

The structure and evolution of stars

Lecture 4: The equations of stellar structure



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Introduction and recap

For our stars – which are isolated, static, and spherically symmetric – there are four basic equations to describe structure. All physical quantities depend on the distance from the centre of the star alone

- 1) **Equation of hydrostatic equilibrium:** at each radius, forces due to pressure differences balance gravity
- 2) **Conservation of mass**
- 3) **Conservation of energy :** at each radius, the change in the energy flux = local rate of energy release
- 4) **Equation of energy transport :** relation between the energy flux and the local gradient of temperature

These basic equations supplemented with

- Equation of state (pressure of a gas as a function of its density and temperature)
- Opacity (how opaque the gas is to the radiation field)
- Core nuclear energy generation rate

Learning Outcomes

The theme of this lecture is to discuss the energy generation in stars and how that energy is transported from the centre. The student will

- 1) Learn how to determine the likely form of energy generation**
- 2) Derive the equation of conservation of energy. Which is formula number (3) of the stellar structure equations**
- 3) Before deriving the final formula, student will learn how to determine how energy is transported in the sun. This will include deriving the criterion for convection to occur.**

Energy generation in stars

So far we have only considered the dynamical properties of the star, and the state of the stellar material. We need to consider the source of the stellar energy.

Let's consider the origin of the energy i.e. the conversion of energy from some form in which it is not immediately available into some form that it can radiate.

How much energy does the sun need to generate in order to shine with it's measured flux ?

$$L_{\text{sun}} = 4 \times 10^{26} \text{ W} = 4 \times 10^{26} \text{ J s}^{-1}$$

Sun has not changed flux in 10^9 yr ($=3 \times 10^{16}$ s)

Sun has radiated 1.2×10^{43} J

$$E = mc^2 \rightarrow \text{mass}_{\text{lost}} = 10^{25} \text{ kg} = 10^{-4} M_{\text{sun}}$$

Source of energy generation:

What is the source of this energy ? Four possibilities :

- Cooling or contraction
- Chemical Reactions
- Nuclear Reactions

Cooling and contraction

These are closely related, so we consider them together. Cooling is simplest idea of all. Suppose the radiative energy of Sun is due to the Sun being much hotter when it was formed, and has since been cooling down .We can test how plausible this is.

Or is sun slowly contracting with consequent release of gravitational potential energy, which is converted to radiation ?

Source of energy generation:

In an ideal gas, the thermal energy of a particle (where $n_f = \text{number of degrees}$

$$= \frac{kT}{2} n_f$$
$$= \frac{3kT}{2}$$

Total thermal energy per unit volume $= \frac{3knT}{2}$
 $N = \text{number of particles per unit volume}$

Now, Virial theorem:

$$3 \int_0^{V_s} P dV + \Omega = 0$$

Assume that stellar material is ideal gas (negligible P_r)

$$\Rightarrow P = nkT$$

$$3 \int_0^{V_s} nkT dV + \Omega = 0$$

Source of energy generation:

Now lets define $U = \text{integral over volume of the thermal energy per unit volume}$

$$\text{thermal energy per unit volume} = \frac{3knT}{2}$$

$$\Rightarrow 2U + \Omega = 0$$

The negative gravitational energy of a star is equal to twice its thermal energy. This means that the time for which the present thermal energy of the Sun can supply its radiation and the time for which the past release of gravitational potential energy could have supplied its present rate of radiation differ by only a factor two. We can estimate the later:

Negative gravitational potential energy of a star is related by the inequality

$$-\Omega > \frac{GM_s^2}{2r_s} \quad \text{as an approximation assume} \quad -\Omega \sim \frac{GM_s^2}{2r_s}$$

Source of energy generation:

Total release of gravitational potential energy would have been sufficient to provide radiant energy at a rate given by the luminosity of the star L_s , for a time

$$t_{th} \sim \frac{GM_s^2}{L_s r_s}$$

Putting in values for the Sun: $t_{\odot thermal} = 3 \times 10^7$ years.

Hence if sun were powered by either contraction or cooling, it would have changed substantially in the last 10 million years. A factor of ~ 100 too short to account for the constraints on age of the Sun imposed by fossil and geological records.

