The structure and evolution of stars

Lecture 8: Polytropes and simple models

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Introduction and recap

In previous lecture we saw how a homologous series of models could describe the main-sequence approximately. These models where not full solutions of the equations of stellar structure, but involved simplifications and assumptions

Before we move on to description of the models from full solutions, we will come up with another simplification method that will allow the first two equations of stellar structure to be solved, without considering energy generation and opacity.

This form was historically very important and used widely by Eddington and Chandrasekhar

Learning Outcomes

- What is a polytrope
- Simplifying assumptions to relate pressure and density
- How to derive the Lane-Emden equation
- How to solve the Lane-Emden equation for various polytropes
- How realistic a polytrope is in describing the structure of the Sun

What is a simple stellar model

- We have seen the seven equations required to be solved to determine stellar structure. Highly non-linear, coupled and need to be solved simultaneously with two-point boundary values.

- Simple solutions (i.e. analytic) rely on finding a property that changes moderately from stellar centre to surface such that it can be assumed only weakly dependent on r or m - difficult, as for example T varies by 3 orders of magnitude and P by >14! Chemical composition is a property that can be assumed uniform (e.g. if stars is mixed by convective processes).

- Polytropic models: method of simplifying the equations. Simple relation between pressure and density (for example) is assumed valid throughout the star. Eqns of hydrostatic support and mass conservation can be solved independently of the other 5.

- Before the advent of computing technology, polytropic models played an important role in the development of stellar structure theory.

Polytropic models

Take the equation for hydrostatic support (in terms of the radius variable r),

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

Multiply by r^2/ρ and differentiating with respect to r, gives

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dM}{dr}$$

Now substitute the equation of mass-conservation on the right-hand side, and we obtain

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

Let us now adopt an equation of state of the form (where is it customary to adopt γ= 1+1/n) . K is a constant and **n is known as the polytropic index**.

$$P = K\rho^{\gamma} = K\rho^{\frac{n+1}{n}}$$

Recall the equations:

We already have the four eqns of stellar structure in terms of mass (m)

$$\frac{dr}{dM} = \frac{1}{4\pi r^2 \rho} \qquad \frac{dL}{dM} = \varepsilon$$

$$\frac{dP}{dM} = -\frac{GM}{4\pi r^4} \qquad \frac{dT}{dM} = \frac{3\kappa_R L}{64\pi^2 r^4 \sigma T^3}$$

With boundary conditions:

R=0, L=0 at M=0

ρ =0, T=0 at M=M_s

And supplemented with the three additional relations for P, rho, eps (assuming that the stellar material behaves as an ideal gas with negligible radiation pressure, and laws of opacity and energy generation can be approximated by power laws)

$$P = \frac{\Re \rho T}{\mu}$$

$$\kappa = \kappa_0 \rho^\alpha T^\beta$$

$$\varepsilon = \varepsilon_0 \rho T^\eta$$

Where α, β, η are constants and κ_0 *and* ε_0 are constants for a given chemical composition.

Polytropic models

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$$\frac{(n+1)K}{4\pi nG}\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho^{\frac{n-1}{n}}}\frac{d\rho}{dr}\right) = -\rho$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

The solution ρ(r) for 0 ≤ r ≤ R is called a **polytrope** and requires two boundary conditions. Hence a polytrope is uniquely defined by three parameters : K, n, and R. This enables calculation of additional quantities as a function of radius, such as pressure, mass or gravitational acceleration.

Now for the solution, it is convenient to define a dimensionless variable θ in the range $0 \le \theta \le 1$ by

$$\rho = \rho_c \theta^n$$

Which allows the derivation of the well-known Lane-Emden equation, of index n

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \qquad \text{where } r = \alpha\xi$$

Solving the Lane-Emden equation

It is possible to solve the equation analytically for only three values of the polytropic index n

$$n = 0, \qquad \theta = 1 - \left(\frac{\xi^2}{6}\right)$$

$$n = 1, \qquad \theta = \frac{\sin\xi}{\xi}$$

$$n = 5, \qquad \theta = \frac{1}{\left(1 + \xi^2/3\right)^{0.5}}$$

Solutions for all other values of n must be solved numerically i.e. we use a computer program to determine θ for values of ξ

