

The structure and evolution of stars

Lecture 5: The equations of stellar structure



Introduction and recap

For our stars – which are isolated, static, and spherically symmetric – there are four basic equations to describe structure. All physical quantities depend on the distance from the centre of the star alone

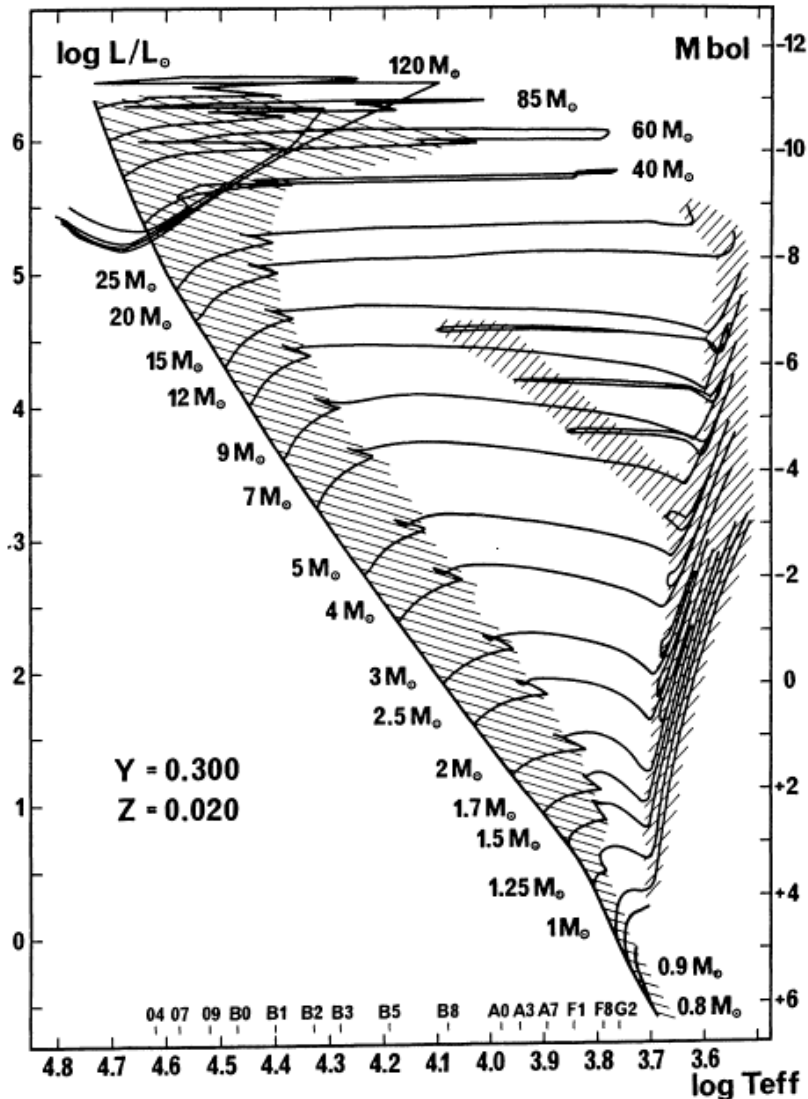
- 1) **Equation of hydrostatic equilibrium:** at each radius, forces due to pressure differences balance gravity
- 2) **Conservation of mass**
- 3) **Conservation of energy :** at each radius, the change in the energy flux = local rate of energy release
- 4) **Equation of energy transport :** relation between the energy flux and the local gradient of temperature

Learning Outcomes

The student will learn:

- Explore ways to solve these equations.
- How to go about constructing models of stellar evolution – how the models can be made to be time variable. You will gain an understanding of what time dependent processes are the most important
- How to come up with the boundary conditions required for the solution of the equations.
- How to consider the effects and influence of convection in stars, when and where it is important, and how it can be included into the structure equations.

Theoretical stellar evolution



In Lecture 9 we will discuss the results of modern stellar evolutionary computations.

The outcome will be this type of theoretical HR-diagram.

At present we are deriving the fundamental physics underlying the calculations - the end point is a diagram like this.

