c_2(G) - \frac{4}{3} N_f c(R) \right) \quad (†)$$

Prendiamo $G = SU(N)$ e $R = N \Rightarrow \begin{cases} c_2(G) = N \\ c(R) = 1/2 \end{cases}$

$$\beta = -\frac{g_r^3}{16\pi^2} \left(\frac{11}{3} N - \frac{2}{3} N_f \right)$$

In QCD $N = 3$ $N_f = 6$ (u, d, s, c, t, b)

$$\beta_{\text{QCD}} = -\frac{7}{16\pi^2} g_r^3 < 0 \Rightarrow \text{ASINTOTICAMENTE LIBERA in UV}$$

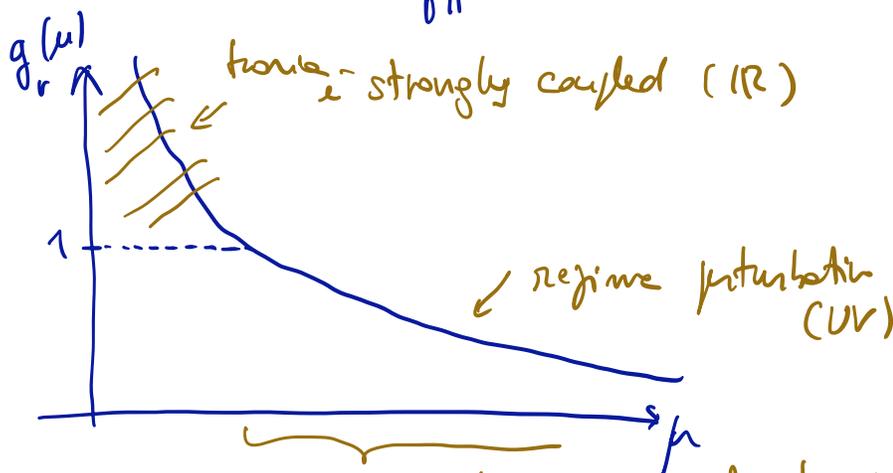
$P_{\text{in}} \text{ YM} \quad N_f = 0$

$$\beta_{\text{YM}} = -\frac{11}{48} C_2(G) g_r^3 < 0 \Rightarrow \text{ASINTOTICAM. LIBERA in UV}$$

(QED ha $\beta_{\text{QED}} > 0 \Rightarrow \text{AS. LIB. in IR}$)

Risoluzione della relazione (†) può essere integrata (*)

$$g_r^2(\mu) = \frac{g_r^2(\mu_0)}{1 + \frac{g_r^2(\mu_0)}{\beta \pi^2} \left(\frac{11}{3} C_2(G) - \frac{4}{3} N_f C(R) \right) \ln \mu / \mu_0}$$



qui possiamo usare la teoria delle perturbazioni

(*) Si può integrare $\beta(g) = \mu \frac{dg}{d\mu}$:

$$\int_{g(\mu_0)}^{g(\mu)} \frac{dg}{\beta(g)} = \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} = \ln(\mu/\mu_0)$$

In regime perturbativo

$$\beta(g) = \beta_0 g^3$$

$$\beta_0 = -\frac{1}{16\pi^2} \left(\frac{11}{3} C_2(G) - \frac{4}{3} N_f C(R) \right) \quad (†)$$

$$\frac{1}{\beta_0} \int_{g(\mu_0)}^{g(\mu)} \frac{dg}{g^3} = \frac{1}{\beta_0} \left[-\frac{1}{2g^2} \right]_{g(\mu_0)}^{g(\mu)} = \frac{1}{2\beta_0 g^2(\mu_0)} - \frac{1}{2\beta_0 g^2(\mu)}$$

$$\Rightarrow \frac{1}{g^2(\mu)} = \frac{1}{g^2(\mu_0)} + \beta_0 \ln\left(\frac{\mu_0^2}{\mu^2}\right)$$

Calcolo dei δZ

$$i\Pi_{\mu\nu}^{cb} = \text{diagram a)} + \text{diagram b)} + \text{diagram c)} + \text{diagram d)}$$

$$\equiv i(q^2 \eta^{\mu\nu} - q^\mu q^\nu) \Pi(q^2) \delta^{cb}$$

a) come in QED con in più il fattore

$$(t_R^a)_{ij} (t_R^b)_{ji} = \text{tr}_R (t_R^a t_R^b) = c(R)$$

quando ci sono N_f fermioni, va moltiplicato per N_f .

