The structure and evolution of stars

Lecture 7: The structure of main-sequence stars: homologous stellar models

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Learning Outcomes

Student will learn:

- How to employ approximate forms for the three equations that supplement the stellar structure equations i.e. opacity, equation of state and energy generation
- How to derive a sequence of homologous stellar models
- Why these homologous sequences are useful
- How the approximate homologous sequence compares to observations of stars

Introduction and recap

- We have 4 differential equations of stellar structure
- Completely accurate expressions for pressure, opacity and energy generation are extremely complicated, but we can find simple approximate forms
- Eqns of stellar structure too complicated to find exact analytical solution, hence must be solved with computer
- But we **can verify position of main-sequence and find mass-luminosity relation without solving eqns completely**.

- We will attempt to simply derive relationships between luminosity, temperature and mass for a population of stellar models. This will allow comparison with observations.

Equation of state of an ideal gas

We have seen that stellar gas is ionized plasma, and although density is so high that typical inter-particle spacing is of the order of an atomic radius, the effective particle size is more like a nuclear radius (10^5) times smaller. Hence material behaves like an **ideal gas**.

$$P_{gas} = nkT$$

Where n is number of particles per cubic meter, k is Boltzmann's constant

But we want this equation in the form:

$$P = P(\rho, T, composition)$$

Following the class derivation, this can be written:

$$P = \frac{\Re \rho T}{\mu}$$

R=k/m_H =the gas constant

mu = mean molecular weight = mean mass of particles in terms of H-atom (m_H)

If radiation pressure is important

$$P = \frac{\Re \rho T}{\mu} + \frac{aT^4}{3}$$

Mean molecular weight

We can derive an expression for the mean molecular weight μ. An exact solution is complex, depending on fractional ionisation of all the elements in all parts of the star. We will assume that all of the material in the star is **fully ionised**.

Justified as H and He are most abundant, and they are certainly fully ionised in stellar interiors (assumption will break down near stellar surface).

X=fraction of material by mass of H

Y=fraction of material by mass of He

Z=fraction of material by mass of all heavier elements

$X + Y + Z = 1$

Hence in $1m^3$ of stellar gas of density ρ there is mass $X\rho$ of H, $Y\rho$ of He, $Z\rho$ of heavier elements. In a fully ionised gas,

H gives 2 particles per m_H

He gives 3/4 particles per m_H (alpha particle, plus two e^-)

Heavier elements give ~1/2 particles per m_H (^{12}C has nucleus plus $6e^-$ = 7/12)

(^{12}O has nucleus plus $8e^-$ = 9/16)

The total number of particles per cubic metre is then given by the sum:

$$n = \frac{2X\rho}{m_H} + \frac{3Y\rho}{4m_H} + \frac{Z\rho}{2m_H}$$

$$n = \frac{\rho}{4m_H}\left(8X + 3Y + 2Z\right) = \frac{\rho}{4m_H}\left(6X + Y + 2\right)$$

Now as before we define $\rho = n\, m_H\, \mu$

$$\mu = \frac{4}{6X + Y + 2}$$

Which is a good approximation to μ except in the cool outer regions of stars. For solar composition, $X_\odot$=0.747, $Y_\odot$=0.236, $Z_\odot$=0.017, resulting in μ ~0.6,

