Introduction

The equations of stellar structure are coupled differential equations which, along with supplementary equations or data (equation of state, opacity and nuclear energy generation) need to be solved numerically. Useful insight can be gained, however, using analytical methods involving some simple assumptions.

- One approach is to assume that stellar models are homologous; that is, all physical variables in stellar interiors scale the same way with the independent variable measuring distance from the stellar centre (the interior mass at some point specified by $M(r)$). The scaling factor used is the stellar mass (M).

- A second approach is to suppose that at some distance r from the stellar centre, the pressure and density are related by $P(r) = K\,\rho^{\gamma}$ where K is a constant and γ is related to some polytropic index n through $\gamma = (n+1)/n$. Substituting in the equations of hydrostatic equilibrium and mass conservation then leads to the Lane-Emden equation which can be solved for a specific polytropic index n.

Equation of State

Stellar gas is an ionised plasma, where the density is so high that the average particle spacing is of the order of an atomic radius (10^{-15} m):

- An **equation of state** of the form

  $$P = P(\rho, T, \mathrm{composition})$$

  defines pressure as needed to solve the equations of stellar structure.

- The effective particle size is more like a nuclear radius (10^5 times smaller) and hence stellar envelope material behaves like an ideal gas and so

  $$P_{\mathrm{gas}} = \frac{k}{m_{\mathrm{H}}\bar{\mu}}\rho T$$

  where as before $\bar{\mu}$ is the mean molecular weight.

- If radiation pressure is significant, the total pressure is

  $$P = \frac{k}{m_{\mathrm{H}}\bar{\mu}}\rho T + \frac{a T^4}{3}.$$

Mean Molecular Weight – I

Mean molecular weight depends on the ionisation fractions of all elements, and their association into molecules, in all parts of a star.

- H and He are the most abundant elements and are fully ionised in stellar interiors.
- All material in the star is therefore assumed to be fully ionised.
- Assumption breaks down near the stellar surface and in cool stars (M dwarfs and brown dwarfs in particular) where association into molecues becomes important.

The following definitions are made:

- X = hydrogen mass fraction,
- Y = helium mass fraction and
- Z = metal (all elements heavier than helium) mass fraction.
- Clearly $X + Y + Z = 1$.

Therefore in $1\,m^3$ of stellar gas at density ρ, there are $X\rho$ kg of H, $Y\rho$ kg of He and $Z\rho$ kg of heavier elements.

Mean Molecular Weight – II

In a fully ionised gas:

- H gives two particles per $m_{\rm H}$ (mass of hydrogen atom taken to be the proton rest mass).
- He gives 0.75 particles per $m_{\rm H}$ (α-particle and two e^-) and
- Metals give ~ 0.5 particles per $m_{\rm H}$ (^{12}C contributes nucleus with six $e^- = 7/12$ and ^{16}O contributes nucleus with eight $e^- = 9/16$).

The total number of particles per unit volume is then

$$n = \frac{2X\rho}{m_{\rm H}} + \frac{3Y\rho}{4m_{\rm H}} + \frac{Z\rho}{2m_{\rm H}}$$

$$n = \frac{\rho}{4m_{\rm H}}(8X + 3Y + 2Z) = \frac{\rho}{4m_{\rm H}}(6X + Y + 2)$$

Now $\bar{\mu} = \rho/(n m_{\rm H})$ and so

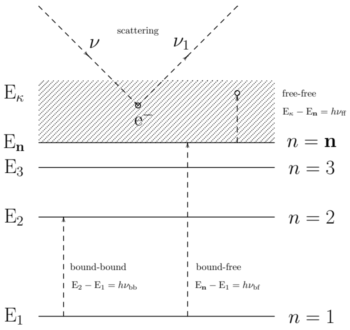
$$\bar{\mu} = \frac{4}{6X + Y + 2}$$

is a good approximation to $\bar{\mu}$ except in the outer regions and in cool dwarfs.

For example, $X_\odot = 0.747$, $Y_\odot = 0.236$ giving $\bar{\mu}_\odot \sim 0.6$ or the mean mass of particles in the solar envelope is $\sim 0.5\, m_{\rm H}$.