Solutions are subject to boundary conditions:

$$\frac{d\theta}{d\xi} = 0, \quad \theta = 1 \quad \text{at} \quad \xi = 0$$

Computational solution of the equation

We start by expressing the Lane-Emden equation in the form:

$$\frac{d^2\theta}{d\xi^2} = -\frac{2}{\xi}\frac{d\theta}{d\xi} - \theta^n$$

The numerical integration technique - step outwards in radius from the centre of the star and evaluate density at each radius (i.e. evaluate θ for each of ξ). At each radius, the value of density θ_{I+1} is given by the density at previous radius, θI plus the change in density over the step (Δξ)

$$\theta_{i+1} = \theta_i + \Delta\xi \frac{d\theta}{d\xi}$$

Now dθ/dξ is unknown, but by same technique we can write

$$\left(\frac{d\theta}{d\xi}\right)_{i+1} = \left(\frac{d\theta}{d\xi}\right)_i + \Delta\xi \frac{d^2\theta}{d\xi^2}$$

Then we can replace the second derivative term in the above by the rearranged form of the Lane-Emden equation:

$$\left(\frac{d\theta}{d\xi}\right)_{i+1} = \left(\frac{d\theta}{d\xi}\right)_{i} - \left(\frac{2}{\xi}\frac{d\theta}{d\xi} + \theta^{n}\right)\Delta\xi$$

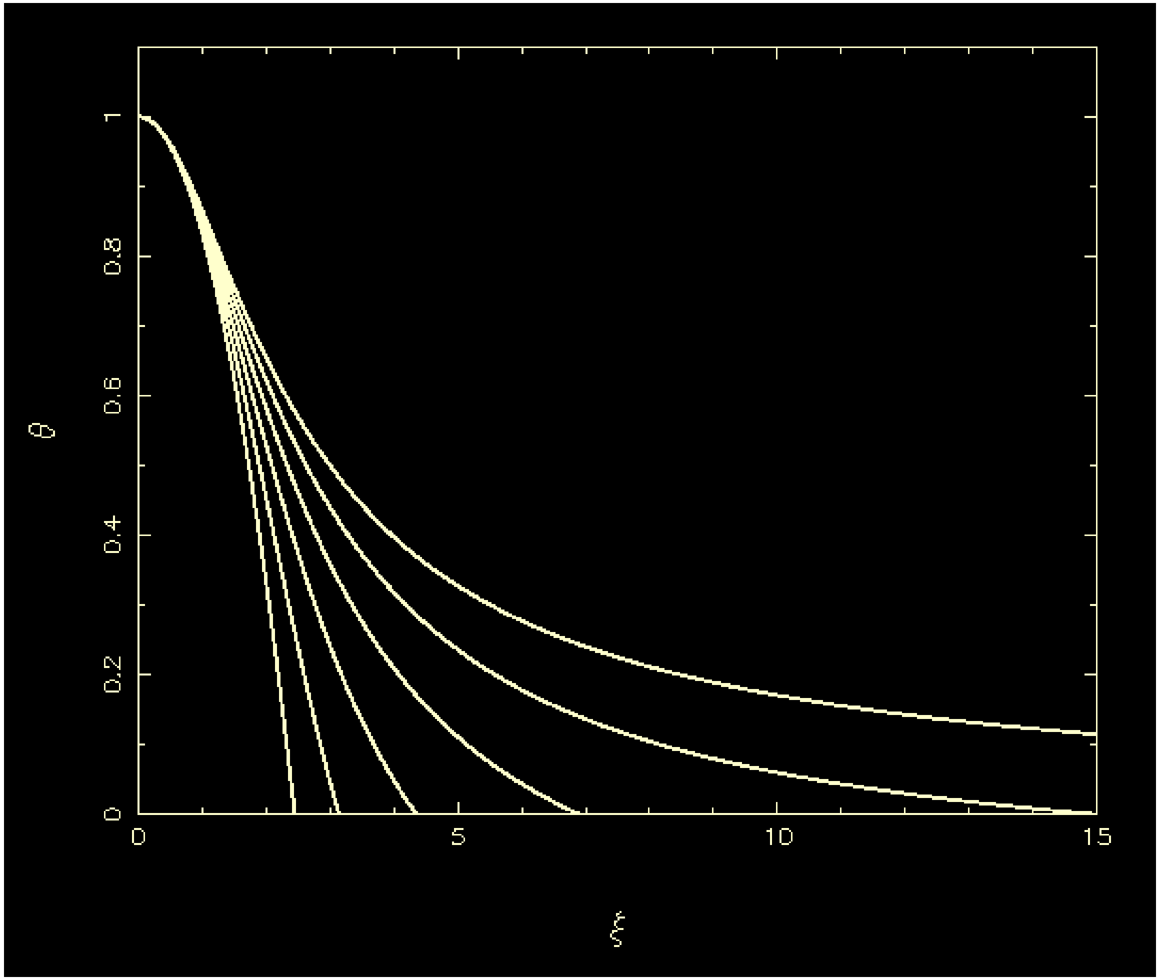
Now we can adopt a value for n and integrate numerically. We have the boundary conditions at the centre.