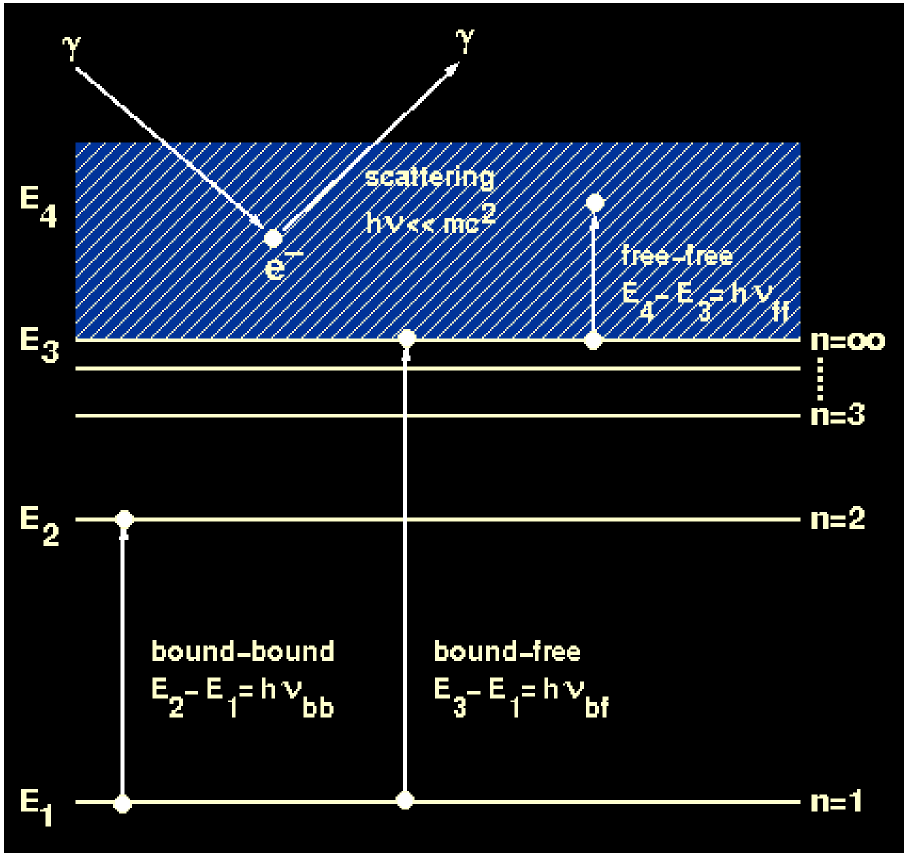
i.e. the mean mass of particles in a star of solar composition is a little over half the mass of the proton

Opacity

Concept of opacity introduced when deriving the equation of radiation transport, and will be discussed extensively in “Corso di Processi radiative”. Opacity is the resistance of material to the flow of radiation through it. In most stellar interiors it is determined by all the processes which scatter and absorb photons

Four processes:

- Bound-bound absorption
- Bound-free absorption
- Free-free absorption
- scattering



Approximate form for opacity

We need an expression for opacity to solve the eqns of stellar structure. For stars in thermodynamic equilibrium with only a slow outward flow of energy, the opacity should have the form

$$\kappa = \kappa(\rho, T, \text{chemical composition})$$

Opacity coefficients may be calculated, taking into account all possible interactions **between the elements and photons of different frequencies**.

This requires an enormous amount of calculation and is beyond the scope of this course. When it has been done, the results are usually approximated by the relatively simple formula :

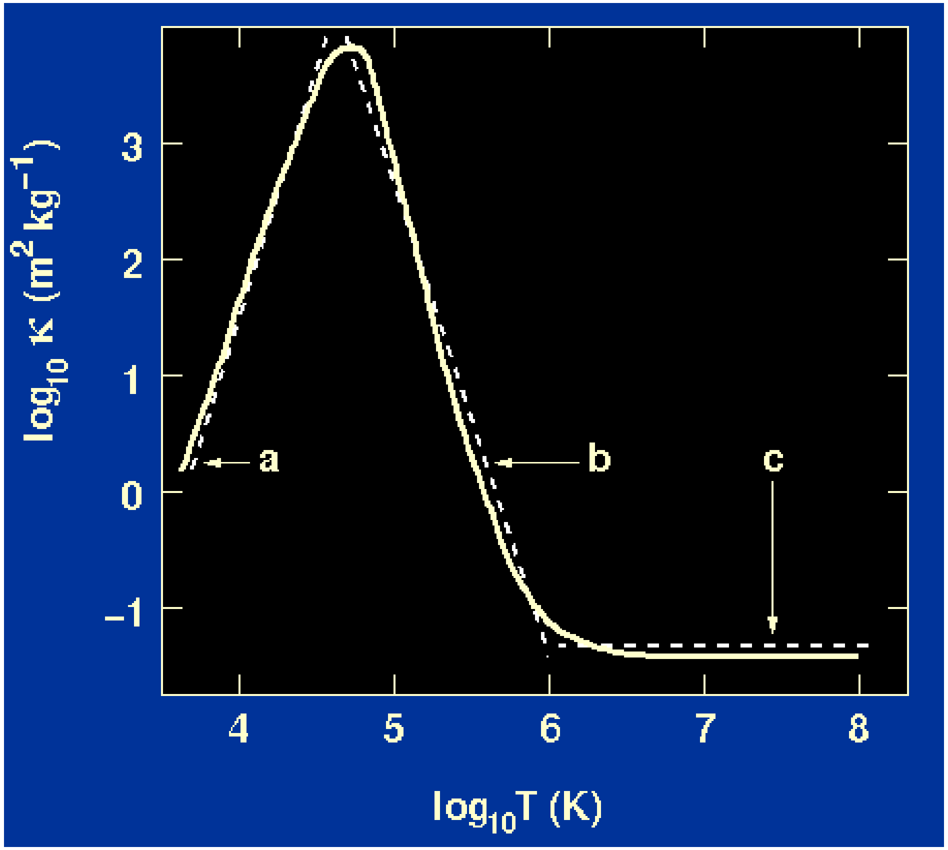
$$\kappa = \kappa_0 \rho^{\alpha} T^{\beta}$$