Opacity – I

Opacity as introduced in the discussion of radiation transport is the resistance of material to the flow of radiation through. In most stellar interiors it is determined through a combination of all processes which scatter and absorb photons; these are illustrated below:



Opacity – II

An expression, or some way of interpolating in pre-compued tables, is needed if the equations of stellar structure are to be solved. For stars in thermodynamic equilibrium with a comparively slow outward flow of energy, opacity should have the form

$$\kappa = \kappa(\rho, T, \text{chemical composition}).$$

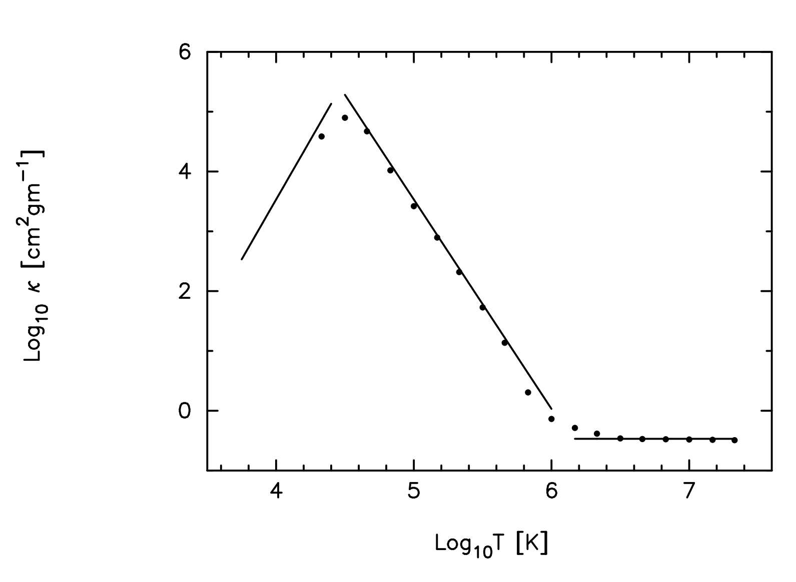
- Opacities may be interpolated in tables computed twenty years ago by the OP and OPAL projects.
- All possible interactions between photons of all frequencies and atoms, ions and molecules need to be taken into account.
- An enormous effort, the OP collaboration involved about thirty man-years of effort.

It turns out that opacity over restricted temperature ranges is well represented by

$$\kappa = \kappa_0 \, \rho^\alpha \, T^\beta,$$

where α and β are dependent on selected density and temperature ranges and κ_0 is a composition dependent constant in a selected temperature range.

Opacity – III



Opacity – IV

Opacity as a function of temperature is shown for a solar composition star at fixed $\rho = 10^{-4}\,\mathrm{gm\,cm^{-3}}$. Points are accurate OPAL calculations. Lines are approximate power-law representations.

- For $T > 10^6\,\mathrm{K}$ most atoms are fully ionised and photon energy is high. Free-free absorption is unlikely and electron scattering is expected to be the only significant source of opacity; this independent of T and so $\alpha = \beta = 0.0$ and $\kappa = \kappa_0$.
- For $10^{4.5}\,\mathrm{K} < T < 10^6\,\mathrm{K}$, κ peaks when bound-free and free-free absorption are very important; it then decreases as T increases. Approximate analytical form is given by $\alpha = 1$ and $\beta = -3.5$.
- For $10^{3.8}\,\mathrm{K} < T < 10^{4.5}\,\mathrm{K}$, κ increases as T increases. Most atoms are not ionised resulting in few electrons to scatter photons or participate in free-free absorption. Approximate analytical form is given by $\alpha = 1/2$ and $\beta = 4$.

Homologous Models – I

Homologous stellar models are defined such that their properties scale in the same way with fractional mass $x = M(r)/M$. That is $x = 0$ at the stellar centre and $x = 1$ at the stellar surface.

