Degeneracy Pressure

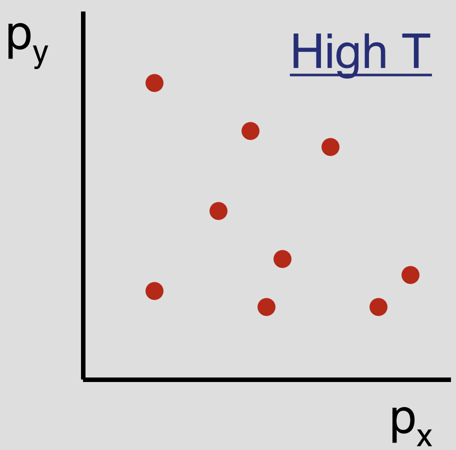
An ordinary classical gas has $P_{gas} \propto T \rightarrow 0$ as $T \rightarrow 0$

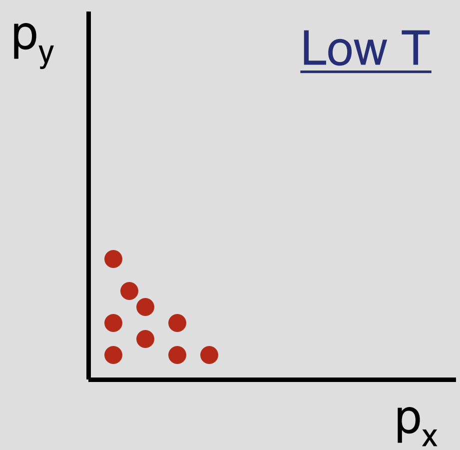
Simultaneously, the mean speed of particles in the gas also goes to zero: $v = \sqrt{2kT/m}$

Since the momentum is given by:

$$p_x = mv_x$$
$$p_y = mv_y$$
$$p_z = mv_z$$

…if we plot the momenta of particles in a 3D space of p_x, p_y and p_z then as T decreases the particles become concentrated near the origin:





At low enough temperatures / high enough densities, the concentration of particles with similar (low) momenta would violate the *Pauli exclusion principle:*

No two electrons can occupy the same quantum state

i.e. have the same momentum, spin, and location.

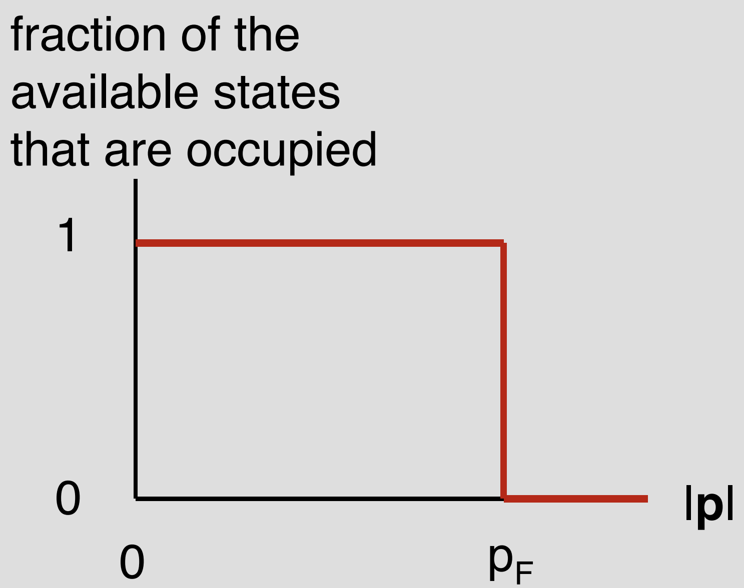
To avoid violating the exclusion principle, electrons in a dense, cold gas must have **larger** momenta than we would predict classically. Since the pressure is given by:

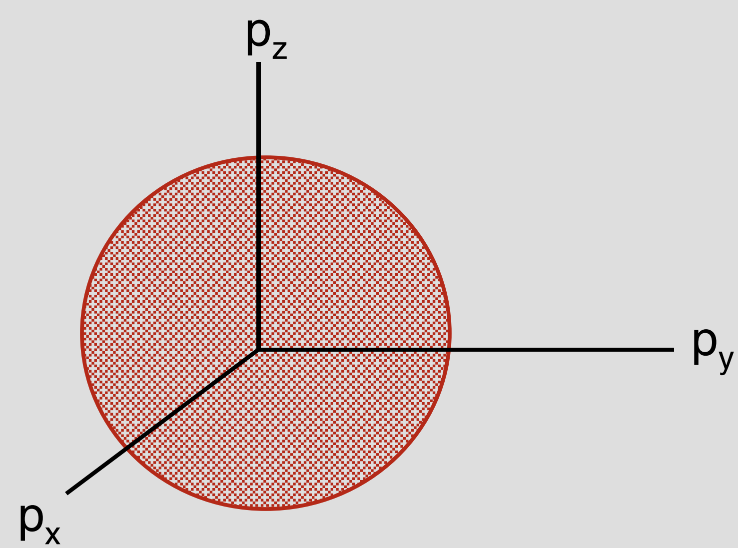
$$P = \frac{1}{3}\int_0 vpn(p)dp$$

…where n(p)dp is the number of particles with momentum between p and p+dp

Larger momentum means higher pressure. This quantum mechanical source of pressure is **degeneracy pressure.**

Simplest case to consider is a gas of electrons (plus protons, but we can ignore those for now) at **zero** temperature. In this case all the quantum states are occupied up to some maximum momentum, but no states with higher |**p**| are occupied:





Since there are three components to the momentum, number of different states with absolute value of **p** between p and p + dp is proportional to $4\pi p^2 dp$