Definition: t_{th} is defined as the *thermal timescale* (or *Kelvin-Helmholtz timescale*)

Chemical Reactions

Can quickly rule these out as possible energy sources for the Sun. We calculated above that we need to find a process that can produce at least 10^{-4} of the rest mass energy of the Sun. Chemical reactions such as the combustion of fossil fuels release $\sim 5 \times 10^{-10}$ of the rest mass energy of the fuel.

Source of energy generation:

Nuclear Reactions

Hence the only known way of producing sufficiently large amounts of energy is through nuclear reactions. There are two types of nuclear reactions, fission and fusion. Fission reactions, such as those that occur in nuclear reactors, or atomic weapons can release $\sim 5 \times 10^{-4}$ of rest mass energy through fission of heavy nuclei (uranium or plutonium).

Class task

Can you show that the fusion reactions can release enough energy to feasibly power a star ?

Assume atomic weight of $H=1.008172$ and $He^4=4.003875$

Hence, we can see that both fusion and fission could in principle power the Sun.

Which is the more likely ?

As light elements are much more abundant in the solar system than heavy ones, we would expect nuclear fusion to be the dominant source.

Given the limits on $P(r)$ and $T(r)$ that we have just obtained - are the central conditions suitable for fusion ? We will return to this later.

Equation of energy production

The third equation of stellar structure:
relation between energy release and the
rate of energy transport

Consider a spherically symmetric star in
which energy transport is radial and in
which time variations are unimportant.

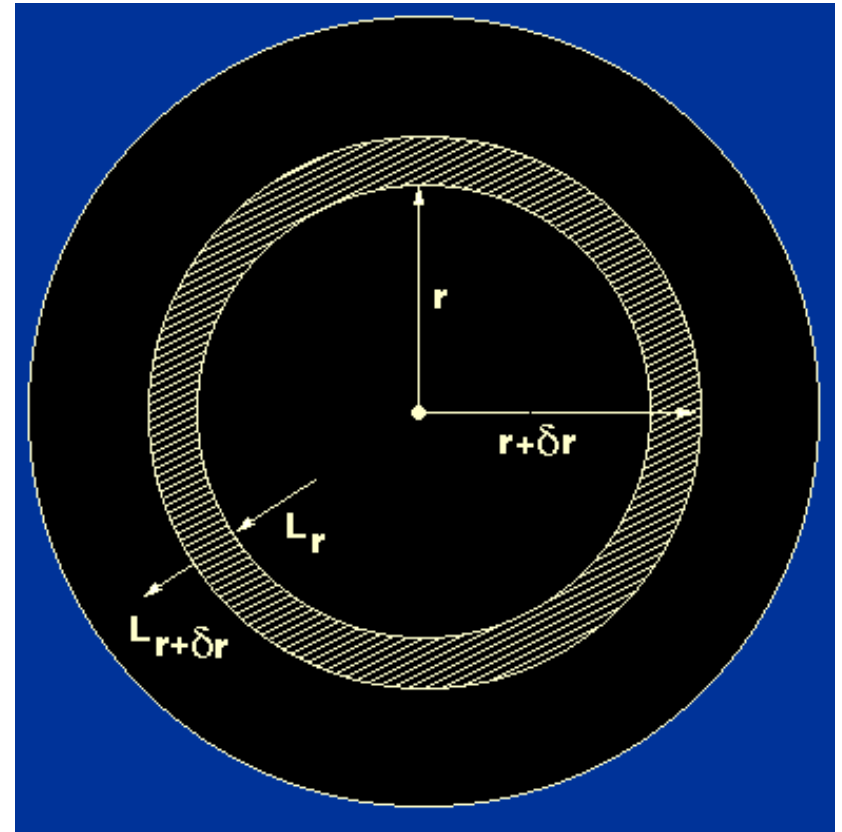
$L(r)$ = rate of energy flow across sphere of
radius r