- For some property $X(x)$ (which may be $T(x)$ or $\rho(x)$ for example), a plot of $X(x)$ against x would be the same for all homologous models.
- Zero-age Main Sequence stars have a uniform chemical composition and homologous models should be a reasonable representation in this case.
- The aim is to recast stellar structure equations to be independent of absolute mass M and depend only on relative mass x.
- Variables of interest are therefore expressed as functions of x with dependencies on M being assumed to be power laws:

Homologous Models – II

$$
\begin{aligned}
r &= M^{a_1}\, r_{\mathrm{s}}(x) & dr &= M^{a_1}\, dr_{\mathrm{s}}(x) \\
\rho(r) &= M^{a_2}\, \rho_{\mathrm{s}}(x) & d\rho(r) &= M^{a_2}\, d\rho_{\mathrm{s}}(x) \\
T(r) &= M^{a_3}\, T_{\mathrm{s}}(x) & dT(r) &= M^{a_3}\, dT_{\mathrm{s}}(x) \\
P(r) &= M^{a_4}\, P_{\mathrm{s}}(x) & dP(r) &= M^{a_4}\, dP_{\mathrm{s}}(x) \\
L(r) &= M^{a_5}\, L_{\mathrm{s}}(x) & dL(r) &= M^{a_5}\, dL_{\mathrm{s}}(x)
\end{aligned}
$$

where the a_i exponents are constants to be determined and the variables $r_{\mathrm{s}}(x)$, $\rho_{\mathrm{s}}(x)$ etc. depend only on the fractional mass x. Also $M(r) = Mx$, $dM(r) = M\,dx$,

$$\kappa(r) = \kappa_0\, \rho^{\alpha}(r)\, T^{\beta}(r) = \kappa_0\, {\rho_{\mathrm{s}}}^{\alpha}(x)\, {T_{\mathrm{s}}}^{\beta}(x)\, M^{\alpha a_2}\, M^{\beta a_3}$$

and

$$\epsilon(r) = \epsilon_0\, \rho(r)\, T^{\eta}(r) = \epsilon_0\, \rho_{\mathrm{s}}(x)\, {T_{\mathrm{s}}}^{\eta}(x)\, M^{a_2}\, M^{\eta a_3}.$$

Homologous Models – III

Substitution into the stellar structure equations now allows these to be expressed in terms of the dimensionless mass x:

- Mass Conservation:

$$\frac{dr}{dM(r)} = \frac{1}{4\,\pi\,r^2\,\rho(r)} \qquad \text{becomes}$$

$$M^{(a_1-1)}\frac{dr_s(x)}{dx} = \frac{1}{4\,\pi\,{r_s}^2(x)\,\rho_s(x)}M^{-(2a_1+a_2)}.$$

  A requirement of the homology condition is that scaling is independent of actual mass; the M exponents on either side of the above equation must then be equal giving

$$3a_1 + a_2 = 1.$$

- Hydrostatic Equilibrium:

$$\frac{dP(r)}{dM(r)} = -\frac{G\,M(r)}{4\,\pi\,r^4} \qquad \text{becomes}$$

$$M^{(a_4-1)}\frac{dP_s(x)}{dx} = -\frac{G\,x}{4\,\pi\,{r_s}^4}M^{(1-4a_1)} \qquad \text{and by the homology condition}$$

$$4a_1 + a_4 = 2.$$

Homologous Models – IV

- Energy Production:

$$\frac{dL(r)}{dM(r)} = \epsilon(r)$$

$$= \epsilon_0\,\rho(r)\,T^{\eta}(r) \qquad \text{becomes}$$

$$M^{a_5-1}\frac{dL_{\rm s}(x)}{dx} = \epsilon_0\rho_{\rm s}(x)T^{\eta}(x)M^{a_2+\eta\,a_3} \qquad \text{and by the homology condition}$$

$$a_2 + \eta\,a_3 + 1 = a_5.$$

- Radiative Transport:

$$\frac{dT(r)}{dM(r)} = -\frac{3\,\bar{\kappa}_{\rm Ross}\,L(r)}{16\,\pi^2\,r^4\,a\,c\,T(r)^3} \qquad \text{becomes}$$

$$M^{(a_3-1)}\frac{dT_{\rm s}(x)}{dx} = -\frac{3(\kappa_0\,\rho_{\rm s}(x)^{\alpha}T_{\rm s}(x)^{\beta})L_{\rm s}(x)}{16\,\pi^2\,r_{\rm s}(x)^4\,a\,c\,T_{\rm s}(x)^3}M^{(a_5+(\beta-3)a_3+\alpha a_2-4a_1)} \qquad \text{and}$$

$$4a_1 + (4-\beta)a_3 = \alpha a_2 + a_5 + 1 \qquad \text{by the homology condition.}$$