Where alpha, beta are slowly varying functions of density and temperature and k_0 is a constant for a given chemical composition

Figure shows opacity as a function of temperature for a star of given ρ (10^{-1} kgm^{-3}). Solid curve is from detailed opacity calculations. Dotted lines are approximate power-law forms.

At high T: k is low and remains constant. Most atoms fully ionised, high photon energy, hence free-free absorption unlikely, Dominant mechanism is electron scattering, independent of T, alpha=beta=0

$$\kappa = \kappa_0 \text{ (curve c)}$$



Opacity is low a low T, and decreases with T. Most atoms not ionised, few electrons available to scatter photons or for free-free absorption. Approx analytical form is alpha=1/2 , beta=4

$$\kappa = \kappa_0 \rho^{1/2} T^4 \text{ (curve a)}$$

At intermediate T, k peaks, when bound-free and free-free absorption are very important, then decreases with T (Kramers opacity law, see Böhm-Vitense Ch. 4)

$$\kappa = \kappa_0 \rho T^{-3.5} \text{ (curve b)}$$

Homologous stellar models

We already have the four eqns of stellar structure in terms of mass (m)

$$\frac{dr}{dM} = \frac{1}{4\pi r^2 \rho} \qquad \frac{dL}{dM} = \varepsilon$$

$$\frac{dP}{dM} = -\frac{GM}{4\pi r^4} \qquad \frac{dT}{dM} = \frac{3\kappa_R L}{64\pi^2 r^4 \sigma T^3}$$

With boundary conditions:

R=0, L=0 at M=0

ρ=0, T=0 at M=M_s

And supplemented with the three additional relations for P, k, ε (assuming that the stellar material behaves as an ideal gas with negligible radiation pressure, and laws of opacity and energy generation can be approximated by power laws)