The characteristic timescales

There are 3 characteristic timescales that aid concepts in stellar evolution

The dynamical timescale

Derived in Lecture 2:

For the Sun $t_d \sim 2000s$

$$t_d = \left(\frac{2r^3}{GM} \right)^{\frac{1}{2}}$$

The thermal timescale

Derived in Lecture 4: time for a star to emit its entire reserve of thermal energy upon contraction provided it maintains constant luminosity (Kelvin-Helmholtz timescale)

For the Sun $t_{th} \sim 30$ Myrs

$$t_{th} \sim \frac{GM^2}{Lr}$$

The nuclear timescale

Time for star to consume all its available nuclear energy (ε = typical nucleon binding energy/nucleon rest mass energy)

For Sun t_{nuc} is larger than age of Universe

$$t_{nuc} \sim \frac{\varepsilon Mc^2}{L}$$

$$\Rightarrow t_d \ll t_{th} \ll t_{nuc}$$

Solving the equations of stellar structure

Hence, we now have four differential equations, which govern the structure of stars (note – in the absence of convection).

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho(r) \varepsilon(r)$$

$$\frac{dT(r)}{dr} = -\frac{3\rho(r)\kappa_R(r)}{64\pi r^2 \sigma T(r)^3} L(r)$$

Where

r = radius

P = pressure at r

M = mass of material within r

ρ = density at r

L = luminosity at r (rate of energy flow across sphere of radius r)

T = temperature at r

κ_R = Rosseland mean opacity at r

ε = energy release per unit mass per unit time

We will consider the quantities:

$P = P(\rho, T, \text{chemical composition})$

$\kappa_R = \kappa_R(\rho, T, \text{chemical composition})$

$\varepsilon = \varepsilon(\rho, T, \text{chemical composition})$

The equation of state

Boundary conditions

Two of the boundary conditions are fairly obvious, at the centre of the star

$$M=0, L=0 \text{ at } r=0$$

At the surface of the star its not so clear, but we use approximations to allow solution. There is no sharp edge to the star, but for the the Sun

ρ (surface) $\sim 10^{-4} \text{ kg m}^{-3}$. Much smaller than mean density $\rho(\text{mean}) \sim 1.4 \cdot 10^3 \text{ kg m}^{-3}$ (which we derived). We know the surface temperature ($T_{\text{eff}}=5780\text{K}$) is much smaller than its minimum mean temperature ($2 \cdot 10^6 \text{ K}$).

Thus, we make two approximations for the surface boundary conditions:

$$\rho = T = 0 \text{ at } r=r_s$$

i.e. that the star does have a sharp boundary with the surrounding vacuum

Use of mass as the independent variable

The above formulae would (in principle) allow theoretical models of stars with a given radius. However, from a theoretical point of view it is the mass of the star which is chosen, the stellar structure equations solved, then the radius (and other parameters) are determined. We observe stellar radii to change by orders of magnitude during stellar evolution, whereas mass appears to remain constant. Hence it is much more useful to rewrite the equations in terms of M rather than r .

If we divide the other three equations by the equation of mass conservation, and invert the latter:

$$\frac{dr}{dM} = \frac{1}{4\pi r^2 \rho}$$

$$\frac{dL}{dM} = \varepsilon$$

$$\frac{dP}{dM} = -\frac{GM}{4\pi r^4}$$

$$\frac{dT}{dM} = -\frac{3\kappa_R L}{64\pi^2 r^4 acT^3}$$

With boundary conditions:

$$r=0, L=0 \text{ at } M=0$$

$$\rho=0, T=0 \text{ at } M=M_s$$

We specify M_s and the chemical composition and now have a well defined set of relations to solve. It is possible to do this analytically if simplifying assumptions are made, but in general these need to be solved numerically on a computer.

Stellar evolution

We have a set of equations that will allow the complete structure of a star to be determined, given a specified mass and chemical composition. However what do these equations not provide us with?

In deriving the equation for hydrostatic support, we have seen that provided the evolution of star is occurring slowly compared to the dynamical time, we can ignore temporal changes (e.g. pulsations)

$$t_d = \left(\frac{2r^3}{GM} \right)^{\frac{1}{2}}$$

And for the Sun for example, this is $t_d \sim 2000s$, hence this is certainly true

And we have also made the assumption that time dependence can be omitted from the equation of energy generation, if the nuclear timescale (the time for which nuclear reactions can supply the stars energy) is greatly in excess of t_{th}

Stellar evolution

If there are no bulk motions in the interior of the star, then any changes of chemical composition are localised in the element of material in which the nuclear reactions occurred. So star would have a chemical composition which is a function of mass M .