$$\frac{d\theta}{d\xi} = 0, \quad \theta = 1 \quad \text{at} \quad \xi = 0$$

So starting at the centre, we determine $\left(\frac{d\theta}{d\xi}\right)_{i+1}$

Which can be used to determine θ_{I+1} . The radius is then incremented by adding $\Delta\xi$ to ξ and the process is repeated until the surface of the star is reached (when θ becomes negative).

In your own time - Fortran program on course website to do these calculations - useful experience.



Numerical solutions to the Lane-Emden equation for (left-to-right) n = 0,1,2,3,4,5

Compare with analytical

$$n=0, \quad \theta = 1 - \left(\frac{\xi^2}{6}\right)$$

$$n=1, \quad \theta = \frac{\sin\xi}{\xi}$$

$$n=5, \quad \theta = \frac{1}{\left(1+\xi^2/3\right)^{0.5}}$$

Solutions decrease monotonically and have θ=0 at $\xi = \xi_R$ (i.e. the stellar radius)

With decreasing polytropic index, the star becomes more centrally condensed.
What does a polytrope of n=5 represent ?

For n < 5 polytropes, the solution for θ drops below zero at a finite value of ξ and hence the radius of the polytrope ξ_R can be determined at this point. In the numerically integrated solutions, a linear interpolation between the points immediately before and after θ becomes negative will give the value for ξ at θ=0. The roots of the equation for a range of polytropic indices are listed below. In the two cases where an analytical solution exists, the solutions are easily derived.

n	ξ_R	$-\left(\frac{d\theta}{d\xi}\right)_{\xi=\xi_R}$
0	2.45	3.33×10^{-1}
1	3.14	1.01×10^{-1}
2	4.35	2.92×10^{-2}
3	6.90	6.14×10^{-3}
4	15.00	5.33×10^{-4}

Recall: $r = \alpha\xi \qquad \rho = \rho_c \theta^n$

How do these polytropic models, compare to the results of a detailed solution of the equations of stellar structure ? To make this comparison we will take an n=3 polytropic model of the Sun (often known as the Eddington Standard Model), with the co-called Standard Solar Model (SSM - Bahcall 1998, Physics Letters B, 433, 1). We need to convert the dimensionless radius ξ and density θ to actual radius (in m) and density (in kg m^{-3}). We must also determine how the mass, pressure and temperature vary with radius:

To determine the scale factor α :

At the surface of the n=3 polytrope (θ=0) , we have

$$\alpha = \frac{R}{\xi_R}$$

Where R=radius of the star (Sun in this case), and ξ_R is the value of ξ at the surface (i.e. the root of the Lane-Emden equation that we listed in the table above)

http://www.sns.ias.edu/~jnb/SNdata/solarmodels.html

SSM data available at:

Next we determine the mass as a function of radius. The rate of change of mass with radius is given by the equation of mass conservation

$$\frac{dM}{dr} = 4\pi r^2 \rho$$

By integrating and substituting $r = \alpha\xi$ and $\rho = \rho_c \theta^n$

$$M = \int_0^R 4\pi r^2 \rho dr = 4\pi\alpha^3 \rho_c \int_0^{\xi_R} \xi^2 \theta^n d\xi$$