$L(r+\delta r)$ = rate of energy flow across
sphere of radius $r + \delta r$

Because shell is thin:

$$\delta V(r) = 4\pi r^2 \delta r$$

$$\text{and } \delta m(r) = 4\pi r^2 \rho(r) \delta r$$



We define $\varepsilon = \text{energy release per unit mass per unit volume}$ (Wkg^{-1})

Hence energy release in shell is written:

$$4\pi r^2 \rho(r) \delta r \varepsilon$$

Conservation of energy leads us to

$$L(r + \delta r) = L(r) + 4\pi r^2 \rho(r) \delta r \varepsilon$$

\Rightarrow

$$\frac{L(r + \delta r) - L(r)}{\delta r} = 4\pi r^2 \rho(r) \varepsilon$$

and for $\delta r \rightarrow 0$

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho(r) \varepsilon$$

This is the *equation of energy production*.

We now have three of the equations of stellar structure. However, we have five unknowns $P(r)$, $M(r)$, $L(r)$, $\varepsilon(r)$, $\rho(r)$. In order, to make further progress we need to consider energy transport in stars.

Method of energy transport

There are three ways energy can be transported in stars

- Convection – energy transport by mass motions of the gas
- Conduction – by exchange of energy during collisions of gas particles (usually e^-)
- Radiation – energy transport by the emission and absorption of photons

Conduction and radiation are similar processes – they both involve transfer of energy by direct interaction, either between particles or between photons and particles.

Which is the more dominant in stars ?

Energy carried by a typical particle $\sim 3kT/2$ is comparable to energy carried by typical photon $\sim hc/\lambda$

But number density of particles is much greater than that of photons. This would imply conduction is more important than radiation.

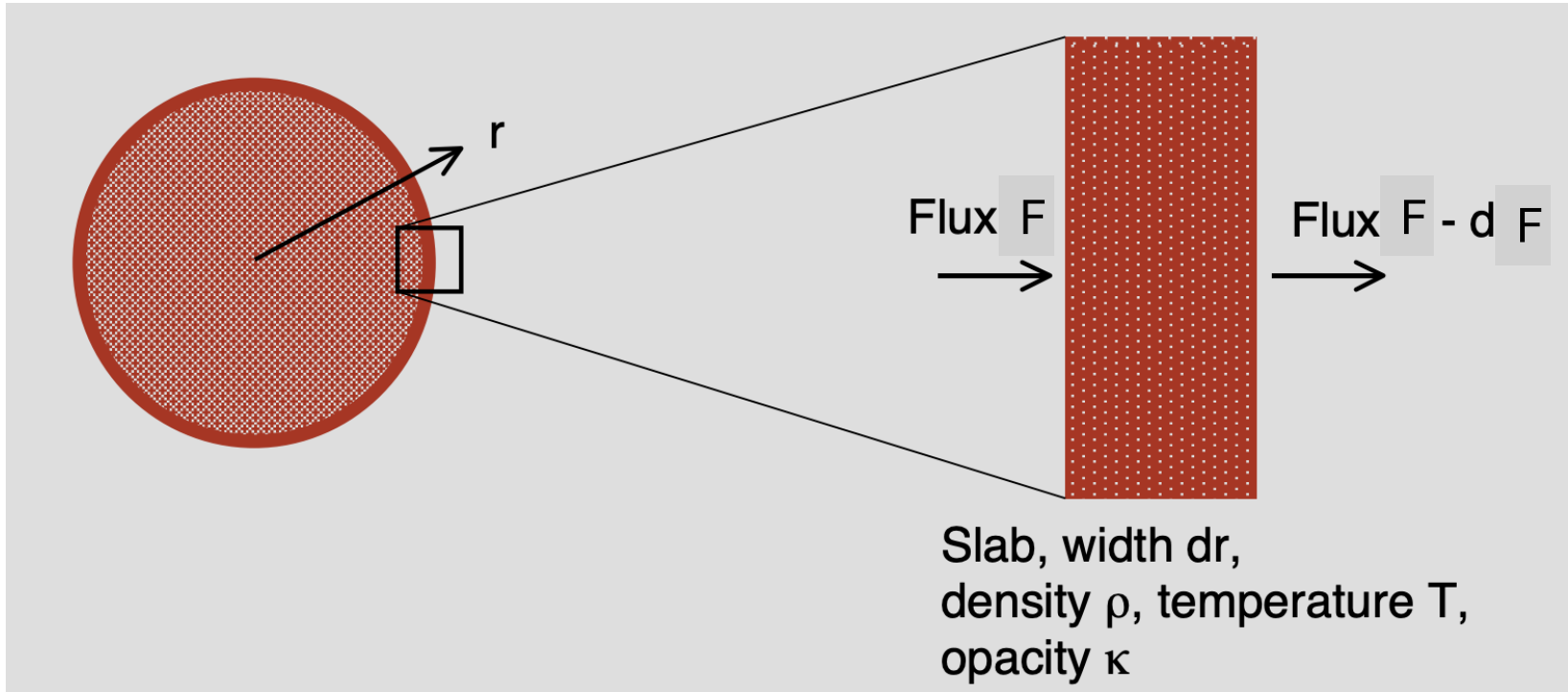
Mean free path of photon $\sim 10^{-2}\text{m}$

