The structure and evolution of stars

Lecture 3: The equations of stellar structure

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Introduction and recap

For our stars – which are isolated, static, and spherically symmetric – there are four basic equations to describe structure. All physical quantities depend on the distance from the centre of the star alone

1) **Equation of hydrostatic equilibrium:** at each radius, forces due to pressure differences balance gravity
2) **Conservation of mass**
3) **Conservation of energy :** at each radius, the change in the energy flux = local rate of energy release
4) **Equation of energy transport :** relation between the energy flux and the local gradient of temperature

These basic equations supplemented with

- Equation of state (pressure of a gas as a function of its density and temperature)
- Opacity (how opaque the gas is to the radiation field)
- Core nuclear energy generation rate

Content of current lecture and learning outcomes

Before deriving the relations for (3) and (4) we will consider several applications of our current knowledge. You will derive mathematical formulae for the following

1) **Minimum value for central pressure of a star**
2) **The Virial theorem**
3) **Minimum mean temperature of a star**
4) **State of stellar material**

In doing this you will learn important assumptions and approximations that allow the values for minimum central pressure, mean temperature and the physical state of stellar material to be derived

Minimum value for central pressure of star

We have only 2 of the 4 equations, and no knowledge yet of material composition or physical state. But can deduce a minimum central pressure :

Why, in principle, do you think there needs to be a minimum value ? given what we know, what is this likely to depend upon ?

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2} \qquad \frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$

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$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2} \qquad \frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$

Divide these two equations: $$\frac{dP}{dr} \Big/ \frac{dM(r)}{dr} \equiv \frac{dP}{dM} = -\frac{GM}{4\pi r^4}$$

Can integrate this to give $$P_c - P_s = \int_0^{M_s} \frac{GM}{4\pi r^4}$$

Lower limit to RHS: $$\int_0^{M_s} \frac{GM}{4\pi r^4} dM > \int_0^{M_s} \frac{GM}{4\pi {r_s}^4} dM = \frac{GM_s^2}{8\pi {r_s}^4}$$

Hence, we have

$$P_c - P_s > \frac{GM_s^2}{8\pi {r_s}^4}$$

We can approximate the pressure at the surface of the star to be zero:

$$P_c > \frac{GM_s^2}{8\pi {r_s}^4}$$

For example, for the Sun:

$P_{c\odot} = 4.5 \times 10^{13}\ \mathrm{Nm}^{-2} = 4.5 \times 10^{8}$ atmospheres

This seems rather large for gaseous material – we shall see that this is not an ordinary gas.

The Virial theorem

Again, let us take the two equations of hydrostatic equilibrium and mass conservation and divide them

$$\frac{dP}{dr} \Big/ \frac{dM(r)}{dr} = \frac{dP}{dM} = -\frac{GM}{4\pi r^4}$$

Now multiply both sides by $4\pi r^2$

$$4\pi r^3 \; dP = -\frac{GM}{r} dM$$

And integrate over the whole star

$$3\int_{P_c}^{P_s} V \; dP = -\int_0^{M_s} \frac{GM}{r} dM$$

Where V = vol contained within radius r

Use integration by parts to integrate LHS

$$3[\mathrm{PV}]_C^S - 3\int_{V_c}^{V_s} P dV = -\int_0^{M_s} \frac{GM}{r} dM$$

At centre, $V_c=0$ and at surface $P_s=0$

Hence we have

$$3\int_0^{V_s} PdV - \int_0^{M_s} \frac{GM}{r}\, dM = 0$$

Now the second term of the eq. is = total gravitational potential energy of the star or it is the energy released in forming the star from its components dispersed to infinity.

Thus we can write the **Virial Theorem :**

$$3\int_0^{V_s} PdV + \Omega = 0$$

This is of great importance in astrophysics and has many applications. We shall see that it relates the gravitational energy of a star to its thermal energy

Minimum mean temperature of a star

We have seen that pressure, *P,* is an important term in the equation of hydrostatic equilibrium and the Virial theorem. We have derived a minimum value for the central pressure ($P_c > 4.5 \times 10^8$ *atmospheres*)

What physical processes give rise to this pressure – which are the most important ?