Homologous Models – V

- Equation of State:
  If gas pressure dominates and we neglect any radial dependence of the mean molecular weight ($\bar{\mu}$):

$$P_{\rm gas}(r) = \frac{\rho(r)\,k\,T(r)}{\bar{\mu}\,m_{\rm H}} \qquad \text{becomes}$$

$$M^{a_4}\,P_{\rm s}(x) = \frac{\rho_{\rm s}(x)\,k\,T_{\rm s}(x)}{\bar{\mu}\,m_{\rm H}}\,M^{a_2+a_3} \qquad \text{and by the homology condition}$$

$$a_2 + a_3 = a_4.$$

  Alternatively, if radiation pressure dominates:

$$P_{\rm rad} = \frac{1}{3}\,a\,T^4 \qquad \text{becomes}$$

$$M^{a_4}\,P_{\rm s}(x) = \frac{1}{3}\,a\,(M^{a_3}T_{\rm s}(x))^4 \qquad \text{and by the homology condition}$$

$$a_4 = 4a_3.$$

Homologous Models – VI

In matrix form, the five equations (assuming $P_{\rm gas} \gg P_{\rm rad}$) for the exponents a_i are

$$\begin{bmatrix} +3 & +1 & & & \\ +4 & & & +1 & \\ & +1 & +\eta & & -1 \\ +4 & -\alpha & +(4-\beta) & & -1 \\ & +1 & +1 & -1 & \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

and which can be solved for given values of α, β and η.

Consider two cases:

- Low-mass stars ($\sim 0.7 \lesssim M \lesssim 2\,M_\odot$) corresponding to spectral types F and later: Adopt Kramers opacity $\kappa \propto \rho T^{-3.5}$ and a nuclear generation rate for PP-Chain $\epsilon \propto T^4$ so that $\alpha = 1$, $\beta = -3.5$ and $\eta = 4$.
- Higher mass stars ($M \gtrsim 2\,M_\odot$) corresponding to spectral types A and earlier: Assume opacity to be dominated by electron scattering and a nuclear generation rate dominated by the CNO cycle with a stronger temperature dependence $\epsilon \propto T^{16}$ so that $\alpha = 0$, $\beta = 0$ and $\eta = 16$.

Homologous Models – VII

Regime	α	β	η	a_1	a_2	a_3	a_4	a_5
Low-Mass	1	-3.5	4	1/13	10/13	12/13	22/13	71/13
Higher-Mass	0	0	16	15/19	$-26/19$	4/19	$-22/19$	3

The system of stellar structure equations in the scaled mass (x) representation may be solved numerically as previously described, subject to the same boundary conditions. But the beauty of the homology approximation is that useful conclusions may be derived analytically from the a_i obtained from the above matrix equation and summarised in the table:

- From the definition of homologous models:

$$L = M^{a_5}\, L_{\rm s}(1) \quad \text{and} \quad R = M^{a_1}\, r_{\rm s}(1)$$

  it follows immediately that

  - Low-mass stars $\quad L \propto M^{71/13} \sim M^{5.5} \qquad R \propto M^{1/13} \sim M^{0.1}$
  - Higher-mass stars $\quad L \propto M^{3} \qquad R \propto M^{15/19} \sim M^{0.8}$

which is not too bad when compared with the actual Main Sequence mass-luminosity relationship.

Homologous Models – VIII

- Secondly, since $L \propto R^2 {T_{\rm eff}}^4$ it also follows that

$$M^{a_5} L_{\rm s}(1) \propto M^{2a_1} r_{\rm s}(1)^2 {T_{\rm eff}}^4 \quad \text{or}$$
$$T_{\rm eff} \propto M^{(a_5-2a_1)/4}.$$

  And so

  - In the low-mass case $T_{\rm eff} \sim M^{1.2}$.
  - In the higher-mass case $T_{\rm eff} \sim M^{0.35}$.

- Combining with the mass-luminosity relationship gives

$$L \propto {T_{\rm eff}}^{4a_5/(a_5-2a_1)}$$

  - which for low-mass stars implies $L \sim {T_{\rm eff}}^{4.5}$,
  - and for higher-mass stars $L \sim {T_{\rm eff}}^{8.5}$.