Mean free path of particle $\sim 10^{-10}\text{ m}$

Photons can move across temperature gradients more easily, hence larger transport of energy.

Conduction is negligible, radiation transport is dominant

Energy transport in stars



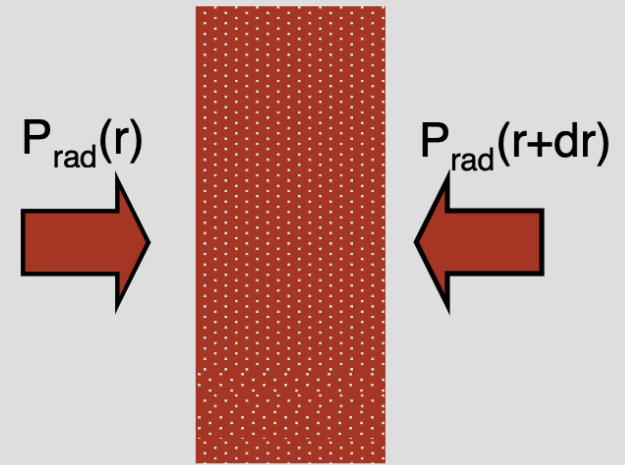
Consider flux of radiation (energy per unit time per unit area) passing through a slab. Change in flux:

$$dF/F = -\rho k dr$$

decrease due to absorption of photons in the slab

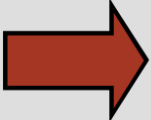
Photons absorbed in the slab impart momentum to the gas: the momentum per unit time per unit area is:

$$\frac{|dF|}{c}$$



Rate of increase of the momentum must be equal to the net force applied to the slab by the radiation field (Newton's second law). This is equal to the *difference* in the radiation pressure between the two faces:

$$P_{\text{rad}}(r) - P_{\text{rad}}(r + dr) = -\frac{dP_{\text{rad}}}{dr} \times dr$$



$$\frac{|dF|}{c} = -\frac{dP_{\text{rad}}}{dr} dr$$

Radiation transport 2

$$P_{rad} = \frac{a}{3} T^4; \quad dPrad = \frac{4a}{3} T^3 dT$$

$$dF/F = -\rho k dr; \quad dF = -\rho k dr F$$

$$F = -\frac{4ac}{3k\rho} T^3 \frac{dT}{dr}$$

$$F = -\frac{4a\lambda c}{3} T^3 \frac{dT}{dr}; \quad \lambda = \frac{1}{\kappa\rho}$$