Using the quantum mechanical result for how close together quantum states can be (e.g. textbook Chapter 3), find that the number of electrons, per unit volume, with momenta in the interval between p and p+dp is:

$$n_e(p)dp = \frac{2}{h^3} 4\pi p^2 dp \qquad p \leq p_F$$
$$= 0 \qquad p > p_F$$

Total number of electrons per unit volume is obtained by integrating over all possible momenta:

$$n_e = \int_0^\infty n_e(p)dp = \frac{8\pi}{h^3} \int_0^{p_F} p^2 dp = \frac{8\pi p_F^3}{3h^3}$$

Rearranging this expression gives the maximum (or Fermi) momentum:

$$p_F = \left(\frac{3h^3 n_e}{8\pi} \right)^{1/3}$$

Calculating the pressure

Use the formula previously derived (c.f. textbook 3.1) to calculate the pressure of a degenerate gas of electrons:

$$P = \frac{1}{3}\int_0^\infty v p n(p) dp$$

$$n_e(p)dp = \frac{2}{h^3} 4\pi p^2 dp \qquad p \le p_F$$
$$= 0 \qquad p > p_F$$

For *non-relativistic* electrons, $v = p / m_e$, where m_e is the electron mass

Result: $$P_{\deg} = \frac{8\pi}{15 m_e h^3} p_F^5 = \underbrace{\frac{h^2}{20 m_e}\left(\frac{3}{\pi}\right)^{2/3}}_{\text{fundamental constants}} n_e^{5/3}$$

density

Degeneracy pressure: $P_{\text{deg}} = K_1 \rho^{5/3}$

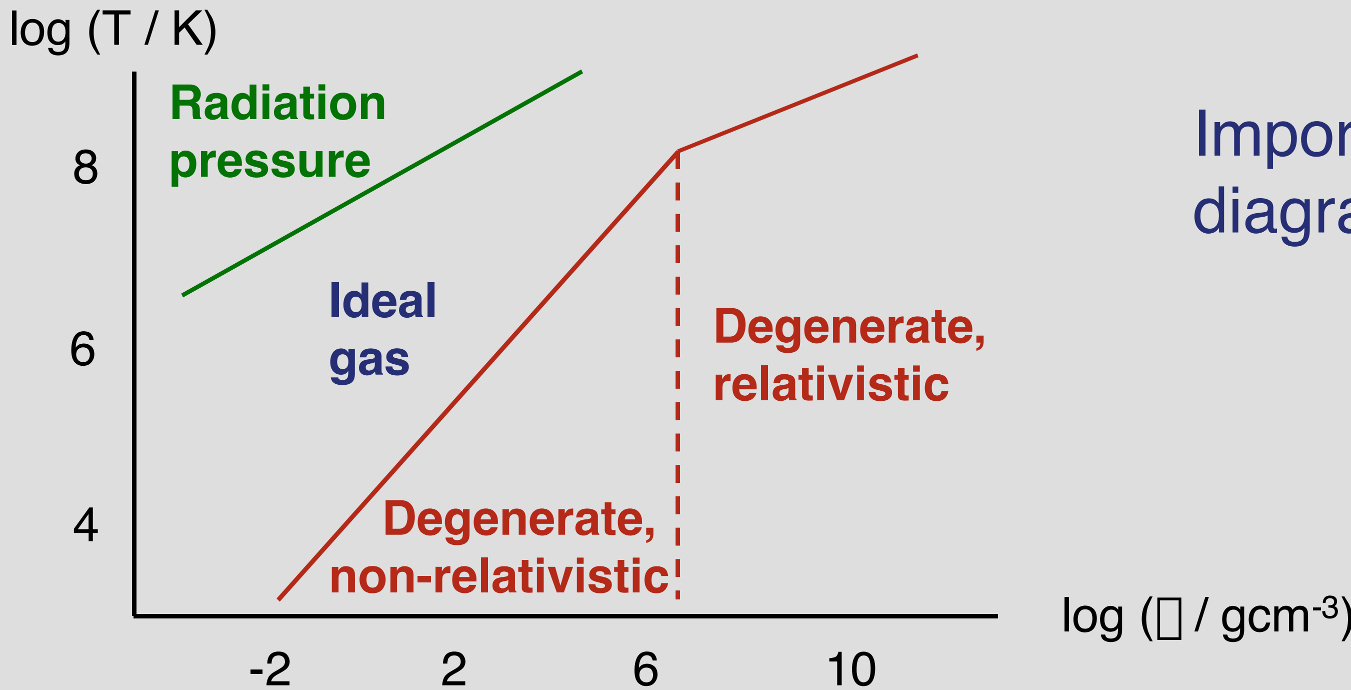
- Scales with density as $\rho^{5/3}$ provided that the electrons remain non-relativistic (speeds $v \ll c$)
- Does not depend upon temperature for low enough T
- Depends upon composition via the relation between n_e and ρ

As ρ (and n_e) becomes larger, $p_F = m_e v$ increases. When $v \sim c$, assumption that electrons are non-relativistic breaks down. Since v can never exceed c, pressure we have calculated is an *overestimate* for this case. If we replace v by c in the pressure integral, get instead:

$$P_{\text{deg}} = K_2 \rho^{4/3}$$

equation of state for relativistic degenerate matter, which applies at high density. This is a \`softer' equation of state, since P rises more slowly with increasing density than for the non-relativistic case.

When do the different pressures matter?



Important diagram!

Different types of star occupy different portions of the plane:

- **Solar-type stars** - ideal gas throughout
- **Massive stars** - radiation pressure
- **White dwarfs** - non-relativistic degeneracy pressure

Relativistic degeneracy implies an unstable equation of state, so no stable stars in that part of the plane.