The qualitative result that there **is** a luminosity-temperature relationship is, in effect, a prediction that a main sequence exists in the HR Diagram.

Convection becomes increasingly important at $M \lesssim 0.7_\odot$ and radiation pressure becomes more important as the stellar mass increases; in these regimes the above homology approximation begins to breakdown.

Lane-Emden Equation – I

- Four stellar structure and three auxilary equations are highly non-linear, coupled and need to be solved simultaneously with two-point boundary values.
- Polytropic models suppose that a simple relation between pressure and density (for example) exists throughout the star; the equations of hydrostatic equilibrium and mass conservation may then be solved independently of the other five.
- Before the advent of computing technology, polytropic models played an important role in the development of stellar structure theory; today they, like homologous models, usefully provide insight.

Take the equation of hydrostatic equilibrium

$$\frac{dP(r)}{dr} = -\frac{G\,M(r)\,\rho(r)}{r^2},$$

multiply by $r^2/\rho(r)$ and differentiate with respect to r gives

$$\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dP(r)}{dr}\right) = -\frac{G\,dM(r)}{dr}.$$

Now substitute the equation of mass-conservation on the right-hand side to obtain

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dP(r)}{dr}\right) = -4\,\pi\,G\,\rho(r).$$

Lane-Emden Equation – II

Adopt an equation of state of the form

$$P(r) = K\,\rho(r)^{\gamma} = K\,\rho(r)^{(n+1)/n} \quad \text{and} \quad \frac{dP(r)}{dr} = K\,\gamma\,\rho(r)^{\gamma-1}\frac{d\rho(r)}{dr}$$

where K is a constant and n (not necessarily an integer) is known as the polytropic index.

Substituting for $dP(r)/dr$ in the hydrostatic equilibrium equation combined with the mass conservation equation, and writing ρ for $\rho(r)$ in order to simplify the notation, gives

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2\,K}{\rho}\gamma\rho^{\gamma-1}\frac{d\rho}{dr}\right) = -4\,\pi\,G\,\rho \quad \text{or}$$
$$\frac{1}{\alpha^2}\frac{1}{\xi^2}\frac{d}{d\xi}\left(\frac{\xi^2\,K}{\rho}\gamma\rho^{\gamma-1}\frac{d\rho}{d\xi}\right) = -4\,\pi\,G\,\rho,$$

where the radial variable r has been rescaled by a constant α^{-1} so that $r = \alpha\xi$.

Suppose a radial density dependence

$$\rho = \rho_{\rm c}\,\theta(\xi)^n \quad \text{and} \quad \frac{d\rho}{d\xi} = \rho_{\rm c}\,n\,\theta(\xi)^{(n-1)}\frac{d\theta(\xi)}{d\xi},$$

where ρ_c is the central density. Writing θ for $\theta(\xi)$ in order to simplify notation, the above equation then becomes

$$\frac{K\,(n+1)}{4\,\pi\,G{\rho_{\rm c}}^{(1-1/n)}\alpha^2}\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\,\frac{d\theta}{d\xi}\right) = -\theta^n.$$

Lane-Emden Equation – III

As α is arbitrary, choose

$$\alpha^2 = \frac{K\,(n+1)}{4\,\pi\,G\,{\rho_c}^{(1-1/n)}}$$

in which case

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n.$$

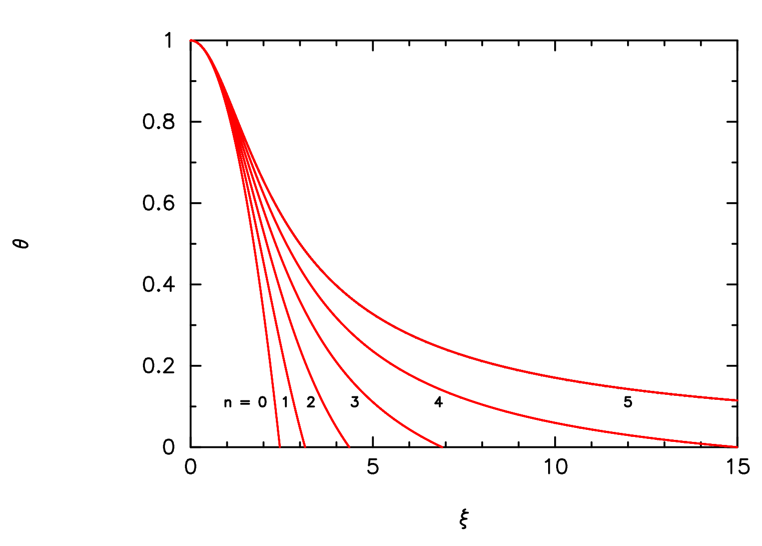
The above equation is known as the **Lane-Emden** equation; it defines the rate of change of density within a stellar interior subject to:

- At the centre of the star where $\xi(0) = 0$, $\theta(0) = 1$ so that $\rho = \rho_c$.
- since $dP/dr \to 0$ as $r \to 0$, $d\theta/d\xi = 0$ at $\xi = 0$.
- The outer boundary (surface) is the first location where $\rho = 0$ or $\theta(\xi) = 0$; this location is referred to as ξ_1.

Solutions of the Lane-Emden equation, known as polytropes, specify $\rho(r)$ although expressed as $\theta(\xi)$. The order of the solution is determined by the index n; in particular it depends only on n and can be scaled by varying P_c (central pressure) and ρ_c to give solutions for stars over a range of total mass and radius. Analytical solutions exist for $n = 0, 1$ and 5; all other solutions need to be obtained numerically.

- For $n = 0$, $\rho(r) = \rho_c$; this is the solution for an incompressible sphere.
- To approximate a fully convective star (such as a M, L or T dwarf) use polytropes having $n = 1 \to 1.5$.
- The Eddington Approximation discussed below corresponds to $n = 3$; it corresponds to a fully radiative star and is a useful approximation for the Sun.

Lane-Emden Equation – Solutions



Comparison Between Polytrope and Real Models – I

Predictions of the $n = 3$ polytropic model for the Sun of mass, density, pressure and temperature variations with radius are needed for comparison with the Standard Solar Model of Bahcall *et al.* (1998 Physics Letters B, 433, 1).

At the surface of the $n = 3$ polytrope where $\theta = 0$

$$\alpha = \frac{R_\odot}{\xi_1} = \frac{7 \times 10^8}{6.9} \text{ m} = 1.01 \times 10^8 \text{ m}.$$

The rate of change of mass with radius is given by the Equation of Mass Conservation

$$\frac{dM(r)}{dr} = 4\,\pi\,r^2\,\rho(r).$$

Integrating and substituting $r = \alpha\xi$ and $\rho = \rho_c\theta^n$ gives

$$M_\odot = \int_0^{R_\odot} 4\,\pi\,r^2\,\rho\,dr = 4\,\pi\,\alpha^3\,\rho_c \int_0^{\xi_1} \xi^2\,\theta^n\,d\xi.$$

The Lane-Emden equation may expressed in the form

$${\xi_1}^2 \left|\frac{d\theta}{d\xi}\right|_{\xi=\xi_1} = -\int_0^{\xi_1} \xi^2\,\theta^n\,d\xi$$

and substituting in the above expression for $M_\odot$ gives

$$M_\odot = -4\,\pi\,\alpha^3\,\rho_c\,{\xi_1}^2 \left|\frac{d\theta}{d\xi}\right|_{\xi=\xi_1}.$$

Comparison Between Polytrope and Real Models – II

The Lane-Emden Equation for $n = 3$ has a solution ($\theta = 0$) relevant to stellar structure at

$$\xi_1 = 6.90 \quad \text{and} \quad \left|\frac{d\theta}{d\xi}\right|_{\xi=\xi_1} = -4.236 \times 10^{-2}.$$

Taking $M_\odot = 2 \times 10^{30}$ kg and the Lane-Emden Equation solution for $n = 3$, the expression for $M_\odot$ above gives an estimate for the central density of the Sun of

$$\rho_\mathrm{c} = 7.66 \times 10^4 \ \mathrm{kg\,m^{-3}}$$

and the dependence of density on radial distance from the solar centre immediately follows from

$$\rho = \rho_\mathrm{c}\, \theta^n$$

since θ varies from $\theta = 1$ at the centre to $\theta = 0$ at the surface.