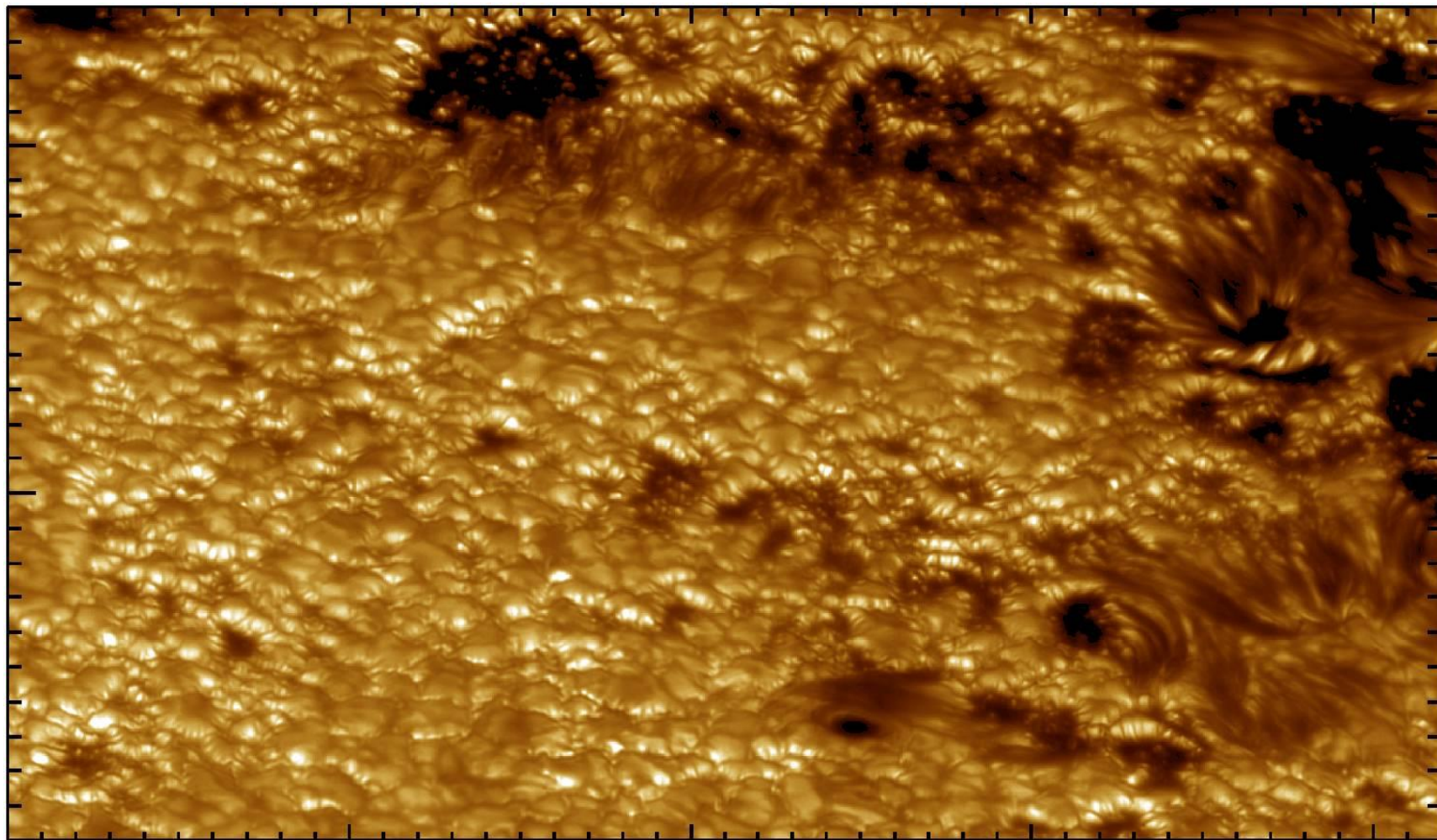
$$\frac{dT}{dr} = -\frac{3k\rho}{4acT^3} \frac{L}{4\pi r^2} = -\frac{3k\rho}{16acT^3} \frac{L}{\pi r^2}$$

Radiation transport 3

$$\frac{dT}{dr} = - \frac{3k\rho}{16acT^3} \frac{L}{\pi r^2}$$

k is the opacity, the probability that a photon is absorbed in a distance dx is $k\rho dx$; k [$\text{cm}^2 \text{gr}^{-1}$]

Solar surface from Swedish Solar Telescope



Resolution $\sim 100\text{km}$

Granule size $\sim 1000\text{km}$

Convection

Convection is the mass motion of gas elements – only occurs when temperature gradient exceeds some critical value. We can derive an expression for this.