$$\Rightarrow \Pi_{1/2-\omega} \supset - \frac{g^2 N_f c(R)}{12 \pi^2 (2-\omega)}$$

b) Ricordiamo $\text{diagram} = -g f^{a_1 a_2 a_3} \{ (k_1 - k_2)_{\mu_3} \eta_{\mu_1 \mu_2} + \text{cyclic} \}$

$$\text{diagram} = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \frac{-i}{p^2} \frac{-i}{(p+q)^2} g^2 \underbrace{f^{abcd}}_{= \text{Tr}(t_{Ad}^a t_{Ad}^b) = c(G) \delta^{ab} = c_2(G) \delta^{ab}}$$

$$\cdot \left\{ (q-p)_\sigma \eta_{\rho\mu} + (p - (-q-p))_\mu \eta_{\rho\sigma} + ((-q-p) - q)_\rho \eta_{\mu\sigma} \right\}$$

$$\cdot \left\{ (-q+p)^\rho \eta_\nu{}^\rho + (-p - (q+p))_\nu \eta^{\sigma\rho} + (q+p - (-q))^\rho \eta_\nu{}^\sigma \right\}$$

• Risolviamo denominatori

$$\frac{1}{p^2} \cdot \frac{1}{(p+q)^2} = \int_0^1 dx \frac{1}{((1-x)p^2 + x(p+q)^2)^2} = \int_0^1 dx \frac{1}{(k^2 - \Delta^2)^2}$$

$$\text{con } k = p + xq \quad \text{e} \quad \Delta = -x(1-x)q^2$$

- Cambio vars. integr. $\int d^d p \rightarrow \int d^d k$
- Riscriviamo numeratore in termini di k, q e x .
- Inoltre eliminiamo termini dispari e sostituiamo

$$k^\mu k^\nu \rightarrow \frac{k^2}{d} \eta^{\mu\nu}$$

$$\left[\int d^d k f(k^2) k^\mu k^\nu = \gamma \eta^{\mu\nu} \quad (\text{by sym.}) \Rightarrow \int d^d k f(k^2) k^2 = d \cdot \gamma \Rightarrow \right. \\ \left. \Rightarrow \gamma = \int d^d k f(k^2) \frac{k^2}{d} . \right]$$

(See Peskin-Schroeder)

Facendo qto, troviamo

$$= \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - \Delta)^2} \cdot \left\{ -\eta_{\mu\nu} k^2 \cdot 6 \left(1 - \frac{1}{d}\right) - \eta_{\mu\nu} q^2 \left[(2-x)^2 + (1-x)^2 \right] \right. \\ \left. + q_\mu q_\nu \left[(2-d)(1-2x)^2 + 2(1+x)(2-x) \right] \right\}$$

• Facciamo rotazione di Wick, $k_0 = ik_4$, e usiamo

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - \Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - d/2}$$

$$\int \frac{d^d k}{(2\pi)^d} \frac{k^2}{(k^2 - \Delta)^n} = \frac{1}{(4\pi)^{d/2}} \cdot \frac{d}{2} \cdot \frac{\Gamma(n - d/2 - 1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - d/2 - 1}$$

Mettendo tutto insieme:

$$\text{diagram} = \frac{i g^2}{(4\pi)^{d/2}} c_2(G) \delta^{ab} \int_0^1 dx \frac{1}{\Delta^{2-d/2}}$$

$$\cdot \left\{ \Gamma(1 - \frac{d}{2}) \eta^{\mu\nu} q^2 \left[\frac{3}{2}(d-1)x(1-x) \right] + \Gamma(2 - \frac{d}{2}) \eta^{\mu\nu} q^2 \left[\frac{1}{2}(2-x)^2 + \frac{1}{2}(1+x)^2 \right] \right. \\ \left. - \Gamma(2 - \frac{d}{2}) q^\mu q^\nu \left[(1 - \frac{d}{2})(1-2x)^2 + (1+x)(2-x) \right] \right\}$$

c)  = $-g^2 c_2(G) \delta^{ab} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2} \eta^{\mu\nu} (d-1) =$

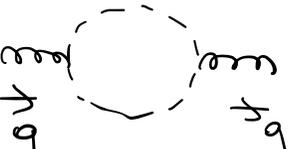
analogous steps as before

$$= \frac{ig^2}{(4\pi)^{d/2}} c_2(G) \delta^{ab} \int_0^1 dx \frac{\eta^{\mu\nu} q^2}{\Delta^{2-d/2}}$$

$$\cdot \left\{ -\Gamma\left(1-\frac{d}{2}\right) \frac{1}{2} d(d-1) x(1-x) - \Gamma\left(2-\frac{d}{2}\right) (d-1) (1-x)^2 \right\}$$

↑
polo a $d=2$
(\leadsto quadratic div. in $d=4$)

↑
polo a $d=4$

d)  = $(-1) \int \frac{d^d p}{(2\pi)^d} \frac{i}{p^2} \frac{i}{(p+q)^2} g^2 f^{dec} f^{cbd} (p+q)^\mu p^\nu c$

$$= \dots = \frac{ig^2}{(4\pi)^{d/2}} c_2(G) \delta^{ab} \int_0^1 dx \frac{dx}{\Delta^{2-d/2}}$$

$$\cdot \left\{ -\Gamma\left(1-\frac{d}{2}\right) \eta^{\mu\nu} q^2 \frac{1}{2} x(1-x) + \Gamma\left(2-\frac{d}{2}\right) q^\mu q^\nu x(1-x) \right\}$$