$$P = \frac{\Re \rho T}{\mu}$$

$$\kappa = \kappa_0 \rho^\alpha T^\beta$$

$$\varepsilon = \varepsilon_0 \rho T^\eta$$

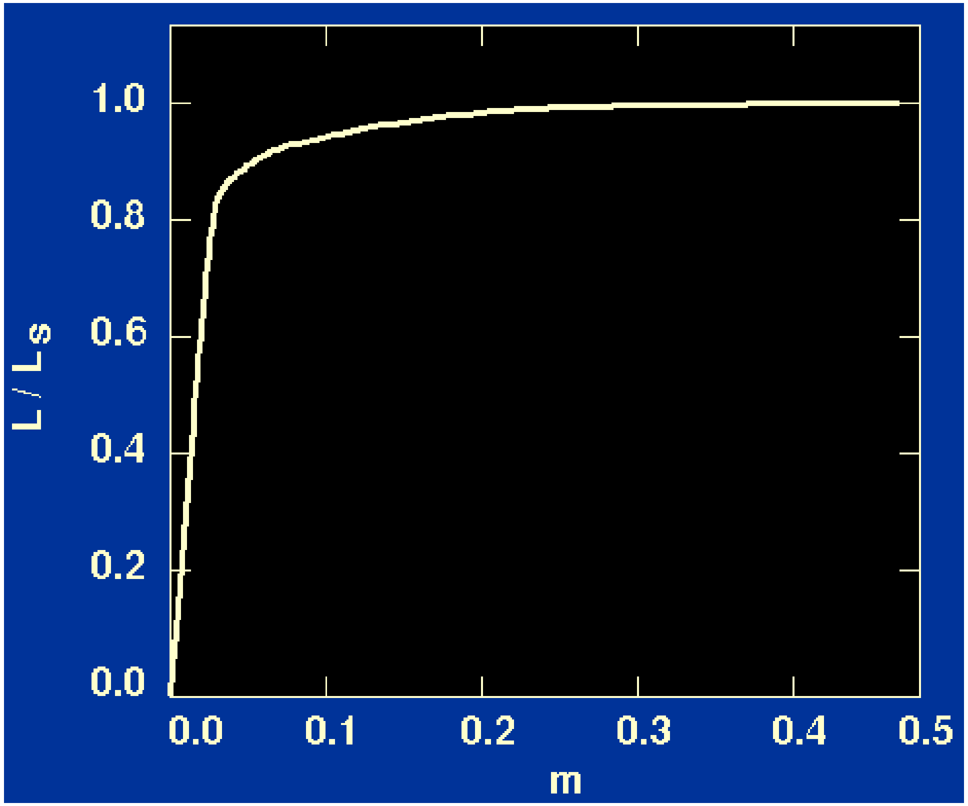
Where alpha, beta, eta are constants and k_0 and ε_0 are constants for a given chemical composition.

Homologous models

We aim to formulate the eqns of stellar structure so that they are independent of mass M_S. Hence, we will assume that the way in which a physical quantity (e.g. L) varies from centre of star to surface is the same for all stars of all masses (only absolute L varies).

Schematic illustration: ratio of luminosity to surface luminosity is plotted against fractional mass (m) , which is defined as the ratio of mass to total mass

$m=M/M_s$



We then assume this curve is the same for ALL stars with the same laws of opacity and energy generation. But that L_S is proportional to some power of M_S, which depend on the values of alpha, beta, eta

The same will also be true for r_s and T_e (effective temperature)

Mathematically expressing this:

$$r = M_s^{a_1} r^*(m)$$

$$\rho = M_s^{a_2} \rho^*(m)$$

$$L = M_s^{a_3} L^*(m)$$

$$T = M_s^{a_4} T^*(m)$$

$$P = M_s^{a_5} P^*(m)$$

Where a_1, a_2, a_3, a_4, a_5, are constants and r^*, rho*, L^*, T^*, P^*, all depend only on fractional mass m

Now we can substitute these expressions into the four stellar structure equations (and the equation of state). Remember our goal is to eliminate the dependence on M in those equations and replace it with m.

So now we have obtained 5 equations for the five constants a_1, a_2, a_3, a_4, a_5 . We also have 5 new equations for stellar structure which are independent of M_S. They are only independent of M_S however if the 5 equations for a_1, a_2, a_3, a_4, a_5 have consistent solutions.

$$4a_1 + a_5 = 2$$

$$3a_1 + a_2 = 1$$

$$a_3 = \eta a_4 + a_2 + 1$$

$$4a_1 + (4 - \beta)a_4 = \alpha a_2 + a_3 + 1$$

$$a_5 = a_2 + a_4$$

These are inhomogeneous algebraic equations (i.e. some contain terms independent of the a values). They can be solved for all reasonable values of α, β, η. The general solution is very complicated, we won't derive it, but will consider solutions with particular values of α, β, η shortly.

Collecting the new equations together

Now we have the 5 new equations

$$\frac{dr^*}{dm} = \frac{1}{4\pi r^{*2} \rho^*} \qquad \frac{dL^*}{dm} = \varepsilon_0 \rho^* T^{*\eta} \qquad P^* = \frac{\Re \rho^* T^*}{\mu}$$

$$\frac{dP^*}{dm} = -\frac{Gm}{4\pi r^{*4}} \qquad \frac{dT^*}{dm} = \frac{-3\kappa_0 \rho^{*\alpha} T^{*(\beta-3)} L^*}{64\pi^2 \sigma r^{*4}}$$

These equations can now be solved to find r*, rho*, L*, T*, and P* in terms of m using the boundary conditions

r*=0, L*=0 at m=0

rho*=0, T*=0 at m=1

Where the centre and surface of the star are at m=0 and m=1 respectively.