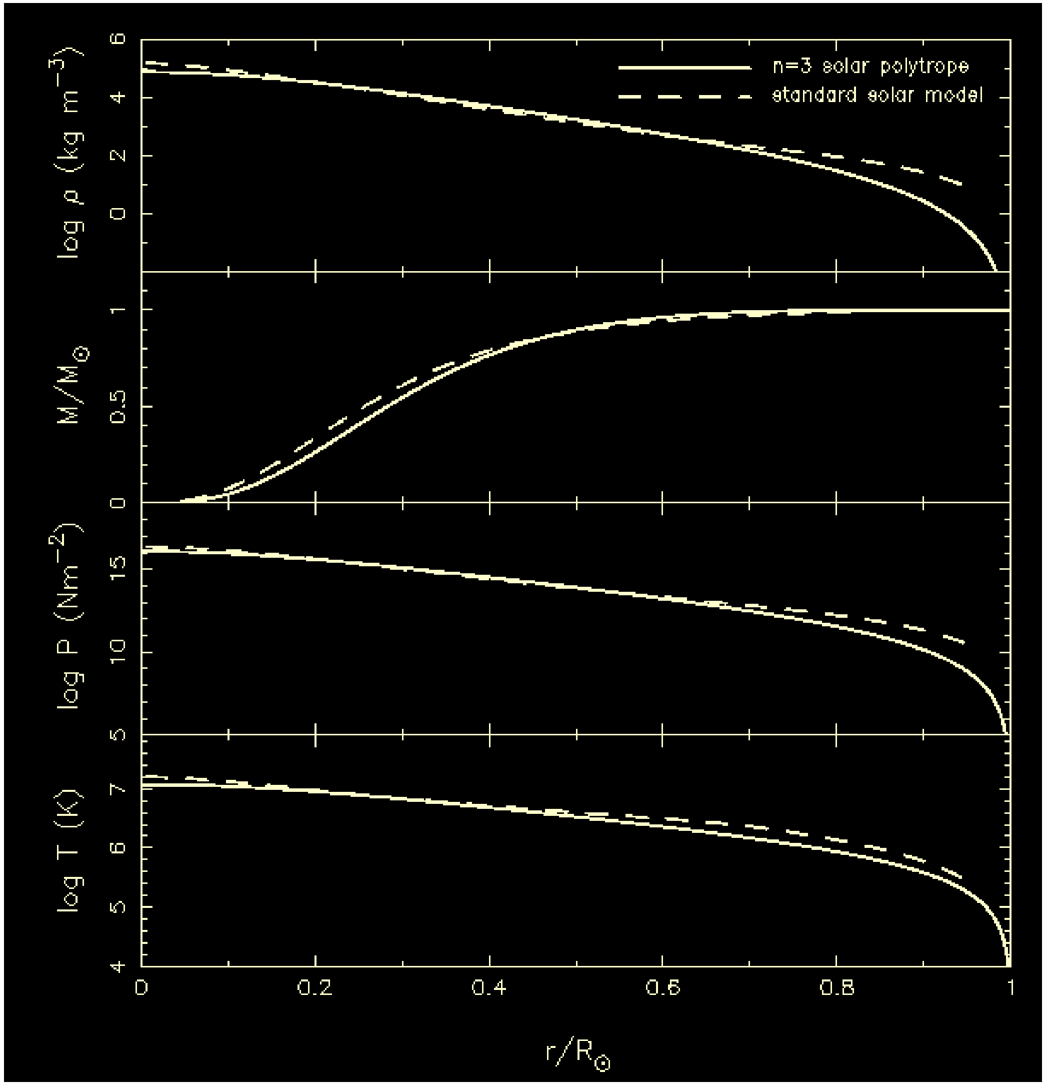
And using the Lane - Emden equation in the form: $\xi^2 \frac{d\theta}{d\xi} = -\int_0^{\xi} \xi^2 \theta^n d\xi$

$\Rightarrow$ Mass of the star as a function of radius $M = -4\pi\alpha^3 \rho_c \xi_R^2 \left[\frac{d\theta}{d\xi}\right]_{\xi = \xi_R}$

So now we assume that we know $M_\odot$ and $R_\odot$ independently, then we can find expressions for the internal structure

$$\rho_{AV} = \frac{3M_{sol}}{4\pi R_{sol}^3} = \frac{3M_{sol}}{4\pi\alpha^3 \xi_{sol}^3} = -3\rho_c \left[\frac{1}{\xi}\frac{d\theta}{d\xi}\right]_{\xi = \xi_{sol}}$$

We can use this equation to determine ρ_c which in turn allows us to determine the variation of M with ξ. This can be transformed to the variation of M with r using $r = \alpha\xi$ (assuming that we know R independently, which we do for the Sun).



Comparison of numerical solution for n=3 polytrope of the Sun versus the Standard Solar Model.