• Combinando b, c, d , il coeff. di $\Gamma\left(1-\frac{d}{2}\right)$ è
 $\sim (3d-3-d^2+d-1) = -2\left(1-\frac{d}{2}\right)(2-d)$

Il polo $\frac{1}{1-\frac{d}{2}}$ della funzione Γ è CANCELLATO
 $(\Rightarrow$ no quadratic div. in $d=4)$

• Potendo rimpiazzare $x \leftrightarrow (1-x)$ in ogni termine e svolgendo i conti, otteniamo

de b, c, d :

$$\frac{ig^2}{(4\pi)^{d/2}} c_2(G) \delta^{ab} \int_0^1 dx \frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}} (\eta^{\mu\nu} q^2 - q^\mu q^\nu) [(1-\frac{d}{2})(1-2x)^2 + 2].$$

$\leftarrow = 1$
 $d=4$

- Estraiamo ora il contributo del polo $\frac{1}{2-\omega}$ ($d=2\omega$), sapendo che $\Gamma(z) \sim \frac{1}{z}$.

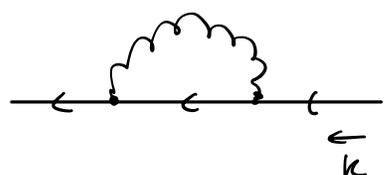
$$\frac{ig^2}{(4\pi)^{d/2}} c_2(G) \delta^{ab} \frac{1}{2-\omega} (\eta^{\mu\nu} q^2 - q^\mu q^\nu) \int_0^1 [1 + 4x - 4x^2] dx$$

$\underbrace{\hspace{10em}}_{5/3}$

- Mettendo assieme anche il contributo da q), otteniamo

$$\begin{aligned} \Pi_{1/\epsilon} &= -\frac{g^2 N_F c(R)}{6\pi^2(2\omega)} - \frac{g^2}{(4\pi)^2} \left(-\frac{5}{3}\right) c_2(G) \frac{1}{2-\omega} = \\ &= -\frac{g^2}{16\pi^2} \left[\frac{4}{3} N_F c(R) - \frac{5}{3} c_2(G) \right] = \delta Z_A \end{aligned}$$

Passiamo a

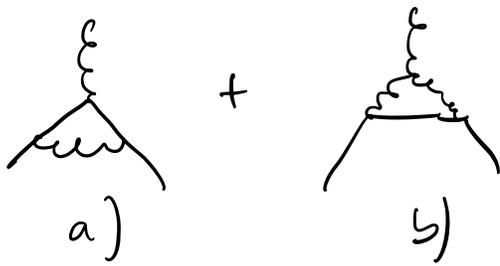


$$= (ig)^2 \int \frac{d^d p}{(2\pi)^d} \gamma^\mu t_R^a \frac{i}{\not{p}-\not{k}} \gamma_\mu t_R^a \frac{-i}{\not{p}^2}$$

- Usiamo $(t_R^a t_R^a)_{ij} = c_2(R) \delta_{ij}$

$$\rightarrow \delta Z_\psi = -\frac{g^2}{16\pi^2} c_2(R) \frac{1}{2-\omega}$$

E ora:



a)

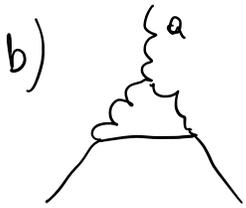
$$= g^3 \int \frac{d^d p}{(2\pi)^d} t_R^b t_R^a t_R^b \gamma^\nu \frac{1}{\not{p} + \not{k}'} \gamma^\mu \frac{1}{\not{p} + \not{k}} \not{p} \cdot \frac{1}{p^2}$$

• come in QED, con l'unica aggiunta del fattore

$$t^b t^a t^b = t^b t^b t^a + t^b [t^a, t^b] =$$

$$= C_2(R) t^a + \underbrace{i t^b f^{abc} t^c}_{if^{abc} \frac{1}{2} [t^b, t^c]} = [C_2(R) - \frac{1}{2} C_2(G)] t^a$$

$$= \frac{1}{2} \underbrace{if^{abc} if^{bad}}_{-C_2(G) \delta^{ab}} t^a$$



• il color factor è

$$f^{abc} t^b t^c = \frac{1}{2} f^{abc} if^{bad} t^a = \frac{i}{2} C_2(G) t^a$$

Mettendo assieme a) e b):

$$\delta Z_4 = -\frac{g^2}{16\pi^2} \frac{1}{2-\omega} (C_2(R) + C_2(A))$$