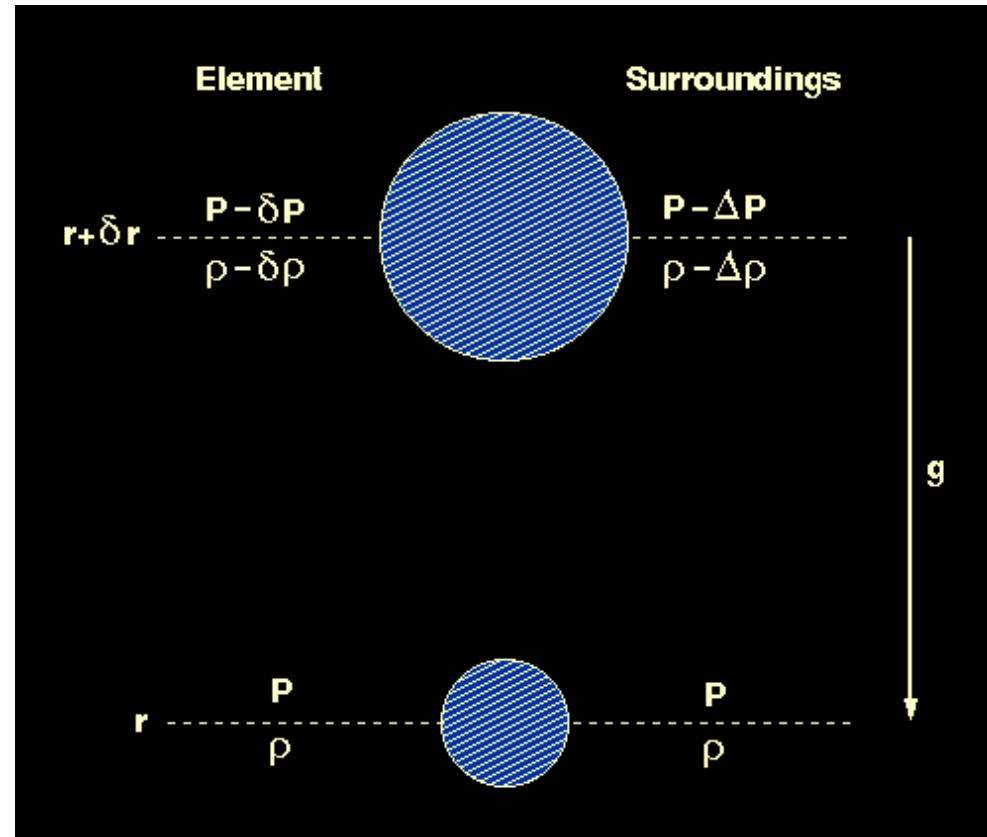
Consider a convective element at distance r from centre of star. Element is in equilibrium with surroundings

Now let's suppose it rises to $r + \delta r$. It expands, $P(r)$ and $\rho(r)$ are reduced to $P - \delta P$ and $\rho - \delta \rho$

But these may not be the same as the same as the new surrounding gas conditions. Define those as $P - \Delta P$ and $\rho - \Delta \rho$

If gas element is denser than surroundings at $r + \delta r$ then will sink (i.e. stable)

If it is less dense then it will keep on rising – *convectively unstable*



The condition for instability is therefore

$$\rho - \delta\rho < \rho - \Delta\rho$$

Whether or not this condition is satisfied depends on two things:

- The rate at which the element expands due to decreasing pressure
- The rate at which the density of the surroundings decreases with height

Let's make two assumptions

1. The element rises *adiabatically*
2. The element rises at a speed much less than the sound speed. During motion, sound waves have time to smooth out the pressure differences between the element and the surroundings. Hence $\Delta P = \delta P$ at all times

The first assumption means that the element must obey the adiabatic relation between pressure and volume

$$PV^\gamma = \text{constant}$$

Where $\gamma = c_p/c_v$ is the specific heat (i.e. the energy in J to raise temperature of 1kg of material by 1K) at constant pressure, divided by specific heat at constant volume