By definition

$$\alpha^2 = \frac{K\,(n+1)}{4\,\pi G\, \rho_\mathrm{c}{}^{(1-1/n)}}$$

and as ρ_c and α are known, $K = 3.85 \times 10^{10}\ \mathrm{Nm\,kg^{-1}}$. It then follows since $P = K\,\rho^\gamma$ that an estimate of the pressure at the centre of the Sun (where $\rho = \rho_\mathrm{c}$) is

$$P_\mathrm{c} = 1.25 \times 10^{16} \quad \mathrm{N}\, m^{-2},$$

and the dependence of gas pressure on radial distance follows directly by substituting the appropriate ρ.

Comparison Between Polytrope and Real Models – III

By a similar argument, the equation of state for a perfect gas

$$P_{\rm gas} = \frac{k}{m_{\rm H}\,\bar{\mu}}\,\rho\,T$$

gives the dependence of T on radial distance (r) on substituting the $P_{\rm gas}(r)$ and adopting $\bar{\mu} \simeq 0.6$ as previously derived. In particular, setting $P_{\rm gas}(r) = P_{\rm c}$ gives a temperature at the solar centre of

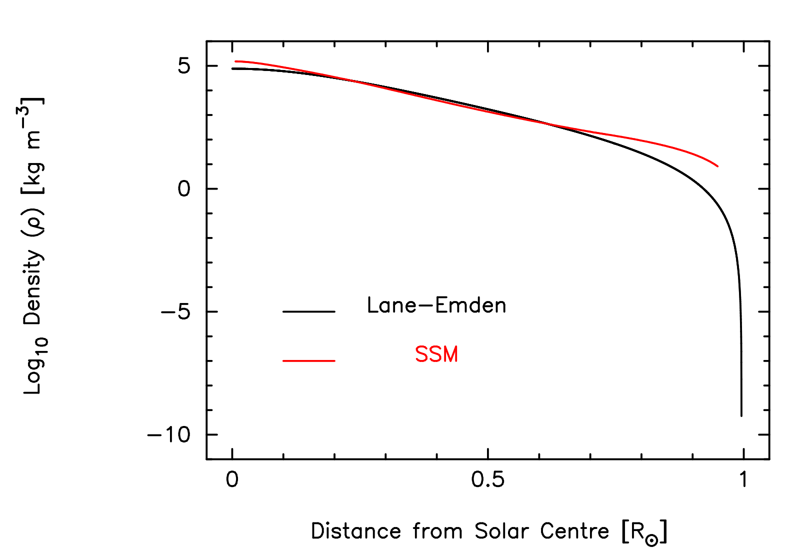
$$T_{\rm c} = 1.19 \times 10^7\ {\rm K}.$$

As previously discussed, the mass $(M(r))$ interior to some distance r from a stellar centre is given by the mass conservation equation, to which the Lane-Emden equation may be applied, to give

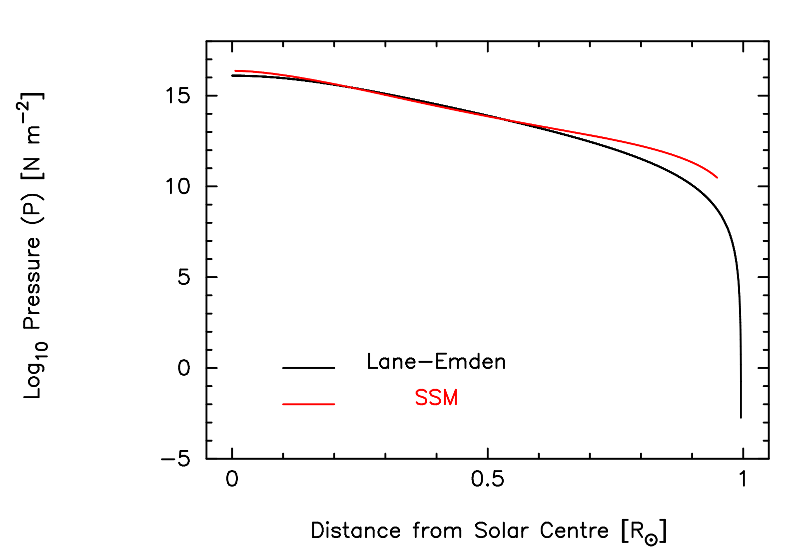
$$M(r) = -4\,\pi\alpha^3\,\rho_{\rm c}\,{\xi_r}^2\,\left|\frac{d\theta}{d\xi}\right|_{\xi=\xi_r}$$

where ξ_r is the scaled radial distance r/α at distance r from the centre of the Sun. Evaluating the right-hand side for successive values of ξ_r gives the mass interior to those points. Comparisons with the Standard Solar Model (SSM) are shown in the plots which follow.