We have derived the variation of M with r

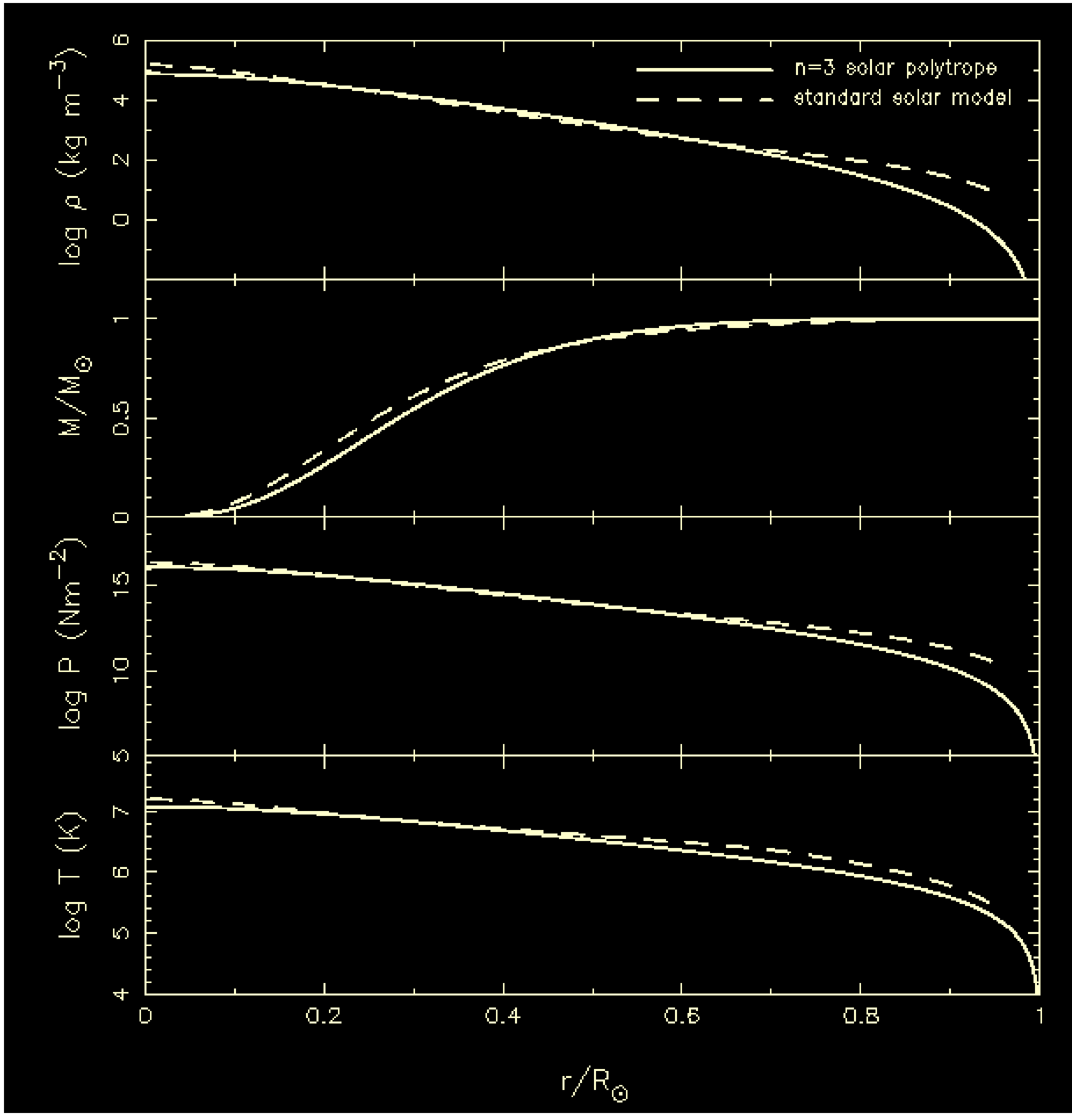
Now straightforward to determine the variation of density, pressure and temperature with r

How does the polytrope compare ?

Polytrope does remarkably well considering how simple the physics is - we have used only the mass and the radius of the Sun and an assumption about the relationship between internal pressure and density as a function of radius.

The agreement is particularly good at the core of the star:

Property	n=3 polytrope	SSM
ρ_c	7.65×10^4 kgm^{-3}	1.52×10^5 kgm^{-3}
P_c	1.25×10^{16} Nm^{-2}	2.34×10^{16} Nm^{-2}
T_c	1.18×10^7 K	1.57×10^7 K



In the outer convective regions the solutions deviate significantly

Summary

- We have defined a method to relate the internal pressure and density as a function of radius - the polytropic equation of state
- We derived the Lane-Emden equation
- We saw how this equation could be numerically integrated in general
- We compared the $n=3$ polytrope with the Standard Solar model, finding quite good agreement considering how simple the input physics was
- Now we are ready to discuss modern computational solutions of the full structure equations