- Gas pressure P_g
- Radiation pressure P_r
- We shall show that P_r is negligible in stellar interiors and pressure is dominated by P_g

To do this we first need to estimate the minimum mean temperature of a star

Consider the Ω term, which is the gravitational potential energy:

$$\Omega = -\int_0^{M_s} \frac{GM}{r}\, dM$$

We can obtain a lower bound on the RHS by noting: at all points inside the star $r<r_s$ and hence $1/r > 1/r_s$

$$\int_0^{M_s} \frac{GM}{r}\, dM > \int_0^{M_s} \frac{GM}{r_s}\, dM = \frac{GM_s}{2r_s}$$

Now $dM=\rho dV$ and the Virial theorem can be written

$$-\Omega = 3\int_0^{V_s} P\,dV = 3\int_0^{M_s} \frac{P}{\rho}\, dM$$

Now pressure is sum of radiation pressure and gas pressure: $P = P_g + P_r$

Assume, for now, that stars are composed of ideal gas with negligible P_r

The eqn of state of ideal gas

$$P = nkT = \frac{k\rho T}{m}$$

where n=number of particles per m^3

m= average mass of particles

k=Boltzmann's constant

Hence, we have

$$-\Omega = 3\int_0^{M_s} \frac{P}{\rho} dM = 3\int_0^{M_s} \frac{KT}{m} dM$$

And we may use the inequality derived above to write

$$-\Omega = 3\int_0^{M_s} \frac{KT}{m} dM > \frac{GM_s^2}{2r_s}$$

$$\Longrightarrow \int_0^{M_s} T dM > \frac{GM_s^2 m}{6kr_s}$$

We can think of the LHS as the sum of the temperatures of all the mass elements *dM* which make up the star

The mean temperature of the star $\bar{T}$ is then just the integral divided by the total mass of the star M_s

$$\Longrightarrow M_s\bar{T} = \int_0^{M_s} T dM \Longrightarrow \bar{T} > \frac{GM_s m}{6kr_s}$$

Minimum mean temperature of the sun

As an example for the Sun we have

$$\bar{T} > 4 \times 10^6 \frac{m}{m_H} \quad \text{where } m_H = 1.67 \times 10^{-27}\,\text{Kg}$$

Now we know that H is the most abundant element in stars and for a fully ionised hydrogen star $m/m_H = 1/2$ (as there are two particles, p + e^-, for each H atom). And for any other element m/m_H is greater

$$\Rightarrow \quad \bar{T}_\odot > 2 \times 10^6 \text{ K}$$

Physical state of stellar material

We can also estimate the mean density of the Sun using:

$$\rho_{av} = \frac{3M_{sun}}{4\pi {r_{sun}}^3} = 1.4 \times 10^3 \text{ Kg m}^{-3}$$

Mean density of the sun is only a little higher than water and other ordinary liquids. We know such liquids become gaseous at T much lower than $\bar{T}_{\odot}$

Also, the average K.E. of particles at $\bar{T}_{\odot}$ is much higher than the ionisation potential of H. Thus, the gas must be highly ionised, i.e. is a plasma.

It can thus withstand greater compression without deviating from an ideal gas.

Note that an ideal gas demands that the distances between the particles are much greater than their sizes, and nuclear dimension is 10^{-15} m compared to atomic dimension of 10^{-10} m

Lets revisit the issue of radiation vs gas pressure. We assumed that the radiation pressure was negligible. The pressure exerted by photons on the particles in a gas is:

$$P_{rad} = \frac{aT^4}{3}$$

Where *a* = radiation density constant

Now compare gas and radiation pressure at a typical point in the Sun

$$\frac{P_{rad}}{P_{gas}} = \frac{\frac{aT^4}{3}}{\frac{kT\rho}{m}} = \frac{maT^3}{3k\rho}$$

Taking $T \sim T_{av} = 2 \times 10^6 K, \rho = \rho_{av} = 1.4 \times 10^3 kgm^{-3}$

$$and\ m = 1.67 \times \frac{10^{-27}}{2} kg$$

Gives $\frac{P_{rad}}{P_{gas}} \sim 10^{-4}$

Hence radiation pressure appears to be negligible at a typical (average) point in the Sun. In summary, with no knowledge of how energy is generated in stars we have been able to derive a value for the Sun's internal temperature and deduce that it is composed of a near ideal gas plasma with negligible radiation pressure

Mass dependency of radiation to gas pressure

However, we shall later see that P_r does become significant in higher mass stars. To give a basic idea of this dependency: replace ρ in the ratio equation above:

$$\frac{P_{rad}}{P_{gas}} = \frac{maT^3}{3k\rho} = \frac{maT^3}{3k\left(\frac{3M_s}{4\pi r_s^3}\right)} = \frac{4\pi\, m\, a\, r_s^3\, T^3}{9kM_s}$$

From the Virial theorem: $\bar{T} \sim \frac{GM_s}{6kr_s}$

$$\frac{P_{rad}}{P_{gas}} \propto M_s^2$$

i.e. P_r becomes more significant in higher mass stars.

Summary and next lecture

With only two of the four equations of stellar structure, we have derived important relations for P_c and mean T

We have derived and used the Virial theorem – this is an important formula and concept in this course, and astrophysics in general. You should be comfortable with the derivation and application of this theorem.

In the next lecture we will explore the energy generation and energy transport in stars to provide the four equations that can be simultaneously solved to provide structural models of stars.