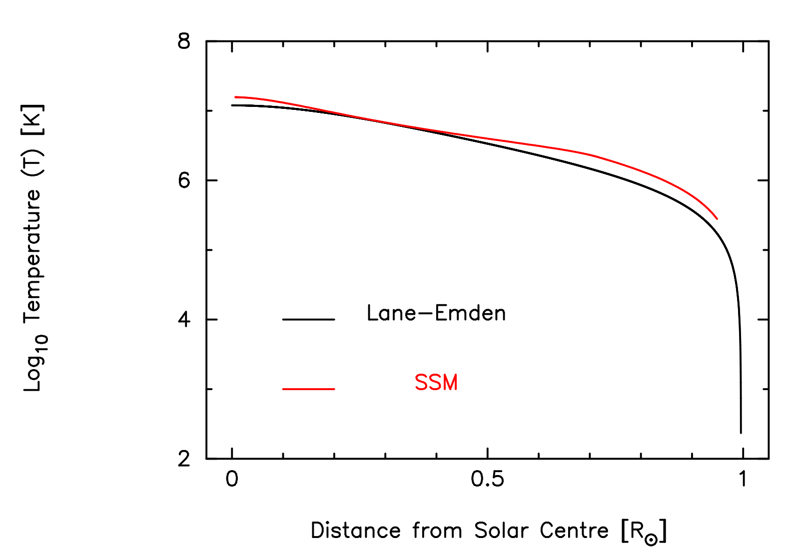
Comparison Between Polytrope and Real Models – IV



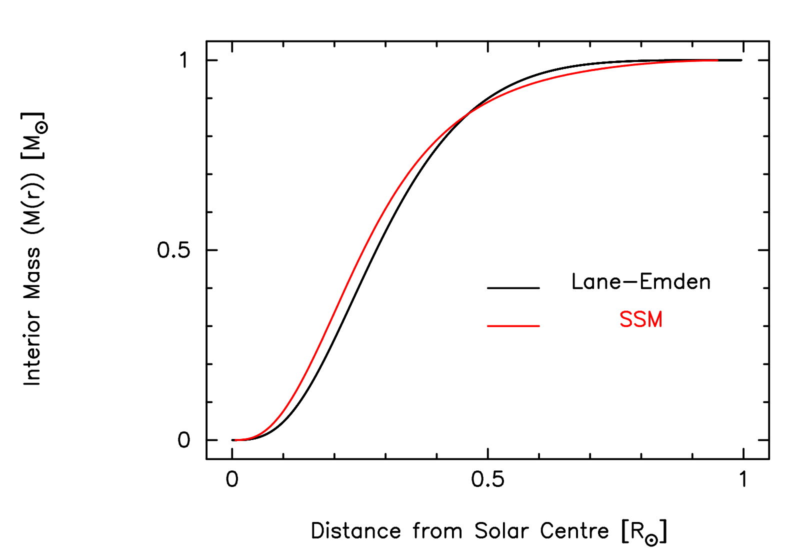
Comparison Between Polytrope and Real Models – V



Comparison Between Polytrope and Real Models – VI



Comparison Between Polytrope and Real Models – VII



Summary

Essential points covered in sixth lecture:

- Simple approximations allow a solution of the stellar structure problem without resorting a computationally expensive full solution of the coupled differential equations of stellar structure.
- In particular, with a polytropic index $n = 3$, an approximate solar model can be obtained using the Lane-Emden equation.
- Agreement between the Lane-Emden solar model and the detailed standard solar model (incorporating the best physics and numerical methods) is remarkably good over much of the solar interior.

Acknowledgement

Material presented in this lecture on homolgous stellar models and polytropes is based almost entirely on slides prepared by S. Smartt (Queen's University of Belfast).