These must be solved on a computer,and then the r*, rho*, L*, T*, and P* quantities can be converted to r, rho, L, T, and P for a star of any given mass, using the relations previously derived.

Homologous models

Such a set of models of stars in which the dependence of the physical quantities on fractional mass m is independent of the total mass of the star is known as a **homologous sequence of stellar models.**

Without even fully solving the homologous equations of stellar structure, we can deduce a mass-luminosity relation for main-sequence stars and also a simple relation between luminosity and effective temperature – this characterises the main-sequence in the HR diagram, so can be compared to observations.

M-L and L-T_e relations

Actually it's trivial to write down a mass-luminosity relation from our definition of the homologous sequence

$$L = M_s^{a_3} L^*(m)$$

$$L_s \propto M_s^{a_3}$$

Now for the luminosity – effective temperature relation, these quantities are related to the radius of a star through:

$$L_s = 4\pi r_s^2 \sigma T^4$$

Combining this with:

$$r = M_s^{a_1} r^*(m)$$

$$L = M_s^{a_3} L^*(m)$$

We can show :

$$L_S \propto T_e^{\frac{4a_3}{a_3 - 2a_1}}$$

This shows that stars lie in the theoretical HR diagram ($\log L_s$ versus $\log T_e$) and this might be identified with the main-sequence

Now although the homologous models do predict a power-law mass-luminosity relation and the existence of a main-sequence type structure in the HR-diagram, we still have not shown that the exponent in these power laws is agreement with the observed values. In order to do this we must solve the 5 algebraic equations :

$$4a_1 + a_5 = 2$$

$$3a_1 + a_2 = 1$$

$$a_3 = \eta a_4 + a_2 + 1$$

$$4a_1 + (4 - \beta)a_4 = \alpha a_2 + a_3 + 1$$

$$a_5 = a_2 + a_4$$

Now the general solution is complex, but we can solve for particular values of α, β, η

In the discussions of stellar opacity, we found that one approximation to the opacity law, which works well at intermediate temperatures is given by α =1 and β =-3.5

And a reasonable approximation of the rate of energy generation by the PP-chain is given with η =4.

Hence:

$$\kappa = \frac{\kappa_0 \rho}{T^{3.5}}$$

$$\varepsilon = \varepsilon_0 \rho T^4$$

And substituting *α=1, β=-3.5 and η=4* into the five algebraic equations, we obtain the simplified set of equations:

$$4a_1 + a_5 = 2$$

$$3a_1 + a_2 = 1$$

$$a_3 = 4a_4 + a_2 + 1$$

$$4a_1 + 7.5a_4 = a_2 + a_3 + 1$$

$$a_5 = a_2 + a_4$$

We now have 5 equations in 5 unknowns – so simply can eliminate each of the a's in turn to obtain a solution for a_3 and a_1. It is left to the student to demonstrate !

So we have the result

a_3=71/13   and

a_1=1/13

Substituting these into the mass-luminosity and luminosity – effective temperature relations we get

$$L_S \propto M_s^{5.46}$$

$$L_S \propto T_e^{4.12}$$

The observed mass-luminosity law is not a simple power law but if the central part of the curve (corresponding to close to a solar mass) is approximated by a power law, it has an exponent of approximately 5. Which is in good agreement with the value of 5.46 above.

Similarly the lower part of the main-sequence on the observed L-T_e diagram (HR diagram) is well represented by a power law of exponent 4.1.

We have therefore verified the observed mass-luminosity relation of main-sequence stars and the existence of the main-sequence on the HR diagram – one of our goals from Lecture 1

Summary and conclusions

Revisit the learning outcomes

- How to employ approximate forms for the three equations that supplement the stellar structure equations i.e. opacity, equation of state and energy generation
- How to derive a sequence of homologous stellar models
- Why these homologous sequences are useful
- How the approximate homologous sequence compares to observations of stars

Next lecture: Another method of simplifying the solution of the stellar structure equations. After that we will move on to discussing the output of full numerical solutions of the equations and realistic predictions of modern theory