Given that V is inversely proportional to ρ , we can write

$$\frac{P}{\rho^\gamma} = \text{constant}$$

Hence equating the term at r and $r+\delta r$:

$$\frac{P - \delta P}{(\rho - \delta\rho)^\gamma} = \frac{P}{\rho^\gamma}$$

If $\delta\rho$ is small we can expand $(\rho - \delta\rho)^\gamma$ using the binomial theorem as follows

$$(\rho - \delta\rho)^\gamma \sim \rho^\gamma - \gamma\rho^{\gamma-1} \delta\rho \quad \text{and combining last two expressions}$$

$$\delta\rho = \frac{\rho}{\gamma P} \delta P$$

Now we need to evaluate the change in density of the surroundings, $\Delta\rho$

Lets consider an infinitesimal rise of δr

$$\Delta\rho = \frac{d\rho}{dr} \delta r$$

And substituting these expressions for $\delta\rho$ and $\Delta\rho$ into the condition for convective instability derived above:

$$\frac{\rho}{\gamma P} \delta P < \frac{d\rho}{dr} \delta r$$

And this can be rewritten by recalling our 2nd assumption that element will remain at the same pressure as its surroundings, so that in the limit

$$\delta r \rightarrow 0, \quad \frac{\delta P}{\delta r} = \frac{dP}{dr}$$

$$\frac{\rho}{\gamma P} \frac{dP}{dr} < \frac{d\rho}{dr}$$

The LHS is the density gradient that would exist in the surroundings if they had an adiabatic relation between density and pressure. RHS is the actual density in the surroundings. We can convert this to a more useful expression, by first dividing both sides by dP/dr . Note that dP/dr is negative, hence the inequality sign must change.

$$\frac{\rho}{\gamma P} < \frac{d\rho}{dr} / \frac{dP}{dr}$$

$$\Rightarrow \frac{\rho}{\gamma P} > \frac{d\rho}{dP}$$

$$\left(\frac{P}{\rho}\right) \frac{d\rho}{dP} < \frac{1}{\gamma}$$

And for an ideal gas in which radiation pressure is negligible (where m is the mean mass of particles in the stellar material)

$$P = \frac{\rho k T}{m}$$

$$\ln P = \ln \rho + \ln T + \text{constant}$$

And can differentiate to give

$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T}$$

And combining this with the equation above gives

Condition for occurrence of convection

$$\frac{P}{T} \frac{dT}{dP} > \frac{\gamma - 1}{\gamma}$$

Which is the condition for the occurrence of convection (in terms of the temperature gradient). A gas is convectively unstable if the actual temperature gradient is steeper than the adiabatic gradient.

If the condition is satisfied, then large scale rising and falling motions transport energy upwards.

The criterion can be satisfied in two ways. The ratio of specific heats γ is close to unity or the temperature gradient is very steep.

For example, if a large amount of energy is released at the centre of a star, it may require a large temperature gradient to carry the energy away. Hence where nuclear energy is being released, convection may occur.

Condition for occurrence of convection

Alternatively in the cool outer layers of a star, gas may only be partially ionised, hence much of the heat used to raise the temperature of the gas goes into ionisation and hence the specific heat of the gas at constant V is nearly the same as the specific heat at constant P , and $\gamma \sim 1$.

In such a case, a star can have a cool outer convective layer. We will come back to the issues of convective cores and convective outer envelopes later.

Convection is an extremely complicated subject, and it is true to say that the lack of a good theory of convection is one of the worst defects in our present studies of stellar structure and evolution. We know the conditions under which convection is likely to occur but don't know how much energy is carried by convection. Fortunately, we will see that we can often find occasions where we can manage without this knowledge.

Useful further reading: Taylor Ch. 3, Pages 64-68, 73-79

Summary

Hence, we have shown that the source of energy in the sun must be nuclear. Presumably, you all knew that anyway!

We have derived the third formula of the equations of stellar structure (the equation of energy production).

We have derived also the 4th equation – the equation of radiative transport and also considered the mode of energy transport in stellar interiors and derived the condition for convection.

We saw that convection may be important in hot stellar cores and cool outer envelopes, but that a good quantitative theory is lacking.