In the case of no bulk motions – the set of equations we derived must be supplemented by equations describing the rate of change of *abundances* of the different chemical elements. Let $C_{X,Y,Z}$ be the chemical composition of stellar material in terms of mass fractions of hydrogen (X), helium, (Y) and metals (Z) [e.g. for solar system $X=0.7, Y=0.28, Z=0.02$]

$$\frac{\partial(C_{X,Y,Z})_M}{\partial t} = f(\rho, T, C_{X,Y,Z})$$

Now lets consider how we could evolve a model

$$(C_{X,Y,Z})_{M, t_0 + \delta t} = (C_{X,Y,Z})_{M, t_0} + \frac{\partial(C_{X,Y,Z})_M}{\partial t}$$

Influence of convection

Ideally we would like to know exactly how much energy is transported by convection – but lack of a good theory makes it difficult to predict exactly. We can obtain an approximate estimate.

Heat is convected by rising elements which are hotter than their surroundings and falling elements which are cooler. Suppose the element differs by δT from its surroundings, because an element is always in pressure balance with its surroundings, it has energy content per kg which differs from surrounding kg of medium of $c_p T$ (where c_p is specific heat at constant pressure).

If material is mono-atomic ideal gas then $c_p = 5k/2m$

Where m = average mass of particles in the gas

Assuming a fraction $\alpha (<1)$ of the material is in the rising and falling columns and that they are both moving at speed $v \text{ ms}^{-1}$ then the rate at which excess energy is carried across radius is:

$$\begin{aligned} L_{conv} &= \text{surface area of sphere} \times \text{rate of transport} \times \text{excess energy} \\ &= 4\pi r^2 \alpha \rho v \frac{5k\delta T}{2m} = \frac{10\pi r^2 \alpha \rho v k \delta T}{m} \end{aligned}$$

Hence putting in known solar values, at a radius halfway between surface and centre, what is the outcome?

The surface luminosity of the sun is $L_{\odot} = 3.86 \times 10^{26} \text{W}$, and at no point in the Sun can the luminosity exceed this value (see eqn of energy production).

What can you conclude from this ?

As the δT and v of the rising elements are determined by the difference between the actual temperature gradient and adiabatic gradient, this suggests that the actual gradient is not greatly in excess of the adiabatic gradient. To a reasonable degree of accuracy we can assume that the temperature gradient has exactly the adiabatic value in a convective region in the interior of a star and hence can rewrite the condition of occurrence of convection in the form

$$\frac{P}{T} \frac{dT}{dP} = \frac{\gamma - 1}{\gamma}$$

Thus IN A CONVECTIVE REGION we must solve the four differential equations, together with equations for ε and P

$$\frac{dr}{dM} = \frac{1}{4\pi r^2 \rho} \quad \frac{dL}{dM} = \varepsilon$$

$$\frac{dP}{dM} = -\frac{GM}{4\pi r^4} \quad \frac{P}{T} \frac{dT}{dP} = \frac{\gamma - 1}{\gamma}$$

The eqn for luminosity due to radiative transport is still true:

$$L_{rad} = \frac{64\pi^2 r^4 acT^3}{3\kappa_R} \frac{dT}{dM}$$

And once the other equations have been solved, L_{rad} can be calculated. This can be compared with L (from $dL/dM = \varepsilon$) and the difference gives the value of luminosity due to convective transport $L_{conv} = L - L_{rad}$

In solving the equations of stellar structure the eqns appropriate to a convective region must be switched on whenever the temperature gradient reaches the adiabatic value, and switched off when all energy can be transported by radiation.

Note – it can break down near the surface of a star (see illustrative example)

Conclusions and summary

We have derived the 4th equation to describe stellar structure, and explored the ways to solve these equations.

As they are not time dependent, we must iterate with the calculation of changing chemical composition to determine short steps in the lifetime of stars. The crucial changing parameter is the H/He content of the stellar core (and afterwards, He burning will become important – to be explored in next lectures).

We have discussed the boundary conditions applicable to the solution of the equations and made approximations, that do work with real models.

We have explored the influence of convection on energy transport within stars and have shown that it must be considered, but only in areas where the temperature gradient approaches the adiabatic value. In other areas, the energy can be transported by radiation alone and convection is not required.

The next lectures will explore stellar interiors and the nuclear reactions.